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# A FAMILY OF GEOMETRIC CONSTANTS ON MORREY SPACES

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*Abstract.* In this paper, we calculate a family of geometric constants for Morrey spaces, small Morrey spaces and discrete Morrey spaces. This family of constants measures uniformly non-squareness of the associated spaces. We obtain that the value this family of constants for the aforementioned spaces is $2^{1-\frac{1}{t}}$ for $1 \leqslant t < \infty$, which means that the spaces are not uniformly non-square. The main results obtained in this paper generalize some existing results in the recent literature.

## 1. Introduction and preliminaries

In recent years, various geometric constants for a Banach space have been defined and studied. In general, the study of the geometric property of a Banach space is not easy. Alternatively one can do this with the help of some certain geometric constants.

For a real Banach space $X$, let $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \leqslant 1\}$ be the unit sphere and the closed unit ball of $X$, respectively. Also, let $\Lambda$ denote the set of all continuous functions $\lambda : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that $\lambda$ is homogeneous of degree 1 and $\lambda(1,1) = 1$.

Recently, Amini-Harandi and Rahimi [2] has introduced the constants $C_{\lambda,t}(X)$ and $C'_{\lambda,t}(X)$ of $X$ by

$$C_{\lambda,t}(X) = \sup\left\{\lambda\left(\|ru+sv\|, \|ru-sv\|\right) : u,v \in S_X,\ r,s \geqslant 0 \text{ and } \|(r,s)\|_t = 1\right\}$$

for each $\lambda \in \Lambda$ and $t \in [1,\infty]$, where

$$\|(r,s)\|_t = \begin{cases} \left(|r|^t + |s|^t\right)^{\frac{1}{t}}, & 1 \leqslant t < \infty, \\ \max\{|r|,|s|\}, & t = \infty \end{cases}$$

and

$$C'_{\lambda,t}(X) = \sup\left\{\frac{\lambda\left(\|u+v\|, \|u-v\|\right)}{2^{\frac{1}{t}}} : u,v \in S_X\right\}.$$

Notice that since for each $r,s \in [0,1]$, we have $\|ru+sv\|, \|ru-sv\|, \|u+v\|, \|u-v\| \in [0,2]$ and $\lambda$ is bounded on $[0,2]\times[0,2]$, then $C_{\lambda,t}(X) < \infty$ and $C'_{\lambda,t}(X) < \infty$.

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For each $r,s \in [0,\infty)$ and $t \in [1,\infty)$, set $\varepsilon(r,s) = \min\{r,s\}$, $\alpha_t(r,s) = \left(\frac{r^t+s^t}{2}\right)^{\frac{1}{t}}$ and $\pi(r,s) = \sqrt{rs}$. It is obvious that the constants $C_{\lambda,t}(X)$ and $C'_{\lambda,t}(X)$ include some known geometric constants such as the generalized von Neumann-Jordan constant $C_{NJ}^{(t)}(X) = 2^{2-t}(C_{\alpha_t,t}(X))^t$ ([4, 5, 6]), the generalized Zbăganu constant $C_Z^{(t)}(X) = 2^{2-t}(C_{\pi,t}(X))^t$ ([17, 18]), the generalized von Neumann-Jordan type constant $C_{-\infty}^{(t)}(X) = 2^{2-t}(C_{\varepsilon,t}(X))^t$ ([7, 14]), the generalized modified von Neumann-Jordan constant $\overline{C}_{NJ}^{(t)}(X) = 2^{2-t}(C'_{\alpha_t,t}(X))^t$ ([16]), the James constant $J(X) = C_{\varepsilon,\infty}(X)$ ([11]), Baronti-Papini's constant $A_{2,t}(X) = 2^{\frac{1}{t}}C_{\alpha_1,t}(X)$ ([3, 8]), Alonso-Llorens-Fuster's constant $T(X) = C'_{\pi,\infty}(X)$ ([1]). It is interesting to remark at this point that for all $1 \leqslant t < \infty$,

$$2^{\frac{1}{2}-\frac{1}{t}} \leqslant C'_{\lambda,t}(X) \leqslant C_{\lambda,t}(X) \leqslant 2^{1-\frac{1}{t}}.$$

