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ON L^p MARKOV TYPE INEQUALITY FOR SOME CUSPIDAL DOMAINS

TOMASZ BEBEROK

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Abstract. The purpose of this paper is to study a Markov type inequality for algebraic polynomials in L^p norm on two-dimensional cuspidal domains.

1. Introduction

In the space $\mathbb{R}^d$ we consider the Euclidean norm: $|\mathbf{x}| := \sqrt{|x_1|^2 + \cdots + |x_d|^2}$, where $\mathbf{x} = (x_1, \ldots, x_d)$. For a nonempty compact set $E \subset \mathbb{R}^d$, $1 \leqslant p < \infty$ and $h : E \to \mathbb{R}$ for which the pth power of the absolute value is Lebesgue integrable, we put

$$\|h\|_{L^p(E)} := \left(\int_E |h(x)|^p \, dx \right)^{1/p}.$$

If two sequences z_n and w_n of real numbers have the property that $w_n \neq 0$ and the sequence $|z_n|/|w_n|$ has finite positive limit as $n \to \infty$, we write $z_n \sim w_n$. Throughout the paper, $\mathscr{P}_n(\mathbb{R}^d)$ denotes the space of real algebraic polynomials of d variables and degree at most n and $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree n associated to parameters α, β. Moreover, $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

DEFINITION 1. Let $1 \leqslant p < \infty$. We say that a compact set $\emptyset \neq E \subset \mathbb{R}^d$ satisfies *L^p Markov type inequality* (or: is a *L^p Markov set*) if there exist $\kappa, C > 0$ such that, for each polynomial $P \in \mathscr{P}_n(\mathbb{R}^d)$ and each $1 \leqslant j \leqslant d$,

$$\left\| \frac{\partial P}{\partial x_j} \right\|_{L^p(E)} \leqslant C n^{\kappa} \|P\|_{L^p(E)}. \tag{1}$$

We denote by $B(\mathbf{a}, r) \subset \mathbb{R}^d$ the closed Euclidean ball with center $\mathbf{a}$ and radius r, and $\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ is the unit sphere. For any $r > 0$, $\mathbf{a} \in \mathbb{R}^d$ and $\mathbf{u} \in \mathbb{S}^{d-1}$ the cylinder $L_{\mathbf{a}}(r, \mathbf{u})$ with center $\mathbf{a}$, radius $r > 0$, and axis $\mathbf{u}$ is given by

$$L_{\mathbf{a}}(r, \mathbf{u}) := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x} - \mathbf{a}|^2 < r^2 + \langle \mathbf{x} - \mathbf{a}, \mathbf{u} \rangle^2\}.$$

Furthermore, $l_{\mathbf{x}}(\mathbf{u})$ will denote the line in $\mathbb{R}^d$ in direction $\mathbf{u} \in \mathbb{S}^{d-1}$ through point $\mathbf{x} \in \mathbb{R}^d$.

Following Kroó [12], we introduce a graph domain with respect to the cylinder $L_{\mathbf{a}}(r, \mathbf{u})$ and a piecewise graph domain.

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DEFINITION 2. K is called a *graph domain* with respect to the cylinder $L_{\mathbf{a}}(r,\mathbf{u})$ if for every $\mathbf{x} \in B(\mathbf{a},r)$ we have that $l_{\mathbf{x}}(\mathbf{u}) \cap K = [A_1(\mathbf{x}), A_2(\mathbf{x})]$ with $A_i(\mathbf{x})$, $i = 1,2$ being continuous for $\mathbf{x} \in B(\mathbf{a},r)$ and

$$\delta_r(\mathbf{a},\mathbf{u}) := \inf_{\mathbf{x}\in B(\mathbf{a},r)} |A_1(\mathbf{x}) - A_2(\mathbf{x})| > 0.$$

Moreover, $K \subset \mathbb{R}^d$ is a *piecewise graph domain* if it can be covered by finite number of cylinders so that K is a graph domain with respect to each of them.

Similarly to [12] $\omega_K(\cdot)$ denotes the modulus of continuity of the boundary of piecewise graph domain K which is defined as the maximum of modula of continuity of all functions $A_i(\cdot)$ involved in the corresponding finite covering by cylinders. If $\varepsilon := \varepsilon_n(K)$ is a solution of the equation

$$2n^2 \omega_K\left(\frac{\varepsilon}{n^2}\right) = 1, \quad n \in \mathbb{N},$$

then the main result of the mentioned paper of Kroó is

THEOREM 1. *Let $K \subset \mathbb{R}^d$ be a cuspidal piecewise graph domain. Then there exists a positive constant B, depending on K and on p, such that for $Q \in \mathscr{P}_n(\mathbb{R}^d)$, $n \in \mathbb{N}$,*

$$\|\nabla Q\|_{L^p(K)} \leqslant B\frac{n^2}{\varepsilon_n}\|Q\|_{L^p(K)}. \tag{2}$$

Here $\nabla Q := \max_{1\leqslant j\leqslant d}\left|\frac{\partial Q}{\partial x_j}\right|$. In particular, if K is Lipγ, $0<\gamma<1$ then

$$\|\nabla Q\|_{L^p(K)} \leqslant B n^{\frac{2}{\gamma}}\|Q\|_{L^p(K)}.$$

The above theorem is a particular result in the general problem of estimating the exponent of the growth rate (with respect to the degree n) of the best comparability constant of the semi-norm $\|\nabla\cdot\|_{L^p(\Omega)}$ and the norm $\|\cdot\|_{L^p(\Omega)}$ acting on the space $\mathscr{P}_n(\mathbb{R}^d)$ for a given compact set Ω. More precisely, the *Markov exponent in L^p-norm of a L^p Markov set K* is defined as the infimum of l as l ranges over all positive numbers such that there exists a constant $C>0$, independent of n, with the property that L^p Markov type inequality (1) holds (with l and C), which we denote by $\mu_p(K)$.

The notion of Markov exponent (in the supremum norm) appears first in [5]. The Markov exponent has many interesting applications in approximation theory, constructive function theory and in analysis (for instance, to Sobolev inequalities or Whitney-type extension problems see [7], [16] and [17]). It is known that $\mu_p(K) \geqslant 2$ for every compact set $K \subset \mathbb{R}^d$ with nonempty interior such that $K = \overline{\operatorname{int} K}$. In [8] it is proved that if K is a locally Lipschitz compact subsets of $\mathbb{R}^d$, then $\mu_p(K) = 2$. See also [3], [10], [9] and [15]. In the case of cuspidal domains, see [11] and [13]. Markov's inequality and its various generalizations were studied in a large number of papers, it is beyond the scope of this paper to give a complete bibliography, an extensive survey of the results is given in [6], [14], [19] and [18].

Theorem 1 gives in general, the best possible Markov type upper bound in L^p, $1 \leqslant p < \infty$ norm for cuspidal graph domains. Indeed, we can show that it is attained for a large family of cuspidal $Lip\gamma$, $0 < \gamma < 1$ domains. If we assume that the domain is imbedded in an affine image of the l_γ ball with one of its vertices being on the boundary ∂K of K. Then we have $\mu_p(K) = \frac{2}{\gamma}$, see [12]. A natural question is whether the upper bound (2) is best possible for cuspidal graph domains that cannot be embedded in an affine image of the l_γ ball. Some special cases of such domains are noteworthy. Below we give a list of examples.

- $\{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : 0 < x \leqslant a, \quad 0 \leqslant y \leqslant x^k \ln(-\ln x)\}$,
- $\{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : 0 < x \leqslant 1, \quad 0 \leqslant y \leqslant -x^k \ln x\}$,
- $\{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : 0 < x \leqslant 1, \quad 0 \leqslant y \leqslant x^k(-\ln x)^c\}$,
- $\{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : 0 < x \leqslant 1, \quad 0 \leqslant y \leqslant x^{k+1} \ln^*(1/x)\}$,
- $\{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : 0 < x \leqslant a, \quad 0 \leqslant y \leqslant x^k \exp((-\ln x)^c(\ln(-\ln x))^{1-c})\}$.

