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INEQUALITIES IN TIME–FREQUENCY ANALYSIS

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Abstract. Different types of Nash inequaliy, Sobolev inequality, Pitt inequality, logarithmic Sobolev inequality and Gross inequality are proved for the short time Fourier transform. Also, several formulations of Beckner's logarithmic uncertainty principle are established for the same transform.

1. Introduction

Uncertainty principles in harmonic analysis state that a nonzero function and its Fourier transform cannot be simultaneously and sharply localized, that is, it's impossible for a nonzero function to be arbitrary small as well as its Fourier transform. There are many different formulations of this general fact where the localization and the smallness have been interpreted by several ways. In the literature, many of these uncertainty principles are formulated as inequalities indeed many authors have showed in the Euclidean case different type of uncertainty inequalities as Sobolev and Pitt's inequalities [2, 3, 4, 29], which were generalized by Ghobber and Omri [11, 32] for the Bessel-Kingman hypergroup and by Ghobber and Soltani in the Dunkl setting [10, 30, 31]. For more detail about uncertainty principles we refer the reader to [9, 16].

It's well known that in signal analysis, the classical Fourier transform provide a global description of the spectrum of a given signal, however this description is unfortunately devoid of any chronology, loosing then the localization of each spectral component. To overcome this strong constraint, many authors introduced in the last decades several time-frequency representations where the temporal and the frequency variables are simultaneously present in the so-called time frequency plane. One of the most important time-frequency representations, called the short time Fourier transform, was introduced by Gabor in the sixties, more precisely for a nonzero function $g \in L^2(\mathbb{R}^d)$ called a window function, the short-time Fourier transform (STFT) is defined on $L^2(\mathbb{R}^d)$ by [14]

$$\forall (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,\ \mathscr{V}_g(f)(x,\omega) = \int_{\mathbb{R}^d} f(z)\overline{g(z-x)}e^{-i\langle z,\omega\rangle}\,d\mu_d(z), \qquad (1.1)$$

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where $\langle.,.\rangle$ is the classical inner product on $\mathbb{R}^d$ defined by $\langle z,\omega\rangle = \sum_{i=1}^{d} z_i\omega_i$ and $d\mu_d(z) = (2\pi)^{-d/2}dz$ is the normalized Lebesgue measure. Many harmonic analysis results related to the STFT will be developed in the second section and for more details about the harmonic analysis of the STFT, we refer the reader to [14].

Uncertainty principles for the STFT say that for a given function $f \in L^2(\mathbb{R}^d)$, $\mathscr{V}_g(f)$ cannot be arbitrary localized, otherwise f is zero. In this context, Lieb, Omri, Lamouchi, Gröchenig, Zimmermann, Bonami, Demange, Jaming, Fernandez, Galbis, Wilczok and recently Ghobber and Oueslati [6, 8, 12, 15, 20, 22, 33] showed many uncertainty principles for the STFT. Moreover, many authors were interested in generalizing different uncertainty inequalities to the Gabor analysis, so the main subject of this paper is a continuity of these works where we shall generalize through this work several uncertainty inequalities already proved in the Euclidean case, for more details about uncertainty principle in Gabor analysis we refer the reader to [7].

Recently, Kubo, Ogawa and Suguro proved different type of logarithmic Sobolev inequalities as well as Shannon inequality in the Euclidean case [19]. A part of these results were generalized very recently by Mejjaoli and Shah in the directional short time Fourier transform setting [25] and for the classical Gabor transform as well [24]. The results showed in this paper are complementary to those obtained by Mejjaoli and Shah in [24], and the only commun result which is the logarithmic uncertainty principle, has been obtained slightly differently.

The paper is organized as follows, in the second section we will recall some harmonic analysis tools connected with the STFT. In the last section we prove the main results of this paper, that are Nash, Sobolev, Pitt, Gross and logarithmic Sobolev inequalities. Also, we deduce different type of Heisenberg and Beckner's uncertainty principles related to the STFT.

2. Harmonic analysis associated with the short time Fourier transform

For every $x,\omega \in \mathbb{R}^d$, we denote by $\mathscr{M}_\omega$ and τ_x respectively the modulation and the shift operator defined by,

$$\mathscr{M}_\omega h(z) = e^{i\langle z,\omega\rangle} h(z), \tag{2.1}$$

and

$$T_x h(z) = h(z-x). \tag{2.2}$$

Then, by (2.1) and (2.2), we deduce that for every $x,\omega \in \mathbb{R}^d$, we have

$$\forall z \in \mathbb{R}^d, \qquad \mathscr{M}_\omega(T_x h)(z) = e^{i\langle z,\omega\rangle} h(z-x), \tag{2.3}$$

and

$$\forall z \in \mathbb{R}^d, \qquad T_x(\mathscr{M}_\omega h)(z) = e^{-i\langle x,\omega\rangle} e^{i\langle z,\omega\rangle} h(z-x). \tag{2.4}$$

Again, by (2.2), the STFT may be expressed by

$$\mathscr{V}_g(f)(x,\omega) = \widehat{f\overline{T_x g}}(\omega), \tag{2.5}$$

and by (1.1) and (2.1), we have

$$\mathscr{V}_g(f)(x,\omega) = \overline{\int_{\mathbb{R}^d} \overline{f(z)} e^{i\langle z,\omega\rangle} g(z-x) d\mu_d(z)} = \overline{\mathscr{M}_\omega \overline{f} * g(x)}, \tag{2.6}$$

where $*$ denotes the classical convolution product on $\mathbb{R}^d$.