Recall that a Banach space $X$ is called uniformly non-square provided that there exists $\delta > 0$ such that either $\|x+y\| \leqslant 2-\delta$ or $\|x-y\| \leqslant 2-\delta$ for all $x,y \in B_X$. In [11] it was proved that uniformly non-square Banach spaces are reflexive. It is worthwhile to mention that $X$ is uniformly non-square if and only if $C_{\lambda,t}(X) < 2^{1-\frac{1}{t}}$ for all $1 < t < \infty$.

The goal of this work is to compute the values of the constants $C_{\lambda,t}(X)$ and $C'_{\lambda,t}(X)$ for Morrey spaces $X = \mathscr{M}_q^p(\mathbb{R}^d)$, small Morrey spaces $X = m_q^p(\mathbb{R}^d)$ and discrete Morrey spaces $X = \ell_q^p(\mathbb{Z}^d)$, where $1 \leqslant p < q < \infty$ and $1 \leqslant t < \infty$. Our main results tell us that all of those spaces are not uniformly non-square. Moreover, the main results obtained in this paper generalize some previous results in the recent literature on this topic.

## 2. Small Morrey spaces

For $1 \leqslant p \leqslant q < \infty$, the small Morrey space $m_q^p = m_q^p(\mathbb{R}^d)$ is the set of all measurable functions $f$ such that

$$\|f\|_{m_q^p} := \sup_{a\in\mathbb{R}^d, R\in(0,1)} |B(a,R)|^{\frac{1}{q}-\frac{1}{p}} \left( \int_{B(a,R)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty,$$

where $|B(a,R)|$ denotes the Lebesgue measure of the open ball $B(a,R)$ in $\mathbb{R}^d$, with center $a$ and radius $R$. Small Morrey spaces are Banach spaces [15]. Note that for $p = q$, the space $m_q^p$ is identical with the space $L_{uloc}^q$ [15].

Our result for small Morrey spaces is presented in the following theorem.

THEOREM 1. *Let $1 \leqslant p < q < \infty$ and $1 \leqslant t < \infty$. Then*

$$C_{\lambda,t}(m_q^p) = C'_{\lambda,t}(m_q^p) = 2^{1-\frac{1}{t}}.$$

*Proof.* Suppose that $1 \leqslant p < q < \infty$ and $1 \leqslant t < \infty$ and let $f(x) = \chi_{(0,1)}(|x|)|x|^{-\frac{n}{q}}$, where $x \in \mathbb{R}^n$ and $|x|$ denotes the Euclidean norm of $x$. Then $f \in m_q^p$. For each $\varepsilon \in$

$(0,1)$, we consider $g(x) = \chi_{(0,\varepsilon)}(|x|)f(x)$, $h(x) = f(x) - g(x)$ and $k(x) = g(x) - h(x)$. Note that $g$ depends on $\varepsilon$, so that $h$ and $k$ also depend on $\varepsilon$. Therefore, we obtain

$$\|f\|_{m_q^p} = \sup_{R\in(0,1)} |B(0,R)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(0,R)} |y|^{-\frac{np}{q}} dy\right)^{\frac{1}{p}} = \left(\frac{C_n}{n}\right)^{\frac{1}{q}} \left(1-\frac{p}{q}\right)^{-\frac{1}{p}},$$

$$\begin{aligned}
\|g\|_{m_q^p} &= \sup_{R\in(0,1)} |B(0,R)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(0,R)} |\chi_{(0,\varepsilon)}(|y|)f(y)|^p dy\right)^{\frac{1}{p}} \\
&= \sup_{R\in(0,\varepsilon)} |B(0,R)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(0,R)} |y|^{-\frac{-np}{q}} dy\right)^{\frac{1}{p}} = \left(\frac{C_n}{n}\right)^{\frac{1}{q}} \left(1-\frac{p}{q}\right)^{-\frac{1}{p}} \\
&= \|f\|_{m_q^p},
\end{aligned}$$