Here $k > 1$, $0 < a \leqslant 1/e$, $0 < c < 1$ and

$$\ln^*(x) := \begin{cases} 0, & \text{if } x \leqslant 1 \\ 1 + \ln^*(\ln x), & \text{if } x > 1 \end{cases}.$$

We note that the domains above are related to the function classes that are frequently encountered in algorithm runtime analysis. One of the purposes of this note is to show that, if $d = 2$, the factor $\frac{n^2}{\varepsilon_n}$ is best possible for larger class of domains (including the above domains) then $Lip\gamma$. Another goal is to prove that for every sequence $\{\varepsilon_n\}$, satisfying certain properties, there exist a compact set $D \subset \mathbb{R}^2$, a constant $M > 0$ and a sequence of polynomials P_n such that

$$\frac{\|\nabla P_n\|_{L^p(D)}}{\|P_n\|_{L^p(D)}} \sim \frac{n^2}{\varepsilon_n} \quad \text{and} \quad \|\nabla Q\|_{L^p(D)} \leqslant M \frac{n^2}{\varepsilon_n} \|Q\|_{L^p(D)}$$

for any real algebraic polynomial Q of two variables and degree at most n. Moreover, for every $\iota \geqslant 1$ and $1 \leqslant p < \infty$ we give an example of connected compact subset E_ι of $\mathbb{R}^2$ such that $\mu_p(E_\iota) = 2\iota$ and the inequality (1) does not hold with the exponent $\mu_p(E_\iota)$.

2. Index of convexity

Let $\phi_r(t) := t^{1/r}$, where $t, r > 0$ throughout this section.

DEFINITION 3. Let $-\infty < a < b < \infty$. Let $f : (a,b) \to (0,\infty)$ be a convex function such that there exists a positive constant r so that $(f)^{\frac{1}{r}}$ is concave. The *index of convexity* of f is defined by

$$I_{conv}(f) := \inf\{r > 0 : \phi_r \circ f \text{ is concave}\}.$$

The following remark shows why we do not consider convex functions with negative index of convexity.

REMARK 1. Let $f:(a,b)\to(0,\infty)$ be a convex and increasing function. Assume further that $\lim\limits_{x\to a^+} f(x)=0$. Then, for every $r>0$, $(f)^{-\frac{1}{r}}$ cannot be concave.

REMARK 2. Let $f:(a,b)\to\mathbb{R}$ and $g:(a,b)\to\mathbb{R}$ where $f((a,b))\subseteq(c,d)$. It is known that the composite function $g\circ f$ is convex on (a,b) when f and g are convex and g is increasing. Hence, the index of convexity cannot be smaller then 1.

EXAMPLE 1. Let $k>1$, $0<c<1$ and $b>0$. Let

- $f_1:(0,b)\to\mathbb{R},\quad f_1(x):=x^k\ln(-\ln x)$,
- $f_2:(0,b)\to\mathbb{R},\quad f_2(x):=x^k(-\ln x)^c$.

Then, for sufficiently small b, we have $I_{conv}(f_1)=I_{conv}(f_2)=k$.

Proof. Fix $k>1$. It is now a tedious computation (or a task for a computer algebra package) to check that there exists $b>0$ such that

$$(f_1)''\geqslant 0 \quad\text{and}\quad (\phi_{k+a}\circ f_1)''\leqslant 0 \quad\text{on}\quad (0,b)$$

for every $a>0$. Hence $I_{conv}(f_1)=k$.

Now fix $0<c<1$ and consider f_2. Direct calculus computations lead to

$$(\phi_{k+a}\circ f_2)''(x)=-(\phi_{k+a}\circ f_2)(x)\frac{c(k-a)\ln\left(\frac{1}{x}\right)+c(a-c+k)+ak\ln^2\left(\frac{1}{x}\right)}{x^2(a+k)^2\ln^2\left(\frac{1}{x}\right)}.$$

Therefore there exists $b>0$ such that $(\phi_{k+a}\circ f_2)''\leqslant 0$ on $(0,b)$ for all $a>0$. Thus $I_{conv}(f_2)=k$. $\square$

The following result will be useful for the proof of Theorem 2 in the next section.

PROPOSITION 1. *Let $f:(0,b)\to(0,\infty)$ be a convex function. Suppose that $I_{conv}(f)=r$. If $\lim\limits_{x\to 0^+} f(x)=0$, then*

$$(r+a)f(x)\geqslant xf'(x)$$

for every $a>0$ and $0<x<b$.

Proof. First, we note that $(r+a)(\phi_{r+a}\circ f)'=(\phi_{r+a-1}\circ f)f'$. Since $I_{conv}(f)=r$, it is clear that $\phi_{r+a}\circ f$ is a concave function for every $a>0$. Therefore

$$(\phi_{r+a}\circ f)(y)\leqslant(\phi_{r+a}\circ f)(x)+(\phi_{r+a-1}\circ f)(x)f'(x)\frac{(y-x)}{r+a}.$$

Now the result follows from the continuity of ϕ_{r+a} and the following equality

$$\frac{\phi_{r+a}\circ f}{\phi_{r+a-1}\circ f}=f. \quad \square$$

Let us now return to the L^p Markov type inequality to explain why we do not consider convex functions $f:(a,b)\to(0,\infty)$ with the property that for each $r>0$ there exists $c\in(a,b)$ so that $(f)^{\frac{1}{r}}$ is convex on (a,c).

PROPOSITION 2. *Let $E:=\{(x,y)\in\mathbb{R}^2: 0<y\leqslant e^{-\frac{1}{x}}, 0<x\leqslant 1\}\cup\{(0,0)\}$ (see [21]). Then, for $1\leqslant p<\infty$, E does not satisfy L^p Markov type inequality (1).*

Proof. If we consider the polynomials $P_k(x,y)=y(1-x)^k$ for $k=1,2,\dots$, then

$$\mathrm{e}_k:=\frac{\int_E|\frac{\partial P_k}{\partial y}(x,y)|^p\,dxdy}{\int_E|P_k(x,y)|^p\,dxdy}=\frac{\int_0^1 e^{-\frac{1}{x}}(1-x)^{kp}\,dx}{\frac{1}{p+1}\int_0^1 e^{-(p+1)\frac{1}{x}}(1-x)^{kp}\,dx}.$$

It is easy to derive that

$$\max_{x\in[0,1]}\{e^{-(p+1)\frac{1}{x}}(1-x)^{kp}\}=e^{-(p+1)\frac{1}{a_k(p)}}(1-a_k(p))^{kp},$$

where $a_k(p)=\frac{\sqrt{1+p}\sqrt{1+p+4kp}-p-1}{2kp}$. Using integration by parts, we obtain

$$\int_0^1 e^{-\frac{1}{x}}(1-x)^{kp}\,dx\geqslant\int_{a_k(p)}^1 e^{-\frac{1}{x}}(1-x)^{kp}\,dx\geqslant e^{-\frac{1}{a_k(p)}}\frac{(1-a_k(p))^{kp+1}}{kp+1}.$$

Therefore,

$$\mathrm{e}_k\geqslant\frac{p+1}{kp+1}e^{\frac{p}{a_k(p)}}(1-a_k(p)). \tag{3}$$

Now we observe that

$$\lim_{k\to\infty}a_k(p)\sqrt{k}=\left(\frac{p}{p+1}\right)^{-1/2}. \tag{4}$$

Combining (3) and (4) leads to

$$\lim_{k\to\infty}k^{-r}\mathrm{e}_k=\infty$$

for every $r>0$. Thus E cannot be L^p Markov set. $\square$

3. A lower bound in L^p Markov type inequality

For a given point $\mathbf{a} \in \mathbb{R}^2$ and a line $l_{\mathbf{b}}(\mathbf{u}) \subset \mathbb{R}^2$, $S_{l_{\mathbf{b}}(\mathbf{u})}(\mathbf{a})$ stands for the point that is symmetric to the point $\mathbf{a}$ with respect to the line $l_{\mathbf{b}}(\mathbf{u})$. The point $\pi_{l_{\mathbf{b}}(\mathbf{u})}(\mathbf{a}) \in \mathbb{R}^2$ is the orthogonal projection of the point $\mathbf{a}$ onto the line $l_{\mathbf{b}}(\mathbf{u})$ i.e., $\pi_{l_{\mathbf{b}}(\mathbf{u})}(\mathbf{a}) := l_{\mathbf{b}}(\mathbf{u}) \cap l_{\mathbf{a}}(\mathbf{w})$, where $\mathbf{w} \perp \mathbf{u}$.

