Mathematical
Inequalities
& Applications
Volume 26, Number 2 (2023), 343–349 doi:10.7153/mia-2023-26-22

TURÁN'S INEQUALITY FOR ULTRASPHERICAL POLYNOMIALS REVISITED

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(*Communicated by I. Perić*)

Abstract. We present a short proof that the normalized Turán determinant in the ultraspherical case is convex or concave depending on whether parameter λ is positive or negative.

1. Introduction and statement of the result

In the 40's of the last century, while studying the zeros of Legendre polynomials $P_n(x)$, P. Turán discovered the inequality

$$P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geqslant 0, \quad -1 \leqslant x \leqslant 1, \tag{1}$$

with equality only for $x = \pm 1$. Since the left-hand side of (1) is representable in determinant form,

$$\Delta_n(x) = \begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{vmatrix}$$

$\Delta_n(x)$ is referred to as *Turán's determinant*.

The result of Turán inspired a considerable interest, and by now there is a vast amount of publications on the so-called *Turán type inequalities*. G. Szegő [15] gave four different proof of (1). As Szegő pointed out in [15], his third proof extends Turán's inequality to other classes of functions including ultraspherical polynomials, Laguerre and Hermite polynomials, Bessel functions, etc. This idea was elaborated further by Skovgaard [13].

Karlin and Szegő [8] posed the problem of characterizing the set of pairs $\{\alpha, \beta\}$ for which the normalized Jacobi polynomials $P_m^{(\alpha,\beta)}(x)/P_m^{(\alpha,\beta)}(1)$ admit a Turán type inequality. Szegő proved that Turán's inequality holds whenever $\beta \geqslant |\alpha|$, $\alpha > -1$. In two subsequent papers G. Gasper [5, 6] improved Szegő's result showing finally that the sought pairs $\{\alpha, \beta\}$ are those satisfying $\beta \geqslant \alpha > -1$.

Mathematics subject classification (2020): Primary 33C45; Secondary 42C05.
Keywords and phrases: Turán inequality, normalized Turán determinant, ultraspherical polynomials.
This research is supported by the Bulgarian National Research Fund under Contract KP-06-N62/4.

Paper MIA-26-22

Our concern here is Turán's inequality in the ultraspherical case. Throughout this paper, $p_n^{(\lambda)}$ stands for the n-th ultraspherical polynomial normalized to assume value 1 at $x=1$,

$$p_n^{(\lambda)}(x)=\frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)}.$$

Let

$$\Delta_{n,\lambda}(x):=\left[p_n^{(\lambda)}(x)\right]^2-p_{n-1}^{(\lambda)}(x)p_{n+1}^{(\lambda)}(x), \tag{2}$$

then Turán's inequality for ultraspherical polynomials reads as

$$\Delta_{n,\lambda}(x)\geqslant 0, \qquad x\in[-1,1]. \tag{3}$$

To the many proofs of (3) (see, e.g. [2, 14, 15, 18, 19]), let us add the one in [10] based on a Hermite interpolation formula, yielding the representation

$$\Delta_{n,\lambda}(x)=\frac{1-x^2}{n(n+2\lambda)}\sum_{k=1}^{n}\ell_k^2(x)(1-x_kx)\left[p_n'(x_k)\right]^2$$

(here, $\{\ell_k\}_{k=1}^n$ are the Lagrange basis polynomials for interpolation at the zeros $\{x_k\}_{k=1}^n$ of $p_n=p_n^{(\lambda)}$).

Since $\Delta_{n,\lambda}(\pm 1)=0$, it is of interest to describe the behavior of the normalized Turán function

$$\varphi_{n,\lambda}(x):=\frac{\Delta_{n,\lambda}(x)}{1-x^2}. \tag{4}$$

Thiruvenkatachar and Nanjundiah [18] have shown that $\varphi_{n,\lambda}$ increases in $[-1,0]$ and decreases in $[0,1]$ when $-1/2<\lambda<0$, and has the opposite behavior when $\lambda>0$. Since $\varphi_{n,\lambda}$ is an even function, it follows that for $x\in[-1,1]$,

$$\begin{aligned}\varphi_{n,\lambda}(1)\leqslant\varphi_{n,\lambda}(x)\leqslant\varphi_{n,\lambda}(0), &\quad -1/2<\lambda<0\\ \varphi_{n,\lambda}(0)\leqslant\varphi_{n,\lambda}(x)\leqslant\varphi_{n,\lambda}(1), &\quad \lambda>0.\end{aligned}$$