$$\begin{aligned}
\|h\|_{m_q^p} &\geqslant \sup_{R\in(0,1)} |B(0,R)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(0,R)} |\chi_{(\varepsilon,1)}(|y|)f(y)|^p dy\right)^{\frac{1}{p}} \\
&= \sup_{R\in(\varepsilon,1)} |B(0,R)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(0,R)\setminus B(0,\varepsilon)} |y|^{-\frac{-np}{q}} dy\right)^{\frac{1}{p}} \\
&= \sup_{R\in(\varepsilon,1)} \left(\frac{C_n}{n}\right)^{\frac{1}{q}-\frac{1}{p}} R^{\frac{n}{q}-\frac{n}{p}} \left(C_n \int_{\varepsilon}^{R} r^{-\frac{-np}{q}+n-1} dr\right)^{\frac{1}{p}} \\
&= \sup_{R\in(\varepsilon,1)} \left(\frac{C_n}{n}\right)^{\frac{1}{q}} \left(1-\frac{p}{q}\right)^{-\frac{1}{p}} \left(1-R^{\frac{-np}{q}-n}\varepsilon^{-\frac{-np}{q}+n}\right)^{\frac{1}{p}} \\
&= \left(1-\varepsilon^{n-\frac{-np}{q}}\right)^{\frac{1}{p}} \|f\|_{m_q^p}
\end{aligned}$$

and

$$\begin{aligned}
\|k\|_{m_q^p} &= \sup_{R\in(0,1)} |B(0,R)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(0,R)} \left|\left(\chi_{(0,\varepsilon)}(|y|) - \chi_{(\varepsilon,1)}(|y|)\right)f(y)\right|^p dy\right)^{\frac{1}{p}} \\
&= \sup_{R\in(0,1)} |B(0,R)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(0,R)} |f(y)|^p dy\right)^{\frac{1}{p}} = \|f\|_{m_q^p},
\end{aligned}$$

where $C_n$ denotes the "area" of the unit sphere in $\mathbb{R}^n$. First, let us compute the constant $C_{\lambda,t}(m_q^p)$. Then, we have

$$\begin{aligned}
C_{\lambda,t}(m_q^p) &\geqslant \frac{1}{\left(\|f\|_{m_q^p}^t + \|k\|_{m_q^p}^t\right)^{\frac{1}{t}}} \lambda\left(\|f+k\|_{m_q^p}, \|f-k\|_{m_q^p}\right) \\
&= \frac{1}{\left(\|f\|_{m_q^p}^t + \|k\|_{m_q^p}^t\right)^{\frac{1}{t}}} \lambda\left(\|2g\|_{m_q^p}, \|2h\|_{m_q^p}\right) \\
&\geqslant \frac{1}{2^{\frac{1}{t}}\|f\|_{m_q^p}} \lambda\left(2\|f\|_{m_q^p}, 2\|f\|_{m_q^p}\left(1-\varepsilon^{n-\frac{-np}{q}}\right)^{\frac{1}{p}}\right).
\end{aligned}$$

Since we may choose $\varepsilon$ to be arbitrary small, it follows that $C_{\lambda,t}(m_q^p) \geqslant 2^{1-\frac{1}{t}}$. Since $C_{\lambda,t}(m_q^p) \leqslant 2^{1-\frac{1}{t}}$, we conclude that $C_{\lambda,t}(m_q^p) = 2^{1-\frac{1}{t}}$.