Let $K \subset \mathbb{R}^2$ be a piecewise graph domain. Suppose that $\mathbf{z} \in K$ is one of the strongest cuspidal point of K i.e., there exists a cylinder $L_{\mathbf{a}}(r,\mathbf{u})$ such that K is a graph domain with respect to it, $\mathbf{z} = A_2(\mathbf{b})$ for some $\mathbf{b} \in B(\mathbf{a},r)$ and for all sufficiently large n,

$$\max\left\{|\mathbf{z} - A_2(\mathbf{g_n})| : \mathbf{g_n} \in B(\mathbf{a},r), |\mathbf{b} - \mathbf{g_n}| \leqslant \frac{\varepsilon_n}{n^2}\right\} = \frac{1}{2n^2}.$$

Let $\mathbf{w} \in \mathbb{S}^1$, $\mathbf{w} \perp \mathbf{u}$. We say that $\mathbf{z}$ is regular if there exist $\mathbf{o} \in l_{\mathbf{b}}(\mathbf{u})$, and a function $f : [\mathbf{o},\mathbf{z}] \to \mathbb{R}^2$ such that $f(\mathbf{z}) = \mathbf{z}$,

$$\begin{aligned}
&[f(\mathbf{x}), \pi_{l_{\mathbf{z}}(\mathbf{u})}(f(\mathbf{x}))] \subset K \cap l_{\mathbf{x}}(\mathbf{w}) \subset [f(\mathbf{x}), S_{l_{\mathbf{z}}(\mathbf{u})}(f(\mathbf{x}))] \quad \text{for all } \mathbf{x} \in [\mathbf{o},\mathbf{z}],\\
&d(t) := \operatorname{dist}(l_{f((1-t)\mathbf{o}+t\mathbf{z})}(\mathbf{u}), l_{\mathbf{z}}(\mathbf{u})) \quad \text{is convex on the interval } (0,1) \text{ and}\\
&I_{conv}(d) = r \quad \text{for some } r.
\end{aligned}$$

THEOREM 2. *Let $K \subset \mathbb{R}^2$ be a piecewise graph domain. Suppose that one of the strongest cuspidal point of K is regular. If ε_n is a solution of the equation $2n^2\omega_K\left(\frac{\varepsilon_n}{n^2}\right) = 1$, $n \in \mathbb{N}$, then there exist $\Psi > 0$ and a sequence of polynomials $P_n \in \mathscr{P}_n(\mathbb{R}^2)$ such that*

$$\|\nabla P_n\|_{L^p(K)} \geqslant \Psi \frac{n^2}{\varepsilon_n} \|P_n\|_{L^p(K)}.$$

Proof. Without loss of generality we may suppose that $K \subset [0,1] \times [-1,1]$, $\mathbf{v} = (1,0)$ is the strongest cuspidal point of K and $\{(x,y) \in \mathbb{R}^2 : \eta \leqslant x \leqslant 1, 0 \leqslant y \leqslant f(x)\} \subset K \subset [0,\eta] \times [-1,1] \cup \{(x,y) \in \mathbb{R}^2 : \eta \leqslant x \leqslant 1, -f(x) \leqslant y \leqslant f(x)\}$ for some $0 < \eta < 1$ and a convex function $f : [\eta,1] \to \mathbb{R}$ with the property that $f(1) = 0$ and $I_{conv}(f) < \infty$. (This can be achieved by shifting the point $\mathbf{o}$ into the origin, rotating around the origin and dilating the space by a proper constant.) Let $x_n = f^{-1}(\varepsilon_n/n^2)$ for all sufficiently large n. Then

$$\begin{aligned}
\iint_K |yP_n^{(\alpha,\beta)}(x)|^p\,dxdy \leqslant \int_0^\eta \int_{-1}^1 |yP_n^{(\alpha,\beta)}(x)|^p\,dydx + \int_\eta^{x_n} \int_{-f(x)}^{f(x)} |yP_n^{(\alpha,\beta)}(x)|^p\,dydx\\
+ \int_{x_n}^1 \int_{-f(x)}^{f(x)} |yP_n^{(\alpha,\beta)}(x)|^p\,dydx. \qquad (5)
\end{aligned}$$

Our plan is to obtain the estimates of each integral on the right side. We start with the last one. It is clear that

$$\int_{x_n}^{1}\int_{-f(x)}^{f(x)}|yP_n^{(\alpha,\beta)}(x)|^p\,dxdy=\frac{2}{p+1}\int_{x_n}^{1}(f(x))^{p+1}|P_n^{(\alpha,\beta)}(x)|^p\,dx$$
$$\leqslant\frac{2}{p+1}\left(\frac{\varepsilon_n}{n^2}\right)^{p+1}\int_{x_n}^{1}|P_n^{(\alpha,\beta)}(x)|^p\,dx. \tag{6}$$

Then the change of variable $x=\cos\theta$ gives us

$$\int_{x_n}^{1}|P_n^{(\alpha,\beta)}(x)|^p\,dx=\int_0^{u_n}|P_n^{(\alpha,\beta)}(\cos\theta)|^p\sin\theta\,d\theta.$$

Here $u_n=\arccos x_n$. Since $\mathbf{v}=(1,0)$ is the strongest cuspidal point of K, it follows that

$$\omega_K\left(\frac{\varepsilon_n}{n^2}\right)=\max\left\{|(f^{-1}(y),y)-(1,0)|:0\leqslant y\leqslant\frac{\varepsilon_n}{n^2}\right\}.$$

The convexity of f and the fact that $f(1)=0$ guarantee

$$\omega_K\left(\frac{\varepsilon_n}{n^2}\right)=|(x_n,f(x_n))-(1,0)|.$$

Hence, by $\omega_K\left(\frac{\varepsilon_n}{n^2}\right)=\frac{1}{2n^2}$,

$$x_n=1-\sqrt{\frac{1}{4n^4}-\frac{\varepsilon_n^2}{n^4}}.$$

By the fact that $\varepsilon_n\to 0$ there exists a natural number n_0 such that $1-\frac{1}{2n^2}\leqslant x_n\leqslant 1-\frac{1}{4n^2}$ for all $n\geqslant n_0$. Hence there exist a natural number n_1 and positive constants a,b such that $\frac{a}{n}\leqslant u_n\leqslant\frac{b}{n}$ for all $n\geqslant n_1$. Applying certain properties of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ verified in [8], (7.32.5), p. 169, we conclude that there exists a natural number n_2 so that

$$\int_0^{u_n}|P_n^{(\alpha,\beta)}(\cos\theta)|^p\sin\theta\,d\theta\leqslant Cn^{\alpha p}\int_0^{u_n}\theta\,d\theta\leqslant\frac{Cb^2}{2}n^{\alpha p-2} \tag{7}$$

for $n\geqslant n_2$ and appropriately adjusted constant C. Then by (6) and (7)

$$\int_{x_n}^{1}\int_{-f(x)}^{f(x)}|yP_n^{(\alpha,\beta)}(x)|^p\,dydx\leqslant\frac{Cb^2}{p+1}\varepsilon_n^{p+1}n^{\alpha p-2p-4} \tag{8}$$

for all sufficiently large n.

Now select $\alpha>-1$ such that $\alpha p+p/2-2>2I_{conv}(f)(p+1)$. It is easy to see that

$$\int_{\eta}^{x_n}\int_{-f(x)}^{f(x)}|yP_n^{(\alpha,\beta)}(x)|^p\,dydx=\frac{2}{p+1}\int_{\eta}^{x_n}(f(x))^{p+1}|P_n^{(\alpha,\beta)}(x)|^p\,dx. \tag{9}$$

Let $\sigma=\arccos\eta$. Proceeding similarly as before, we obtain

$$\frac{2}{p+1}\int_{\eta}^{x_n}(f(x))^{p+1}|P_n^{(\alpha,\beta)}(x)|^p\,dx\leqslant\frac{2\Lambda n^{-p/2}}{(p+1)}\int_{u_n}^{\sigma}(f(\cos\theta))^{p+1}\theta^{-\alpha p-p/2}\sin\theta\,d\theta \tag{10}$$

for appropriately adjusted constant Λ and all sufficiently large n. Since $\sin x \leqslant x$ for $x \geqslant 0$, we have

$$\frac{2}{p+1}\int_{\eta}^{x_n}(f(x))^{p+1}|P_n^{(\alpha,\beta)}(x)|^p\,dx \leqslant \frac{2\Lambda n^{-p/2}}{(p+1)}\int_{u_n}^{\sigma}(f(\cos\theta))^{p+1}\theta^{-\alpha p-p/2+1}\,d\theta. \tag{11}$$