These inequalities together with

$$\begin{aligned}\varphi_{n,\lambda}(0)&=\Delta_{n,\lambda}(0),\\ \varphi_{n,\lambda}(1)&=-\frac{\Delta_{n,\lambda}'(1)}{2}=1/(2\lambda+1)\end{aligned}$$

imply the following two-sided estimates for $\Delta_{n,\lambda}(x)$ when $x\in[-1,1]$.

$$\begin{aligned}\frac{1-x^2}{2\lambda+1}\leqslant\Delta_{n,\lambda}(x)\leqslant\Delta_{n,\lambda}(0)(1-x^2), &\quad -1/2<\lambda<0\\ \Delta_{n,\lambda}(0)(1-x^2)\leqslant\Delta_{n,\lambda}(x)\leqslant\frac{1-x^2}{2\lambda+1}, &\quad \lambda>0.\end{aligned} \tag{5}$$

Here we make this observation more precise by proving the following:

THEOREM 1. *The normalized Turán function $\varphi_{n,\lambda}$ is concave or convex on $\mathbb{R}$ depending on whether $-1/2 < \lambda < 0$ or $\lambda > 0$.*

(Note that $\varphi_{n,0} \equiv 1$.) Theorem 1 reproduces one of the inequalities in (5) and both sharpens and extends to the whole real line the other one. More precisely, Theorem 1 implies immediately

COROLLARY 1. *(i) If $-1/2 < \lambda < 0$, then*

$$\Delta_{n,\lambda}(x) \leqslant \Delta_{n,\lambda}(0)(1-x^2), \quad x \in [-1,1],$$
$$\Delta_{n,\lambda}(x) \geqslant \Big[(1-|x|)\Delta_{n,\lambda}(0) + \frac{|x|}{2\lambda+1}\Big](1-x^2), \quad x \in \mathbb{R}.$$

(ii) If $\lambda > 0$, then

$$\Delta_{n,\lambda}(x) \geqslant \Delta_{n,\lambda}(0)(1-x^2), \quad x \in [-1,1],$$
$$\Delta_{n,\lambda}(x) \leqslant \Big[(1-|x|)\Delta_{n,\lambda}(0) + \frac{|x|}{2\lambda+1}\Big](1-x^2), \quad x \in \mathbb{R}.$$

The proof of Theorem 1 is given in the next section. The last section contains some remarks and comments.

2. Proof of Theorem 1

We shall work with the renormalized ultraspherical polynomials

$$p_n^{(\lambda)}(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1),$$

and for the simplicity sake we omit the superscript $^{(\lambda)}$, writing $p_n := p_n^{(\lambda)}$. The next two identities readily follow from [16, equation (4.7.28)]:

$$p_n(x) = -\frac{1}{n+2\lambda}\, x\, p_n'(x) + \frac{1}{n+1}\, p_{n+1}'(x),$$
$$p_{n+1}(x) = -\frac{1}{n+2\lambda}\, p_n'(x) + \frac{1}{n+1}\, x\, p_{n+1}'(x).$$

These identities are used for deriving representations of p_{n+1} and p_{n-1} in terms of p_n and p_n':

$$p_{n+1}(x) = x\, p_n(x) - \frac{1-x^2}{n+2\lambda}\, p_n'(x),$$
$$p_{n-1}(x) = x\, p_n(x) + \frac{1-x^2}{n}\, p_n'(x).$$

By replacing p_{n+1} and p_{n-1} in $\Delta_{n,\lambda} = p_n^2 - p_{n-1}p_{n+1}$ we obtain

$$\Delta_{n,\lambda}(x) = \frac{1-x^2}{n(n+2\lambda)}\Big[n(n+2\lambda)p_n^2(x) - 2\lambda\, x\, p_n(x)p_n'(x) + (1-x^2)\big[p_n'(x)\big]^2\Big],$$

hence

$$\varphi_{n,\lambda}(x) = \frac{1}{n(n+2\lambda)}\Big[n(n+2\lambda)p_n^2(x) - 2\lambda\, x\, p_n(x)p_n'(x) + (1-x^2)\big[p_n'(x)\big]^2\Big]. \qquad (6)$$

Differentiating (6) and using the differential equation

$$(1-x^2)y'' - (2\lambda+1)xy' + n(n+2\lambda)y = 0, \qquad y = p_n(x), \qquad (7)$$

we find

$$\begin{aligned}\varphi_{n,\lambda}'(x) &= \frac{2\lambda}{n(n+2\lambda)}\Big[x\big[p_n'(x)\big]^2 - p_n(x)p_n'(x) - x\,p_n(x)p_n''(x)\Big]\\ &= -\frac{2\lambda}{n(n+2\lambda)}\,p_n^2(x)\left(\frac{x\,p_n'(x)}{p_n(x)}\right)'.\end{aligned} \qquad (8)$$