Next, we move to the constant $C'_{\lambda,t}(m_q^p)$. Due to $\|f\|_{m_q^p} = \|k\|_{m_q^p}$, we consider $\frac{f}{\|f\|_{m_q^p}}$ and $\frac{k}{\|f\|_{m_q^p}}$. Hence, we have

$$\begin{aligned}
C'_{\lambda,t}(m_q^p) &\geqslant \frac{1}{2^{\frac{1}{t}}\|f\|_{m_q^p}}\lambda\left(\|f+k\|_{m_q^p}, \|f-k\|_{m_q^p}\right) \\
&= \frac{1}{2^{\frac{1}{t}}\|f\|_{m_q^p}}\lambda\left(\|2g\|_{m_q^p}, \|2h\|_{m_q^p}\right) \\
&\geqslant \frac{1}{2^{\frac{1}{t}}\|f\|_{m_q^p}}\lambda\left(2\|f\|_{m_q^p}, 2\|f\|_{m_q^p}\left(1-\varepsilon^{n-\frac{-np}{q}}\right)^{\frac{1}{p}}\right).
\end{aligned}$$

By using similar arguments as before, we conclude that $C'_{\lambda,t}(m_q^p) = 2^{1-\frac{1}{t}}$. $\square$

COROLLARY 1. *Let $1 \leqslant p < q < \infty$ and $1 \leqslant t < \infty$. Then*

$$\begin{aligned}
C_{NJ}^{(t)}(m_q^p) &= \bar{C}_{NJ}^{(t)}(m_q^p) = C_Z^{(t)}(m_q^p) = C_{-\infty}^{(t)}(m_q^p) = J(m_q^p) \\
&= A_{2,t}(m_q^p) = T(m_q^p) = 2.
\end{aligned}$$

REMARK 1. Corollary 1 generalizes and improves existing results in [10, 12, 13].

## 3. Morrey spaces

For $1 \leqslant p \leqslant q < \infty$, the (classical) Morrey space $\mathscr{M}_q^p = \mathscr{M}_q^p(\mathbb{R}^d)$ is the set of all measurable functions $f$ such that

$$\|f\|_{\mathscr{M}_q^p} := \sup_{a\in\mathbb{R}^d, R>0} |B(a,R)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,R)} |f(y)|^p dy\right)^{\frac{1}{p}} < \infty,$$

where $|B(a,R)|$ denotes the Lebesgue measure of the open ball $B(a,R)$ in $\mathbb{R}^d$, with center $a$ and radius $R$. Morrey spaces are Banach spaces [15]. Note that for $p = q$, the space $\mathscr{M}_q^p$ is identical with the space $L^q = L^q(\mathbb{R}^d)$, the space of $q$-th power integrable functions on $\mathbb{R}^d$. Note that for all $p$ and $q$, the small Morrey spaces properly contain the Morrey spaces.

Our result for Morrey spaces is stated in the following theorem.

THEOREM 2. *Let $1 \leqslant p < q < \infty$ and $1 \leqslant t < \infty$. Then*

$$C_{\lambda,t}(\mathscr{M}_q^p) = C'_{\lambda,t}(\mathscr{M}_q^p) = 2^{1-\frac{1}{t}}.$$

*Proof.* Suppose that $1 \leqslant p < q < \infty$ and $1 \leqslant t < \infty$ and let $f(x) = |x|^{-\frac{n}{q}}$, where $x \in \mathbb{R}^n$ and $|x|$ denotes the Euclidean norm of $x$. Then $f \in \mathscr{M}_q^p$. Now, we consider $g(x) = \chi_{(0,1)}(|x|)f(x)$, $h(x) = f(x) - g(x)$ and $k(x) = g(x) - h(x)$. One may observe that

$$\|f\|_{\mathscr{M}_q^p} = \|g\|_{\mathscr{M}_q^p} = \|h\|_{\mathscr{M}_q^p} = \|k\|_{\mathscr{M}_q^p} = \left(\frac{C_n}{n}\right)^{\frac{1}{q}} \left(1 - \frac{p}{q}\right)^{-\frac{1}{p}}.$$