Integration by parts gives us

$$\int_{u_n}^{\sigma}(f(\cos\theta))^{p+1}\theta^{-\alpha p-p/2+1}\,d\theta = \left[\frac{(f(\cos\theta))^{p+1}\theta^{-\alpha p-p/2+2}}{-\alpha p-p/2+2}\right]_{u_n}^{\sigma} + \int_{u_n}^{\sigma}\frac{(p+1)(f(\cos\theta))^{p}\theta^{-\alpha p-p/2+2}}{-\alpha p-p/2+2}f'(\cos\theta)\sin\theta\,d\theta. \tag{12}$$

If $-1 \leqslant x \leqslant 1$, then $\sqrt{1-x^2}\arccos x \leqslant 2(1-x)$. Hence

$$-\lambda f'(\cos\lambda)\sin\lambda \leqslant -2(1-\cos\lambda)f'(\cos\lambda) \tag{13}$$

whenever $\lambda \in (0,\sigma]$. By Proposition 1, for each $\delta > 0$, we have

$$(I_{conv}(f)+\delta)f(x) \geqslant -f'(x)(1-x) \tag{14}$$

for all $x \in [\eta, 1]$. Then, by (13) and (14),

$$\int_{u_n}^{\sigma}\frac{(p+1)(f(\cos\theta))^{p}\theta^{-\alpha p-p/2+2}}{-\alpha p-p/2+2}f'(\cos\theta)\sin\theta\,d\theta \leqslant \int_{u_n}^{\sigma}\frac{2(p+1)(I_{conv}(f)+\delta)(f(\cos\theta))^{p+1}\theta^{-\alpha p-p/2+1}}{\alpha p+p/2-2}\,d\theta. \tag{15}$$

Thus, by (12) and (15),

$$\int_{u_n}^{\sigma}(f(\cos\theta))^{p+1}\theta^{-\alpha p-p/2+1}\,d\theta \leqslant \frac{(f(\cos u_n))^{p+1}u_n^{-\alpha p-p/2+2}}{\alpha p+p/2-2-2(p+1)(I_{conv}(f)+\delta)} \leqslant \frac{\varepsilon_n^{p+1}a^{-\alpha p-p/2+2}n^{\alpha p+p/2-2p-4}}{\alpha p+p/2-2-2(p+1)(I_{conv}(f)+\delta)}$$

whenever $\alpha p+p/2-2 > 2(p+1)(I_{conv}(f)+\delta)$. Together with (9), (10) and (11), this last estimate implies that for every $\alpha > -1$ such that $\alpha p+p/2-2 > 2(p+1)I_{conv}(f)$ there exists a constant $C_1 > 0$, independent of n, with

$$\int_{\eta}^{x_n}\int_{-f(x)}^{f(x)}|yP_n^{(\alpha,\beta)}(x)|^p\,dydx \leqslant C_1\varepsilon_n^{p+1}n^{\alpha p-2p-4}. \tag{16}$$

It now remains to prove that there exists a positive constant C_2, independent of n, such that

$$\int_{0}^{\eta}\int_{-1}^{1}|yP_n^{(\alpha,\beta)}(x)|^p\,dydx \leqslant C_2\varepsilon_n^{p+1}n^{\alpha p-2p-4}.$$

It is easy to verify that

$$\int_0^\eta \int_{-1}^1 |yP_n^{(\alpha,\beta)}(x)|^p \, dydx = \frac{2}{p+1} \int_0^\eta |P_n^{(\alpha,\beta)}(x)|^p \, dx.$$

In a similar way as before, we can show that

$$\int_0^\eta |P_n^{(\alpha,\beta)}(x)|^p \, dx \leqslant \Lambda_1 n^{-p/2} \int_\sigma^{\pi/2} \theta^{-\alpha p - p/2 + 1} \, d\theta$$

for appropriately adjusted constant Λ_1 and all sufficiently large n. Hence

$$\int_0^\eta \int_{-1}^1 |yP_n^{(\alpha,\beta)}(x)|^p \, dydx \leqslant \frac{2\Lambda_1 n^{-p/2} \sigma^{-\alpha p - p/2 + 2}}{(\alpha p + p/2 - 2)(p+1)}. \tag{17}$$

Now let $f(\eta) := w$. For every $\delta > 0$ define $h_\delta(x) := (1-x)^{I_{conv}(f)+\delta} \frac{w}{(1-\eta)^{I_{conv}(f)+\delta}}$. Then $f(\eta) = h_\delta(\eta)$ and $f(1) = h_\delta(1)$. By the definition of $I_{conv}(f)$, $(f)^{\frac{1}{I_{conv}(f)+\delta}}$ is concave. Hence

$$(f(x))^{\frac{1}{I_{conv}(f)+\delta}} \geqslant w^{\frac{1}{I_{conv}(f)+\delta}} \frac{1-x}{1-\eta} = (h_\delta(x))^{\frac{1}{I_{conv}(f)+\delta}}$$

for all $x \in [\eta, 1]$. Thus

$$\begin{aligned} \frac{\varepsilon_n}{n^2} &= f(x_n) \geqslant f\left(1 - \frac{1}{4n^2}\right) \geqslant h_\delta\left(1 - \frac{1}{4n^2}\right) \\ &\geqslant \frac{w}{(1-\eta)^{I_{conv}(f)+\delta}(2n)^{2I_{conv}(f)+2\delta}} \end{aligned} \tag{18}$$

for all sufficiently large n. Now if α is selected so that $2I_{conv}(f)(p+1) + 2 - p/2 < \alpha p$, then, by (17) and (18), there exists $C_2 > 0$ such that

$$\int_0^\eta \int_{-1}^1 |yP_n^{(\alpha,\beta)}(x)|^p \, dxdy \leqslant C_2 \varepsilon_n^{p+1} n^{\alpha p - 2p - 4} \tag{19}$$

for all $n \in \mathbb{N}$. Now let $M = Cb^2/2 + C_1 + C_2$ and use the inequalities (5), (8), (16) and (19) to obtain

$$\iint_K |yP_n^{(\alpha,\beta)}(x)|^p \, dxdy \leqslant M \varepsilon_n^{p+1} n^{\alpha p - 2p - 4}. \tag{20}$$

By our assumption on K it follows that

$$\iint_K |P_n^{(\alpha,\beta)}(x)|^p \, dxdy \geqslant \int_{1-\frac{1}{2n^2}}^1 \int_0^{f(x)} |P_n^{(\alpha,\beta)}(x)|^p \, dydx = \int_{1-\frac{1}{2n^2}}^1 f(x) |P_n^{(\alpha,\beta)}(x)|^p \, dx. \tag{21}$$

By making the change of variable $x=1-\frac{z^2}{2n^2}$, we obtain

$$\int_{1-\frac{1}{2n^2}}^{1} f(x)|P_n^{(\alpha,\beta)}(x)|^p\,dx=\frac{1}{n^2}\int_0^1 zf(g_n(z))\,|P_n^{(\alpha,\beta)}(g_n(z))\,|^p\,dz, \tag{22}$$

where $g_n(z)=1-\frac{z^2}{2n^2}$. Again certain properties of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ play a role. By the formula of Mehler-Heine type (see [20], Theorem 8.1.1.)

$$|P_n^{(\alpha,\beta)}(g_n(z))\,|^p \geqslant \frac{n^{\alpha p}}{4^p}\left[4\left(\frac{z}{2}\right)^{-\alpha}J_\alpha(z)-\frac{1}{\Gamma(\alpha+2)}\right]^p$$

for all sufficiently large n. Here $J_\alpha(z)$ is the Bessel functions of the first kind. Since

$$\min_{z\in[0,1]}\{(z/2)^{-\alpha}J_\alpha(z)\}\geqslant \min_{z\in[0,1]}\left\{\frac{1}{\Gamma(\alpha+1)}-\frac{z^2}{4\Gamma(\alpha+2)}\right\}=\frac{4\alpha+3}{4\Gamma(\alpha+2)},$$

we have

$$\frac{1}{n^2}\int_0^1 zf(g_n(z))\left|P_n^{(\alpha,\beta)}(g_n(z))\right|^p dz \geqslant \left(\frac{4\alpha+2}{4\Gamma(\alpha+2)}\right)^p n^{\alpha p-2}\int_0^1 zf(g_n(z))\,dz. \tag{23}$$