Let $x_1 < x_2 < \cdots < x_n$ be the zeros of p_n, they form a symmetric set with respect to the origin, therefore

$$\frac{p_n'(x)}{p_n(x)} = \sum_{k=1}^{n}\frac{1}{x-x_k} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{x-x_k}+\frac{1}{x+x_k}\right) = x\sum_{k=1}^{n}\frac{1}{x^2-x_k^2}.$$

Consequently,

$$\left(\frac{x\,p_n'(x)}{p_n(x)}\right)' = -2x\sum_{k=1}^{n}\frac{x_k^2}{(x^2-x_k^2)^2},$$

and (8) implies

$$\varphi_{n,\lambda}'(x) = \frac{4\lambda\, x}{n(n+2\lambda)}\sum_{k=1}^{n}x_k^2\, q_{n,k}^2(x), \qquad q_{n,k}(x) = \frac{p_n(x)}{x^2-x_k^2}. \qquad (9)$$

Now (9) shows that $\operatorname{sign}\varphi_{n,\lambda}'(x) = \operatorname{sign}\lambda\, x$, a result already obtained by Thiruvenkatachar and Nanjundiah [18]. In fact, (9) implies more than that, namely, we have

$$\operatorname{sign}\varphi_{n,\lambda}^{(r)}(x) = \operatorname{sign}\lambda, \qquad x > x_n,\ \ r = 1,2,\ldots,2n-2. \qquad (10)$$

Indeed, $\varphi_{n,\lambda}'$ is a sum of polynomials with leading coefficients of the same sign as λ and with all their zeros being real and located in $[x_1,x_n]$. By Rolle's theorem, the derivatives of these polynomials inherit the same properties, hence they have no zeros in (x_n,∞) and therefore have the same sign as λ therein. In particular, (10) implies

$$\operatorname{sign}\varphi_{n,\lambda}''(x) = \operatorname{sign}\lambda, \qquad x \in (x_n,\infty)$$

and to prove Theorem 1 we need to show that $\operatorname{sign}\varphi_{n,\lambda}''(x) = \operatorname{sign}\lambda$ for $x \in (0,x_n]$. In view of (8), this is equivalent to prove that the function

$$\psi_{n,\lambda}(x) := \big[p_n'(x)\big]^2 - p_n(x)p_n'(x) - x\,p_n(x)p_n''(x) \qquad (11)$$

satisfies

$$\psi'_{n,\lambda}(x) > 0, \qquad x \in (0, x_n]. \tag{12}$$

We differentiate (11) and make use of the differential equations (7) and

$$(1-x^2)y''' - (2\lambda+3)xy'' + (n-1)(n+2\lambda+1)y' = 0, \qquad y = p_n(x),$$

to obtain a representation of $\psi'_{n,\lambda}(x)$ as a quadratic form of p'_n and p''_n:

$$\begin{aligned} n(n+2\lambda)(1-x^2)\psi'_{n,\lambda}(x) =& (2\lambda+1)(n-1)(n+2\lambda+1)x^2\left[p'_n(x)\right]^2 \\ &- (2\lambda+1)x\left[1+2(\lambda+1)x^2\right]p'_n(x)p''_n(x) \\ &+ (1-x^2)\left[2+(2\lambda+1)x^2\right]\left[p''_n(x)\right]^2 \end{aligned}$$

The discriminant D of this quadratic form equals

$$D(x) = (2\lambda+1)x^2\left[2\lambda+3-(2\lambda+1)(1-x^2)\right]D_1(x),$$

where

$$D_1(x) = (2\lambda+1)\frac{\left[2\lambda+3-(2\lambda+2)(1-x^2)\right]^2}{2\lambda+3-(2\lambda+1)(1-x^2)} - 4(n-1)(n+2\lambda+1)(1-x^2).$$