First, we calculate the constant $C_{\lambda,t}(\mathscr{M}_q^p)$. Hence, we have

$$\begin{aligned} C_{\lambda,t}(\mathscr{M}_q^p) &\geqslant \frac{1}{\left(\|f\|^t_{\mathscr{M}_q^p} + \|k\|^t_{\mathscr{M}_q^p}\right)^{\frac{1}{t}}} \lambda\left(\|f+k\|_{\mathscr{M}_q^p}, \|f-k\|_{\mathscr{M}_q^p}\right) \\ &= \frac{1}{\left(\|f\|^t_{\mathscr{M}_q^p} + \|f\|^t_{\mathscr{M}_q^p}\right)^{\frac{1}{t}}} \lambda\left(\|2g\|_{\mathscr{M}_q^p}, \|2h\|_{\mathscr{M}_q^p}\right) \\ &\geqslant \frac{1}{2^{\frac{1}{t}}\|f\|_{\mathscr{M}_q^p}} \lambda\left(2\|g\|_{\mathscr{M}_q^p}, 2\|h\|_{m_q^p}\right) \\ &= 2^{1-\frac{1}{t}}. \end{aligned}$$

So $C_{\lambda,t}(\mathscr{M}_q^p) \geqslant 2^{1-\frac{1}{t}}$. Since $C_{\lambda,t}(\mathscr{M}_q^p) \leqslant 2^{1-\frac{1}{t}}$, we conclude that $C_{\lambda,t}(\mathscr{M}_q^p) = 2^{1-\frac{1}{t}}$.

Next, for the constant $C'_{\lambda,t}(\mathscr{M}_q^p)$, we consider $\frac{f}{\|f\|_{\mathscr{M}_q^p}}$ and $\frac{k}{\|f\|_{\mathscr{M}_q^p}}$ as $\|f\|_{\mathscr{M}_q^p} = \|k\|_{\mathscr{M}_q^p}$. Then, we have

$$\begin{aligned} C'_{\lambda,t}(\mathscr{M}_q^p) &\geqslant \frac{1}{2^{\frac{1}{t}}\|f\|_{\mathscr{M}_q^p}} \lambda\left(\|f+k\|_{\mathscr{M}_q^p}, \|f-k\|_{\mathscr{M}_q^p}\right) \\ &= \frac{1}{2^{\frac{1}{t}}\|f\|_{\mathscr{M}_q^p}} \lambda\left(\|2g\|_{\mathscr{M}_q^p}, \|2h\|_{\mathscr{M}_q^p}\right) \\ &\geqslant \frac{1}{2^{\frac{1}{t}}\|f\|_{\mathscr{M}_q^p}} \lambda\left(2\|g\|_{\mathscr{M}_q^p}, 2\|h\|_{\mathscr{M}_q^p}\right) \\ &= 2^{1-\frac{1}{t}}. \end{aligned}$$

By applying the same arguments as above, we conclude that $C'_{\lambda,t}(\mathscr{M}_q^p) = 2^{1-\frac{1}{t}}$. $\square$

COROLLARY 2. *Let $1 \leqslant p < q < \infty$ and $1 \leqslant t < \infty$. Then*

$$\begin{aligned} C_{NJ}^{(t)}(\mathscr{M}_q^p) &= \bar{C}_{NJ}^{(t)}(\mathscr{M}_q^p) = C_Z^{(t)}(\mathscr{M}_q^p) = C_{-\infty}^{(t)}(\mathscr{M}_q^p) = J(\mathscr{M}_q^p) \\ &= A_{2,t}(\mathscr{M}_q^p) = T(\mathscr{M}_q^p) = 2. \end{aligned}$$

REMARK 2. Corollary 2 generalizes and improves existing results in [10, 13].