Applying integration by parts yields

$$\int_0^1 zf(g_n(z))\,dz=\left[\frac{1}{2}z^2f(g_n(z))\right]_0^1+\frac{1}{2n^2}\int_0^1 z^3f'(g_n(z))\,dz. \tag{24}$$

From the inequality (14), it follows that, for all $\delta>0$ and sufficiently large n, it must be that

$$\frac{z^2}{2n^2}f'(1-\frac{z^2}{2n^2})\geqslant -(I+\delta)f\left(1-\frac{z^2}{2n^2}\right).$$

Hence

$$\frac{1}{2n^2}\int_0^1 z^3f'(g_n(z))\,dz\geqslant -(I+\delta)\int_0^1 zf(g_n(z))\,dz. \tag{25}$$

Thus, by (24), (25), and by the fact that $f(1)=f(g_n(0))=0$, we have

$$(I_{conv}(f)+1+\delta)\int_0^1 zf(g_n(z))\,dz\geqslant \frac{1}{2}f(g_n(1)). \tag{26}$$

Since f is convex and nonnegative on $[\eta,1]$ with the property that $f(1)=0$,

$$\eta\leqslant x\leqslant y\leqslant 1\Longrightarrow f(x)\geqslant f(y).$$

Therefore

$$\frac{1}{2}f(g_n(1))=f\left(1-\frac{1}{2n^2}\right)\geqslant \frac{1}{2}f(x_n)=\frac{\varepsilon_n}{2n^2} \tag{27}$$

for all sufficiently large n. If n is large enough, then by (21), (22), (23), (26) and (27) there exists a positive constant Υ, independent of n, for which

$$\iint_K |P_n^{(\alpha,\beta)}(x)|^p \, dxdy \geqslant \Upsilon \varepsilon_n n^{\alpha p-4}. \tag{28}$$

Finally, using the inequalities (20) and (28), we obtain

$$\iint_K |P_n^{(\alpha,\beta)}(x)|^p \, dxdy \geqslant \frac{\Upsilon n^{2p}}{M\varepsilon_n^p} \iint_K |yP_n^{(\alpha,\beta)}(x)|^p \, dxdy.$$

This completes the proof. □

4. An application of Theorem 2

We will now focus on the application of the above theorem. Let $0<\eta<1$. Let $f:[\eta,1]\to\mathbb{R}$ be a strictly convex function with the property that $f(1)=f'(1)=0$ and $I_{conv}(f)=k$. Then $D:=\{(x,y)\in\mathbb{R}^2 : \eta\leqslant x\leqslant 1, 0\leqslant y\leqslant f(x)\}\cup\{(x,y)\in\mathbb{R}^2 : 0\leqslant x<\eta, 0\leqslant y\leqslant f(\eta)\}$ is a graph domain with respect to the cylinder $L_{(0,\frac{\eta}{2})}(\frac{\eta}{2},(1,0))$. Moreover,

$$l_{(x_1,x_2)}((1,0))\cap D=\{(x,y)\in\mathbb{R}^2 : y=x_2, 0\leqslant x\leqslant f^{-1}(x_2)\}=[(0,x_2),(f^{-1}(x_2),x_2)]$$

for every $(x_1,x_2)\in B((0,\frac{\eta}{2}),\frac{\eta}{2})$. Hence, by the definition of $\omega_D(t)$,

$$\omega_D(t)=\max\left\{|(f^{-1}(x_2),x_2)-(f^{-1}(y_2),y_2)| : 0\leqslant x_2,y_2\leqslant\eta,\ |x_2-y_2|\leqslant t\right\}.$$

By the convexity of f and the assumption that $f(1)=0$, we can write

$$\omega_D(t)=\max\left\{|(f^{-1}(s),s)-(1,0)| : 0\leqslant s\leqslant\min\{\eta,t\}\right\}.$$

Since f is strictly convex, we conclude that

$$\omega_D(t)=\max\left\{\sqrt{(f^{-1}(s)-1)^2+s^2} : s=\min\{\eta,t\}\right\}.$$

Since $f'(1)=0$, we have

$$\lim_{t\to 0}\frac{\omega_D(t)}{t}=+\infty.$$

Hence if $\omega_D(\frac{\varepsilon_n}{n^2})=\frac{1}{2n^2}$ then $\varepsilon_n\to 0$ when n goes to $+\infty$. Therefore, for n large enough

$$\omega_D\left(\frac{\varepsilon_n}{n^2}\right)=\sqrt{\left(f^{-1}\left(\frac{\varepsilon_n}{n^2}\right)-1\right)^2+\left(\frac{\varepsilon_n}{n^2}\right)^2}.$$

Now assume that ε_n is the solution of the equation $\omega_D(\frac{\varepsilon_n}{n^2})=\frac{1}{2n^2}$. Thus, for n large enough,

$$\frac{\varepsilon_n}{n^2}=f\left(1-\sqrt{\frac{1}{4n^4}-\frac{\varepsilon_n^2}{4n^4}}\right) \tag{29}$$

and $0 < 2\varepsilon_n < 1$. By the fact that f is strictly convex and $f(0) = 0$, we have

$$f\left(1-\sqrt{\frac{1}{4n^4}-\frac{1}{16n^4}}\right) \leqslant f\left(1-\sqrt{\frac{1}{4n^4}-\frac{\varepsilon_n^2}{4n^4}}\right) \leqslant f\left(1-\frac{1}{2n^2}\right). \tag{30}$$

On the other hand, by $I_{conv}(f) = k$, the function $f^{\frac{1}{k+\delta}}$ is concave for every $\delta > 0$. Hence if $0 \leqslant c \leqslant 1$ then

$$f^{\frac{1}{k+\delta}}(1-\frac{c}{2n^2}) \geqslant c f^{\frac{1}{k+\delta}}\left(1-\frac{1}{2n^2}\right).$$

Thus

$$f\left(1-\sqrt{\frac{1}{4n^4}-\frac{1}{16n^4}}\right) \geqslant \left(\frac{\sqrt{3}}{2}\right)^{k+\delta} f\left(1-\frac{1}{2n^2}\right). \tag{31}$$

By (29), (30) and Theorem 2, there exist $\Psi > 0$ and a sequence of polynomials $Q_n \in \mathscr{P}_n(\mathbb{R}^2)$ such that

$$\|\nabla Q_n\|_{L^p(D)} \geqslant \frac{\Psi}{f(1-\frac{1}{2n^2})}\|Q_n\|_{L^p(D)}.$$

Furthermore, by Theorem 1, (29) and (31), there exists a positive constant B such that for $P \in \mathscr{P}_n(\mathbb{R}^d)$, $n \in \mathbb{N}$,

$$\|\nabla P\|_{L^p(D)} \leqslant \frac{B}{f(1-\frac{1}{2n^2})}\|P\|_{L^p(D)}.$$

EXAMPLE 2. Define a function φ_ι on the interval $[0,1]$ as follows:

$$\varphi(t) = \begin{cases} \frac{t}{1+\ln(1/t)}, & \text{if } t \in (0,1] \\ 0, & \text{for } t = 0. \end{cases}$$

Let $E_\iota = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant \varphi((1-x)^\iota)\}$. Then there exist a positive constant B_ι and a sequence of polynomials P_n such that

$$\frac{\|\nabla P_n\|_{L^p(D)}}{\|P_n\|_{L^p(D)}} \sim n^{2\iota}(1+\iota\ln(2n^2)) \quad \text{and} \quad \|\nabla Q\|_{L^p(E_\iota)} \leqslant B_\iota n^{2\iota}(1+\iota\ln(2n^2))\,\|Q\|_{L^p(E_\iota)} \tag{32}$$

for any $Q \in \mathscr{P}_n(\mathbb{R}^2)$.

Using (32), one can see that $\mu_p(E_\iota) = 2\iota$ and L^p Markov type inequality on E_ι does not hold with the exponent $\mu_p(E_\iota) = 2\iota$. This generalizes (to the L^p norm) Proposition 2.6 of [4].

5. A growth rate

Now a similar proof to that of the last theorem gives the following proposition:

PROPOSITION 3. *Let α, β, p be positive real numbers and $0 < \upsilon \leqslant 1$. Let f be a bounded real-valued function defined on the interval $[0,1]$. Suppose that $f(1) = 0$, $f\left(1 - \frac{\upsilon}{n^2}\right) = \frac{\varepsilon_n}{n^2}$ and there exists $0 < \eta < 1$ such that $f|_{[\eta,1]}$ is convex with the property that $I_{conv}(f|_{[\eta,1]}) < \infty$. If $\alpha p \geqslant 2I_{conv}(f|_{[\eta,1]}) + 2 - p/2$, then*

$$\int_0^1 |f(x)| \left|P_n^{(\alpha,\beta)}(x)\right|^p dx \sim \varepsilon_n n^{\alpha p - 4}. \tag{33}$$

It is worth noting that the above result provides a refinement and generalization of Theorem 7.34. from [20].