Our goal is to prove that

$$D_1(x) < 0, \qquad x \in (0, x_n], \tag{13}$$

which implies $D(x) < 0$ and consequently $\psi'_{n,\lambda}(x) > 0$ in $(0, x_n]$. It is readily verified that

$$\frac{\left[2\lambda+3-(2\lambda+2)(1-x^2)\right]^2}{2\lambda+3-(2\lambda+1)(1-x^2)} \leqslant 2\lambda+3-(2\lambda+5/2)(1-x^2), \qquad x \in [-1,1],$$

therefore

$$\begin{aligned} D_1(x) &\leqslant (2\lambda+1)\left[2\lambda+3-(2\lambda+5/2)(1-x^2)\right] - 4(n-1)(n+2\lambda+1)(1-x^2) \\ &= (2\lambda+1)(2\lambda+3) - \left[4(n+\lambda)^2-(\lambda+3/2)\right](1-x^2), \qquad x \in [-1,1]. \end{aligned}$$

Hence, to prove (13), it suffices to show that

$$1-x^2 > \frac{(2\lambda+1)(2\lambda+3)}{4(n+\lambda)^2-\lambda-3/2}, \qquad x \in (0, x_n)$$

or, equivalently,

$$x_n^2 < 1 - \frac{(2\lambda+1)(2\lambda+3)}{4(n+\lambda)^2-\lambda-3/2}. \tag{14}$$

Thus, we need an upper bound for x_n, the largest zero of the ultraspherical polynomial $P_n^{(\lambda)}$. Amongst the numerous upper bounds for x_n in the literature, we use the one from [9, Lemma 6] (see also [4, p. 1801]):

$$x_n^2 < \frac{(n+\lambda)^2-(\lambda+1)^2}{(n+\lambda)^2+3\lambda+5/4+3(\lambda+1/2)^2/(n-1)}. \tag{15}$$

The comparison of the right-hand sides of (14) and (15) (we have used *Wolfram Mathematica* for this purpose) shows that the latter is the smaller one, hence (14) holds true. With this (12) is proved, hence $\operatorname{sign} \varphi''_{n,\lambda}(x) = \operatorname{sign} \lambda$ for $x \in (0, x_n]$ and consequently

$$\operatorname{sign} \varphi''_{n,\lambda}(x) = \operatorname{sign} \lambda, \qquad x \in (0, \infty).$$

Since $\varphi''_{n,\lambda}$ is an even function, this accomplishes the proof of Theorem 1.

3. Remarks

1. There are also some results concerning concavity of $\Delta_{n,\lambda}$. In the classical Turán case, $\lambda = 1/2$, Madhava Rao and Thiruvenkatachar [12] have proved that

$$\frac{d^2}{dx^2} \Delta_n(x) = -\frac{2}{n(n+1} \left[P''_n(x)\right]^2,$$

showing that Δ_n is a concave function. Venkatachaliengar and Lakshmana Rao [19] extended this result by proving that $\Delta_{n,\lambda}$ is a concave function in $[-1,1]$ provided $\lambda \in (0, 1/2]$. Generally, $\Delta_{n,\lambda}$ is neither convex nor concave if $\lambda \notin [0, 1/2]$.

2. Szász [14] proved the following pair of bounds for $\Delta_{n,\lambda}(x)$:

$$\frac{\lambda\left(1 - [p_n^{(\lambda)}(x)]^2\right)}{(n+\lambda-1)(n+2\lambda)} < \Delta_{n,\lambda}(x) < \frac{n+\lambda}{\lambda+1} \, \frac{\Gamma(n)\Gamma(2\lambda+1)}{\Gamma(n+2\lambda+1)}, \qquad \lambda \in (0,1).$$

3. In a recent paper [11] we gave both an analytical and a computer proof of the following refinement of Turán's inequality:

$$|x| \left[p_n^{(\lambda)}(x)\right]^2 - p_{n-1}^{(\lambda)}(x) p_{n+1}^{(\lambda)}(x) \geqslant 0, \qquad x \in [-1,1], \ -1/2 < \lambda \leqslant 1/2,$$

with the equality occurring only for $x = \pm 1$ and, if n is even, $x = 0$. This inequality provides another lower bound for $\Delta_{n,\lambda}(x)$ in the case $-1/2 < \lambda \leqslant 1/2$. A computer proof of the Legendre case ($\lambda = 1/2$) was given earlier by Gerhold and Kauers [7].

4. In [18] the authors proved also monotonicity of $\Delta_{n,\lambda}(x)$, $x \in [-1,1]$ fixed, with respect to n. We refer to [1, 17] for some general condition on the sequences defining the three-term recurrence relation for orthogonal polynomials, which ensure the monotonicity of the associated Turán determinants.

5. For a higher order Turán inequalities and a discussion on the interlink between the Turán type inequalities and the Riemann hypothesis or the recovery of the orthogonality measure, we refer to [3] and the references therein.

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(Received February 2, 2022)

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