## 4. Discrete Morrey spaces

Let $\omega := \mathbb{N} \cup \{0\}$ and $m := (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$. Define

$$S_{m,N} := \left\{ k \in \mathbb{Z}^d \; : \; \|k - m\|_\infty \leqslant N \right\},$$

where $N \in \omega$ and $\|m\|_\infty = \max\left\{ |m_i| \; : \; 1 \leqslant i \leqslant d \right\}$. Then $|S_{m,N}| = (2N+1)^d$ denotes the cardinality of $S_{m,N}$ for each $m \in \mathbb{Z}^d$ and $N \in \omega$. Let $1 \leqslant p \leqslant q < \infty$ and define discrete Morrey spaces $\ell^p_q = \ell^p_q(\mathbb{Z}^d)$ as the set of all functions (sequences) $x : \mathbb{Z}^d \to \mathbb{R}$ such that

$$\|x\|_{\ell^p_q} := \sup_{m \in \mathbb{Z}^d,\, N \in \omega} |S_{m,N}|^{\frac{1}{q} - \frac{1}{p}} \left( \sum_{k \in S_{m,N}} |x(k)|^p \right)^{\frac{1}{p}} < \infty.$$

The discrete Morrey space $\ell^p_q$ with the above norm is a Banach space [9]. Note that for $p = q$, the space $\ell^p_q$ is identical with the space $\ell^q$.

Our result for discrete Morrey spaces is presented in the following theorem.

THEOREM 3. *Let* $1 \leqslant p < q < \infty$ *and* $1 \leqslant t < \infty$. *Then*

$$C_{\lambda,t}(\ell^p_q) = C'_{\lambda,t}(\ell^p_q) = 2^{1 - \frac{1}{t}}.$$

*Proof.* Suppose that $1 \leqslant p < q < \infty$ and $1 \leqslant t < \infty$. Let us first consider the case where $d = 1$. Assume that $n \in \mathbb{Z}$ be an even number with $n > 2^{\frac{q}{q-p}} - 1$, which can be written as $(n+1)^{\frac{1}{q} - \frac{1}{p}} < 2^{-\frac{1}{p}}$. Therefore $(n+1)^{\frac{1}{q} - \frac{1}{p}} 2^{\frac{1}{p}} < 1$. Consider the sequence $(x_k)_{k \in \mathbb{Z}}$ defined by

$$x_0 = x_n = 1 \text{ and } x_k = 0 \text{ for all } k \notin \{0, n\}$$

and the sequence $(y_k)_{k \in \mathbb{Z}}$ defined by

$$y_0 = 1,\; y_n = -1 \text{ and } y_k = 0 \text{ for all } k \notin \{0, n\}.$$

Hence, we have

$$\begin{aligned}
\|x\|_{\ell^p_q} &= \sup_{m \in \mathbb{Z}^d, N \in w} |S_{m,N}|^{\frac{1}{q} - \frac{1}{p}} \left( \sum_{k \in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}} \\
&= \max\left\{ 1, |S_{\frac{n}{2},\frac{n}{2}}|^{\frac{1}{q} - \frac{1}{p}} \left( \sum_{\frac{n}{2},\frac{n}{2}} |x_k|^p \right)^{\frac{1}{p}} \right\} \\
&= \max\left\{ 1, (n+1)^{\frac{1}{q} - \frac{1}{p}} 2^{\frac{1}{p}} \right\} = 1.
\end{aligned}$$

Similarly, we can show that $\|y\|_{\ell^p_q} = 1$. Moreover, we may observe that $\|x + y\|_{\ell^p_q} = 2$ and $\|x - y\|_{\ell^p_q} = 2$.