We shall show that, with suitable hypotheses, there is something like the complement of Theorem 2.

THEOREM 3. *Let $\{\varepsilon_n\}$ be a sequence of real numbers such that $0 < \varepsilon_{n+1} \leqslant \varepsilon_n$, $\lim_{n\to\infty} \varepsilon_n = 0$ and there exist constants $I > 0$ and C_n with the property that (for all n and m)*

$$\frac{\varepsilon_n}{n^2} - \frac{\varepsilon_m}{m^2} \geqslant -C_m \frac{\varepsilon_m}{2}(1/m^2 - 1/n^2), \tag{34}$$

$$\frac{\varepsilon_n}{n^2} - \frac{\varepsilon_m}{m^2} = -C_m \frac{\varepsilon_m}{2}(1/m^2 - 1/n^2) \Rightarrow C_n\varepsilon_n = C_m\varepsilon_m, \tag{35}$$

$$\lim_{n\to\infty} \frac{n^{-I}}{\varepsilon_n} = 0, \quad \text{and} \quad \sup\{C_n : n \in \mathbb{N}\} < \infty. \tag{36}$$

Then there exist a compact set $D \subset \mathbb{R}^2$, a constant $M > 0$ and a sequence of polynomials $P_n \in \mathscr{P}_n(\mathbb{R}^2)$ such that

$$\frac{\|\nabla P_n\|_{L^p(D)}}{\|P_n\|_{L^p(D)}} \sim \frac{n^2}{\varepsilon_n} \quad \text{and} \quad \|\nabla Q\|_{L^p(D)} \leqslant M\frac{n^2}{\varepsilon_n}\|Q\|_{L^p(D)}$$

for any $Q \in \mathscr{P}_n(\mathbb{R}^2)$.

Before we provide proof, we will discuss the nature of conditions (34)–(36). If we define

$$C := \left\{1 - \frac{1}{2n^2} : n \in \mathbb{N}\right\} \cup \{1\},$$

$$f : C \to \mathbb{R}, \quad f\left(1 - \frac{1}{2n^2}\right) = \frac{\varepsilon_n}{n^2}, \quad f(1) := 0,$$

$$G : C \to \mathbb{R}, \quad G\left(1 - \frac{1}{2n^2}\right) = -C_n\varepsilon_n, \quad G(1) := 0,$$

then the conditions (34) and (35) coincide with condition (C) and (CW^1) taken from [1], respectively. Thus, by Theorem 1.10 of [1], f has a convex, C^1 extension F to $\mathbb{R}$, with $F' = G$ on C, if and only if (f,G) satisfies conditions (34) and (35). Let $M := \sup\{C_n : n \in \mathbb{N}\}$. Then the conditions (34) and (36) guarantee that there exists $0 < \eta < 1$ such that, for every $x \in [\eta, 1]$,

$$MF(x) \geqslant -F'(x)(1-x). \tag{37}$$

The details for accomplishing this will be provided later. The inequality (37) is similar to (14) used in the proof of Theorem 2. If $\lim_{n\to\infty} \frac{n^{-I}}{\varepsilon_n} = 0$, then there exists a positive constant L such that

$$\frac{1}{n^I} \leqslant L\varepsilon_n \qquad (n \in \mathbb{N})$$

The above inequality can be treated as an equivalent of the inequality (18). Thus, for any sequence ε_n satisfying the assumptions of Theorem 3, there exists a C^1 convex function $F : \mathbb{R} \to \mathbb{R}$ such that $\varepsilon_n = n^2 F(1 - \frac{1}{2n^2})$.

On the other hand, if $f : [0,1] \to \mathbb{R}$ is differentiable and strictly convex with the property that $f(1) = f'(1) = 0$ then the sequence $\varepsilon_n := n^2 f(1 - \frac{1}{2n^2})$ is monotonically decreasing to zero and satisfies (34) and (35) with $C_m = \frac{-f'(1-\frac{1}{2n^2})}{n^2 f(1-\frac{1}{2n^2})}$. This follows from the fact that a differentiable function of one variable is convex on an interval if and only if its graph lies above all of its tangents:

$$f(x) - f(y) \geqslant f'(y)(x-y)$$

and the three chords inequality:

$$\frac{f(x)-f(a)}{x-a} \leqslant \frac{f(b)-f(a)}{b-a} \leqslant \frac{f(b)-f(x)}{b-x}$$

whenever $a < x < b$. If we assume, in addition, that $I_{conv}(f\big|_{(\eta,1)}) = k$ for some $0 < \eta < 1$ then, by Proposition 1,

$$\sup\{C_n : n \in \mathbb{N}\} < \infty.$$

The property $I_{conv}(f\big|_{(\eta,1)}) = k$ implies that there exists a positive constant L such that

$$\frac{1}{n^I} \leqslant L\varepsilon_n.$$

The proof is similar to that of (18). Thus, in order to give an example of a sequence ε_n that satisfies the assumptions of the above theorem, it is enough to take $n^2 f(1 - \frac{1}{2n^2}) = \varepsilon_n$ for any strictly convex function f defined on $[0,1]$ having the properties $f'(1) = f(1) = 0$ and $I_{conv}(f) < \infty$.

We now turn to the proof of Theorem 3.

Proof. In the first part of the proof, we will define the domain D and derive the formula for the modulus of continuity of the boundary of D. Let C, f, G and M be as above. Using Theorem 1.10 from [1], there exists a continuously differentiable function $F:\mathbb{R}\to\mathbb{R}$ such that F is convex and $F=f$, $F'=G$ on C. Now define

$$\tilde{f}(x) := \begin{cases} \varepsilon_1 & \text{for} \quad x\in[0,1/2] \\ F(x) & \text{for} \quad x\in(1/2,1]. \end{cases} \tag{38}$$

Since F is convex and $F(1)=0$ it follows that $\tilde{f}$ is strictly decreasing on the interval $[1/2,1]$.

If we define

$$D := \{(x,y)\in\mathbb{R}^2 : 0\leqslant x\leqslant 1,\, 0\leqslant y\leqslant \tilde{f}(x)\}, \tag{39}$$

then D is a graph domain with respect to the cylinder $L_{\mathbf{a}}(\varepsilon_1/2,\mathbf{u})$, where $\mathbf{a}=(0,\varepsilon_1/2)$ and $\mathbf{u}=(1,0)$. By the definition of $\omega_D(t)$,

$$\omega_D(t) = \max\left\{|(F^{-1}(x_2),x_2)-(F^{-1}(y_2),y_2)| : 0\leqslant x_2,y_2\leqslant\varepsilon_1,\, |x_2-y_2|\leqslant t\right\}.$$

In order to prove that

$$\omega_D(t) = \max\left\{|(F^{-1}(s),s)-(1,0)| : 0\leqslant s\leqslant \min\{\varepsilon_1,t\}\right\}, \tag{40}$$

it suffices to prove that if $\tilde{f}(y)=\tilde{f}(x_1)-\tilde{f}(x_2)$, then

$$\frac{\tilde{f}(y)}{1-y} \leqslant \frac{\tilde{f}(x_1)-\tilde{f}(x_2)}{x_2-x_1} \tag{41}$$

for any $1/2\leqslant x_1<x_2\leqslant 1$ and $1/2\leqslant y<1$. Since $\tilde{f}(y)=\tilde{f}(x_1)-\tilde{f}(x_2)$, we have $y\geqslant x_1$. If $x_1<x_2\leqslant y$, then, by the mean value theorem, there exist $\xi\in(x_1,x_2)$ and $\eta\in(y,1)$ such that

$$\frac{-\tilde{f}(y)}{1-y} = \tilde{f}'(\eta), \qquad \frac{\tilde{f}(x_2)-\tilde{f}(x_1)}{x_2-x_1} = \tilde{f}'(\xi).$$

Hence, (using the fact that differentiable function of one variable is convex on an interval if and only if its derivative is monotonically non-decreasing on that interval)

$$\frac{\tilde{f}(y)}{1-y} = -\tilde{f}'(\eta) \leqslant -\tilde{f}'(\xi) = \frac{\tilde{f}(x_1)-\tilde{f}(x_2)}{x_2-x_1}.$$

For the case $x_1\leqslant y<x_2$, let

$$S(t,r) := \frac{F(t)-F(r)}{t-r}.$$

It is known that F is convex if and only if $S(t,r)$ is monotonically non-decreasing in t, for every fixed r. Therefore

$$\frac{F(x_1)-F(x_2)}{x_1-x_2} = S(x_2,x_1) \leqslant S(1,x_1) = S(x_1,1) \leqslant S(y,1) = \frac{F(y)-F(1)}{y-1}.$$

Since $F=\tilde{f}$ on the interval $[1/2,1]$, the inequality (41) holds when $x_1 \leqslant y < x_2$.

From the formula (40) it follows that

$$\omega_D(t) = |(F^{-1}(t),t)-(1,0)| = \sqrt{(1-F^{-1}(t))^2+t^2}$$

whenever $t \leqslant \varepsilon_1$. Hence, by $F'(1)=F(1)=0$,

$$\lim_{t\to 0^+} \frac{\omega_D(t)}{t} = +\infty. \tag{42}$$

Moreover,

$$\omega_D\left(\frac{\varepsilon_n}{n^2}\right) = \sqrt{\frac{1}{4n^4}+\left(\frac{\varepsilon_n}{n^2}\right)^2}. \tag{43}$$

Let s_n be the solution of the equation $\omega_D(\frac{s_n}{n^2}) = \frac{1}{2n^2}$. In this part of the proof, we shall show that if $n\in\mathbb{N}$ is large enough, then there exist $m_n\in\mathbb{N}$, $0<\xi<1$ for which

$$\frac{\varepsilon_n}{n^2}\xi \leqslant \frac{\varepsilon_{m_n}}{m_n^2} \leqslant \frac{s_n}{n^2} \leqslant \frac{\varepsilon_n}{n^2}. \tag{44}$$

To establish (44), select $i,\tau\in\mathbb{N}$ so that $\eta := \frac{\tau}{i} < 1$ and $1-\frac{M}{2}+\frac{M}{2}\eta^2 > 0$. For each $n\in\mathbb{N}$ let $l_n\in\mathbb{N}_0$ be such that $n = l_n\tau + s_n$ for some $s_n\in\{0,1,2,\dots,\tau-1\}$. Define $m_n := l_n i + n - l_n\tau$. It is clear that

$$\lim_{n\to\infty} \frac{m_n}{n} = \frac{1}{\eta} > 1. \tag{45}$$

Now choose n_0 so large that,

$$4\varepsilon_{m_n}^2 < \frac{m_n^4}{n^4} - 1$$

whenever $n \geqslant n_0$. Hence, by (43),

$$\omega_D\left(\frac{\varepsilon_{m_n}}{m_n^2}\right) = \sqrt{\frac{1}{4m_n^4}+\frac{\varepsilon_{m_n}^2}{m_n^4}} < \frac{1}{2n^2}.$$

Thus if $\omega_D\left(\frac{s_n}{n^2}\right) = \frac{1}{2n^2}$, then

$$\frac{\varepsilon_{m_n}}{m_n^2} < \frac{s_n}{n^2} < \frac{\varepsilon_n}{n^2}. \tag{46}$$

By (34), we have

$$\frac{\varepsilon_n}{n^2}\left(1-\frac{C_n}{2}+\frac{C_n}{2}\frac{n^2}{m_n^2}\right) \leqslant \frac{\varepsilon_{m_n}}{m_n^2}. \tag{47}$$

Take $\delta > 0$ so that $\xi := 1 - \frac{M}{2} + \frac{M}{2}(\eta^2 - \delta) > 0$, then there exists $n_1 \in \mathbb{N}$ such that

$$\frac{\varepsilon_n}{n^2}\xi \leqslant \frac{\varepsilon_n}{n^2}\left(1 - \frac{C_n}{2} + \frac{C_n}{2}(\eta^2 - \delta)\right) \leqslant \frac{\varepsilon_{m_n}}{m_n^2} \tag{48}$$

whenever $n \geqslant n_1$. This follows from the crucial fact that $\sup\{C_n : n \in \mathbb{N}\} = M < \infty$, the limit (45) and (47). Thus, from (46) and (48), we conclude (44).

Since (42), the domain D is a cuspidal piecewise graph domain. An application of Theorem 1 shows that there exists a constant $B > 0$ such that

$$\|\nabla Q\|_{L^p(D)} \leqslant B\frac{n^2}{s_n}\|Q\|_{L^p(D)} \tag{49}$$

for any $Q \in \mathscr{P}_n(\mathbb{R}^2)$. Hence, by (44),

$$\|\nabla Q\|_{L^p(D)} \leqslant \frac{B}{\xi}\frac{n^2}{\varepsilon_n}\|Q\|_{L^p(D)}. \tag{50}$$

Before we go on to show that the factor $\frac{n^2}{\varepsilon_n}$ is best possible, we will prove the inequality (37). Let $\nu := \max\{n_0, n_1\}$. If $x \in [1 - \frac{1}{2\nu^2}, 1)$, then there exists $\varsigma \in \mathbb{N}$ such that

$$x \in \left[1 - \frac{1}{2\varsigma^2}, 1 - \frac{1}{2(\varsigma+1)^2}\right] \subset \left[1 - \frac{1}{2\varsigma^2}, 1 - \frac{1}{2m_\varsigma^2}\right].$$

Hence, by the fact that F is strictly decreasing on the interval $[1/2, 1]$,

$$\frac{C}{2\xi}F(x) \geqslant \frac{C}{2\xi}F\left(1 - \frac{1}{2m_\varsigma^2}\right) = \frac{C\varepsilon_{m_\varsigma}}{2\xi m_\varsigma^2}. \tag{51}$$

Now by (48) and the definition of C,

$$\frac{C\varepsilon_{m_\varsigma}}{2\xi m_\varsigma^2} \geqslant \frac{C\varepsilon_\varsigma}{2\varsigma^2} \geqslant \frac{C_\varsigma\varepsilon_\varsigma}{2\varsigma^2} = -F'\left(1 - \frac{1}{2\varsigma^2}\right)\frac{1}{2\varsigma^2}. \tag{52}$$

Since $-F'(x)(1-x)$ is nonincreasing on $(1/2, 1)$,

$$\frac{C_\varsigma\varepsilon_\varsigma}{2\varsigma^2} = -F'\left(1 - \frac{1}{2\varsigma^2}\right)\frac{1}{2\varsigma^2} \geqslant -F'(x)(1-x). \tag{53}$$

Hence

$$\frac{C}{2\xi}F(x) \geqslant -F'(x)(1-x) \tag{54}$$

for $x \in [1 - \frac{1}{2\nu^2}, 1)$, which is (37) with $\eta = 1 - \frac{1}{2\nu^2}$ and $M = \frac{C}{2\xi}$.

In the last part we will consider the following family of polynomials

$$P_n(x,y) := yP_n^{(\alpha,\beta)}(x).$$

A simple computation reveals that

$$\|P_n\|_{L^p(D)}^p = \int_0^{1-\frac{1}{2\nu^2}} \int_0^{\tilde{f}(x)} |yP_n^{(\alpha,\beta)}(x)|^p \, dydx + \int_{1-\frac{1}{2\nu^2}}^{1-\frac{1}{2n^2}} \int_0^{\tilde{f}(x)} |yP_n^{(\alpha,\beta)}(x)|^p \, dydx$$
$$+ \int_{1-\frac{1}{2n^2}}^{1} \int_0^{\tilde{f}(x)} |yP_n^{(\alpha,\beta)}(x)|^p \, dydx \tag{55}$$

for $n > \nu$. It is easy to conclude that

$$\int_{1-\frac{1}{2n^2}}^{1} \int_0^{\tilde{f}(x)} |yP_n^{(\alpha,\beta)}(x)|^p \, dydx \leqslant \frac{2}{p+1} \left(\frac{\varepsilon_n}{n^2}\right)^{p+1} \int_{1-\frac{1}{2n^2}}^{1} |P_n^{\alpha}(x)|^p \, dx. \tag{56}$$