Now we shall consider the general case where $d \geqslant 1$. Assume that $n \in \mathbb{Z}$ be an even number with $n > 2^{\frac{q}{d(q-p)}} - 1$, which can be written as $(n+1)^{d(\frac{1}{q}-\frac{1}{p})} < 2^{-\frac{1}{p}}$. Therefore $(n+1)^{d(\frac{1}{q}-\frac{1}{p})} 2^{\frac{1}{p}} < 1$. Define the function $x : \mathbb{Z}^d \to \mathbb{R}$ by

$$x(k) = \begin{cases} 1, & k = (0,0,\ldots,0),(n,0,\ldots,0) \\ 0, & \text{otherwise} \end{cases}$$

and also define the function $y : \mathbb{Z}^d \to \mathbb{R}$ by

$$y(k) = \begin{cases} 1, & k = (0,0,\ldots,0), \\ -1, & k = (n,0,\ldots,0), \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we have

$$\begin{aligned} \|x\|_{\ell_q^p} &= \sup_{m \in \mathbb{Z}^d, N \in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k \in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}} \\ &= \max \left\{ 1, |S_{\frac{n}{2},\frac{n}{2}}|^{d(\frac{1}{q}-\frac{1}{p})} \left( \sum_{\frac{n}{2},\frac{n}{2}} |x_k|^p \right)^{\frac{1}{p}} \right\} \\ &= \max \left\{ 1, (n+1)^{d(\frac{1}{q}-\frac{1}{p})} 2^{\frac{1}{p}} \right\} = 1. \end{aligned}$$

By the same argument, we can show that $\|y\|_{\ell_q^p} = 1$. Moreover, we may observe that $\|x+y\|_{\ell_q^p} = 2$ and $\|x-y\|_{\ell_q^p} = 2$.

First, let us compute the constant $C_{\lambda,t}(m_q^p)$. Then, we obtain

$$\begin{aligned} C_{\lambda,t}(\ell_q^p) &\geqslant \frac{1}{\left(\|x\|_{\ell_q^p}^t + \|y\|_{\ell_q^p}^t\right)^{\frac{1}{t}}} \lambda\left(\|x+y\|_{\ell_q^p}, \|x-y\|_{\ell_q^p}\right) \\ &= \frac{1}{2^{\frac{1}{t}}} \lambda(2,2) = 2^{1-\frac{1}{t}}. \end{aligned}$$

So $C_{\lambda,t}(\ell_q^p) \geqslant 2^{1-\frac{1}{t}}$. Since $C_{\lambda,t}(\ell_q^p) \leqslant 2^{1-\frac{1}{t}}$, we conclude that $C_{\lambda,t}(\ell_q^p) = 2^{1-\frac{1}{t}}$.

Next, we move to the constant $C'_{\lambda,t}(\ell_q^p)$. Hence, we get

$$\begin{aligned} C'_{\lambda,t}(\ell_q^p) &\geqslant \frac{1}{2^{\frac{1}{t}}} \lambda\left(\|x+y\|_{m_q^p}, \|x-y\|_{m_q^p}\right) \\ &= \frac{1}{2^{\frac{1}{t}}} \lambda(2,2) = 2^{1-\frac{1}{t}}. \end{aligned}$$

By using similar arguments as before, we conclude that $C'_{\lambda,t}\ell_q^p) = 2^{1-\frac{1}{t}}$. $\square$

COROLLARY 3. *Let* $1 \leqslant p < q < \infty$ *and* $1 \leqslant t < \infty$. *Then*

$$C_{NJ}^{(t)}(\ell_q^p) = \bar{C}_{NJ}^{(t)}(\ell_q^p) = C_Z^{(t)}(\ell_q^p) = C_{-\infty}^{(t)}(\ell_q^p) = J(\ell_q^p) = A_{2,t}(\ell_q^p) = T(\ell_q^p) = 2.$$

REMARK 3. Corollary 3 generalizes and improves existing results in [10, 14].

As a consequence of Theorems 1, 2 and 3, we obtain the following result.

COROLLARY 4. *Morrey spaces* $\mathscr{M}_q^p$, *small Morrey spaces* $m_q^p$ *and discrete Morrey spaces* $\ell_q^p$ *with* $1 \leqslant p < q < \infty$ *are not uniformly non-square.*

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