An argument similar to the one we gave for $\int_{x_n}^1 |P_n^{(\alpha\beta)}(x)|^p \, dx$ shows that there exists $\vartheta > 0$ such that

$$\int_{1-\frac{1}{2n^2}}^{1} |P_n^{(\alpha,\beta)}(x)|^p \, dx \leqslant \vartheta n^{\alpha p - 2} \tag{57}$$

for all sufficiently large n. Thus by (56) and (57),

$$\int_{1-\frac{1}{2n^2}}^{1} \int_0^{\tilde{f}(x)} |yP_n^{(\alpha,\beta)}(x)|^p \, dydx \leqslant \frac{2\vartheta}{p+1} \varepsilon_n^{p+1} n^{\alpha p - 2p - 4}. \tag{58}$$

Using the methods similar to ones used in the proof of Theorem 2, applying (54) instead of (14), we have

$$\upsilon \varepsilon_n n^{\alpha p - 4} \leqslant \int_{1-\frac{1}{2n^2}}^{1} \int_0^{\tilde{f}(x)} |P_n^{(\alpha\beta)}(x)|^p \, dxdy, \tag{59}$$

$$\int_{1-\frac{1}{2\nu^2}}^{1-\frac{1}{2n^2}} \int_0^{\tilde{f}(x)} |yP_n^{(\alpha,\beta)}(x)|^p \, dydx \leqslant \vartheta_1 \varepsilon_n^{p+1} n^{\alpha p - 2p - 4} \tag{60}$$

for $\alpha p + p/2 - 2 > (p+1)\frac{C}{\xi}$, appropriately adjusted constants υ, ϑ_1 and all sufficiently large n. By the definition of $\tilde{f}$, we have

$$\int_0^{1-\frac{1}{2\nu^2}} \int_0^{\tilde{f}(x)} |yP_n^{(\alpha,\beta)}(x)|^p \, dydx = \frac{\varepsilon_1^{p+1}}{p+1} \int_0^{1-\frac{1}{2\nu^2}} |P_n^{(\alpha,\beta)}(x)|^p \, dx. \tag{61}$$

Let $u = \arccos(1 - \frac{1}{2\nu^2})$. Using the change of variables $x = \cos\theta$, we have

$$\int_0^{1-\frac{1}{2\nu^2}} |P_n^{(\alpha,\beta)}(x)|^p \, dx = \int_u^{\frac{\pi}{2}} |P_n^{(\alpha,\beta)}(\cos\theta)|^p \sin\theta \, d\theta. \tag{62}$$

Applying certain properties of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ verified in [20], $(7.32.5)$, p. 169, we conclude that there exists a natural number n_2 so that

$$\int_u^{\frac{\pi}{2}} |P_n^{(\alpha,\beta)}(\cos\theta)|^p \sin\theta\, d\theta \leqslant Cn^{-\frac{p}{2}} \int_u^{\frac{\pi}{2}} \theta^{-\alpha p-\frac{p}{2}+1}\, d\theta \tag{63}$$

for $n \geqslant n_2$ and appropriately adjusted constant C. Hence, by (61)–(63),

$$\lim_{n\to\infty} \int_0^{1-\frac{1}{2v^2}} \int_0^{\tilde{f}(x)} |yP_n^{(\alpha,\beta)}(x)|^p\, dydx = 0. \tag{64}$$

If $\alpha p-2p-4 \geqslant I(p+1)$, then by (36),

$$\lim_{n\to\infty} \varepsilon_n^{p+1} n^{\alpha p-2p-4} = \infty. \tag{65}$$

Thus, by (64) and (65), for each α so that $\alpha p-2p-4 \geqslant I(p+1)$, there exists a positive constant ϑ_2 with the property that

$$\int_0^{1-\frac{1}{2v^2}} \int_0^{\tilde{f}(x)} |yP_n^{(\alpha,\beta)}(x)|^p\, dydx \leqslant \vartheta_2 \varepsilon_n^{p+1} n^{\alpha p-2p-4}. \tag{66}$$

Putting together (55), (58), (59), (60) and (66), we find that

$$\frac{\left\|\frac{\partial P_n}{\partial y}\right\|_{L^p(D)}}{\|P_n\|_{L^p(D)}} \geqslant \vartheta_3 \frac{n^2}{\varepsilon_n}, \tag{67}$$

where $\vartheta_3 > 0$ is a constant independent of n. Finally, (50) and (67) yield that

$$\frac{\|\nabla P_n\|_{L^p(D)}}{\|P_n\|_{L^p(D)}} \sim \frac{n^2}{\varepsilon_n}. \quad \square$$

REFERENCES

[1] D. AZAGRA AND C. MUDARRA, *Whitney extension theorems for convex functions of the classes C^1 and $C^{1,\omega}$*, Proc. London Math. Soc., **3**, (2017), 133–158.

[2] M. BARAN, *Markov inequality on sets with polynomial parametrization*, Ann. Polon. Math., **60**, (1994), 69–79.

[3] M. BARAN, *New approach to Markov inequality in L^p norms*, in: Approximation theory. In memory of A.K. Varma, I. Govil et al. – editors, M. Dekker, Inc., New York-Basel-Hong Kong, (1998), 75–85.

[4] M. BARAN, L. BIAŁAS-CIEŻ AND B. MILÓWKA, *On the best exponent in Markov's inequality*, Potential Anal., **38**, (2013), 635–651.

[5] M. BARAN AND W. PLEŚNIAK, *Markov's exponent of compact sets in $\mathbb{C}^n$*, Proc. Am. Math. Soc., **123**, (1995), 2785–2791.

[6] P. BORWEIN AND T. ERDÉLYI, *Polynomials and Polynomial Inequalities*, Springer, 1995.

[7] L. P. BOS AND P. D. MILMAN, *Sobolev-Gagliardo-Nirenberg and Markov type inequalities on subanalytic domains*, Geom. Funct. Anal., **5**, (1995), 853–923.

[8] P. GOETGHELUCK, *Markov's inequality on Locally Lipschitzian compact subsets of $\mathbb{R}^n$ in L^p-spaces*, J. Approx. Theory, **49**, (1987), 303–310.

[9] P. GOETGHELUCK, *On the problem of Sharp Exponents in Multivariate Nikolski-Type Inequality*, J. Approx. Theory, **(77)**, (1994), 167–178.

[10] P. GOETGHELUCK, *Polynomial Inequalities on General Subsets of* $\mathbb{R}^N$, Coll. Math., **57**, 1 (1989), 127–136.
[11] A. KROÓ, *On Bernstein-Markov-type inequalities for multivariate polynomials in* L^q*-norm*, J. Approx. Theory, **159**, (2009), 85–96.
[12] A. KROÓ, *Sharp* L^p *Markov type inequality for cuspidal domains in* $\mathbb{R}^d$, J. Approx. Theory, **250**, (2020), 105336.
[13] A. KROÓ AND J. SZABADOS *Bernstein-Markov type inequalities for multivariate polynomials on sets with cusps*, J. Approx. Theory, **102**, (2000), 72–95.
[14] G. V. MILOVANOVIC, D. S. MITRINOVIĆ AND TH. M. RASSIAS, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Sci., Singapore, 1994.
[15] S. M. NIKOLSKII, *On a certain method of covering domains and inequalities for multivariate polynomials*, Mathematica, **8**, (1966), 345–356.
[16] W. PAWŁUCKI AND W. PLEŚNIAK, *Markov's inequality and* C^∞ *functions on sets with polynomial cusps*, Math. Ann., **275**, (1986), 467–480.
[17] W. PLEŚNIAK, *Markov's inequality and the existence of an extension operator for* C^∞ *functions*, J. Approx. Theory, **61**, (1990), 106–117.
[18] W. PLEŚNIAK, *Recent progress in multivariate Markov inequality*, Approximation theory, Monogr. Textbooks Pure Appl. Math., Dekker, New York, (1998), 449–464.
[19] Q. I. RAHMAN AND G. SCHMEISSER, *Analytic Theory of Polynomials*, in London Math. Soc. Monogr. (N.S.) Oxford Univ. Press **26**, Oxford 2002.
[20] G. SZEGŐ, *Orthogonal Polynomials*, AMS Colloquium Publications, v.XXIII, Providence, Rhode Island, 1939.
[21] M. ZERNER *Développement en séries de polynômes orthonormaux des fonctions indéfiniment différentiables* C. R. Acad. Sci. Paris Sér. I, **268**, (1969), 218–220.

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Tomasz Beberok
Department of Applied Mathematics
University of Agriculture in Krakow, Poland
e-mail: t_beberok@pwsztar.edu.pl