According to Gröchenig [14], it's well known that for every $f,g \in L^2(\mathbb{R}^d)$, the function $\mathscr{V}_g(f)$ is uniformly continuous and bounded on the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and satisfies,

$$\|\mathscr{V}_g(f)\|_{\infty,2d} \leqslant \|f\|_{2,d}\|g\|_{2,d}. \tag{2.7}$$

Moreover, we have the following orthogonality property.

THEOREM 2.1. *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$ such that $g_1 \neq 0$ and $g_2 \neq 0$.*
Then, the functions $\mathscr{V}_{g_1}(f_1)$ and $\mathscr{V}_{g_2}(f_2)$ belong to $L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and we have the following orthogonality relation,

$$\big\langle \mathscr{V}_{g_1}(f_1), \mathscr{V}_{g_2}(f_2)\big\rangle_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} = \langle f_1, f_2\rangle_{\mathbb{R}^d} \overline{\langle g_1, g_2\rangle}_{\mathbb{R}^d}, \tag{2.8}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ (resp. $\langle\cdot,\cdot\rangle_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}$) is the usual inner product on $L^2(\mathbb{R}^d)$ (resp. $L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$). In particular, for every $f,g \in L^2(\mathbb{R}^d)$ such that $g \neq 0$, we have the following Plancherel's formula

$$\|\mathscr{V}_g(f)\|_{2,2d} = \|f\|_{2,d}\|g\|_{2,d}. \tag{2.9}$$

THEOREM 2.2. *Let g be a window function and $f \in L^2(\mathbb{R}^d)$. Then,*

i) For every $2 \leqslant p < +\infty$, $\mathscr{V}_g(f) \in L^p(\mathbb{R}^d \times \widehat{\mathbb{R}^d})$ and we have

$$\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} \left|\mathscr{V}_g(f)(x,\omega)\right|^p d\mu_{2d}(x,\omega) \leqslant \left(\frac{2}{p}\right)^d \|f\|_{2,d}^p \|g\|_{2,d}^p. \tag{2.10}$$

2i) For every $1 \leqslant p \leqslant 2$, we have

$$\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} \left|\mathscr{V}_g(f)(x,\omega)\right|^p d\mu_{2d}(x,\omega) \geqslant \left(\frac{2}{p}\right)^d \|f\|_{2,d}^p \|g\|_{2,d}^p. \tag{2.11}$$

PROPOSITION 2.3. *i) Let $1 < p,q < +\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$ and let $g \in L^2(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$ be a window function. Then, for every function $f \in L^2(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$, we have*

$$\|\mathscr{V}_g(f)\|_{\infty,2d} \leqslant \|f\|_{p,d}\|g\|_{q,d}. \tag{2.12}$$

2i) Let $g \in L^\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ be a window function. Then, for every function $f \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, we have

$$\|\mathscr{V}_g(f)\|_{\infty,2d} \leqslant \|f\|_{1,d}\|g\|_{\infty,d}. \tag{2.13}$$

PROPOSITION 2.4. *Let $g \in L^2(\mathbb{R}^d)$ be a window function. Then, for every $f \in L^2(\mathbb{R}^d)$, we have*

$$\forall (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,\ \mathscr{V}_g(f)(x,\omega) = e^{-i\langle \omega, x\rangle} \mathscr{V}_{\hat{g}}(\hat{f})(\omega, -x). \tag{2.14}$$

In the following we denote by $\mathscr{M}(\mathbb{R}^d)$ the vector space of measurable functions $f : \mathbb{R}^d \longrightarrow \mathbb{C}$. For every $\lambda > 0$ and $f \in \mathscr{M}(\mathbb{R}^d)$, we denote by f_λ the dilate of f defined on $\mathbb{R}^d$, by

$$f_\lambda(x) = \lambda^{\frac{d}{2}} f(\lambda x). \tag{2.15}$$

PROPOSITION 2.5. *For every $\lambda > 0$ and $f \in L^2(\mathbb{R}^d)$, the dilate f_λ belongs to $L^2(\mathbb{R}^d)$ and we have*

$$\|f_\lambda\|_{2,d} = \|f\|_{2,d}. \tag{2.16}$$

PROPOSITION 2.6. *Let $g \in L^2(\mathbb{R}^d)$ be a window function. Then, for every $\lambda > 0$ and for every $f \in L^2(\mathbb{R}^d)$, we have*

$$\forall (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,\ \mathscr{V}_{g_\lambda}(f_\lambda)(x,\omega) = \mathscr{V}_g(f)\left(\lambda x, \frac{\omega}{\lambda}\right) \tag{2.17}$$

Proof. See [14]. □

THEOREM 2.7. *Let $g \in S(\mathbb{R}^d)$ be a window function. Then, the following assertions are equivalents:*

(i) $f \in S(\mathbb{R}^d)$.

(ii) $\mathscr{V}_g(f) \in S\left(\mathbb{R}^d \times \widehat{\mathbb{R}}^d\right)$.

Proof. See [15]. □

In signal analysis, the short time Fourier transform is closely related to other common and well known time frequency distributions as the radar ambiguity function and the cross Wigner transform.

DEFINITION 2.8. Let $g \in L^2(\mathbb{R}^d)$ be a window function and $f \in L^2(\mathbb{R}^d)$. The radar ambiguity function $\mathscr{A}(f,g)$ is defined by

$$\forall (x,\omega) \in \mathbb{R}^d \times \hat{\mathbb{R}}^d, \quad \mathscr{A}(f,g)(x,\omega) = \int_{\mathbb{R}^d} f\left(t + \frac{x}{2}\right) \overline{g\left(t - \frac{x}{2}\right)} e^{-i\langle t,\omega\rangle} d\mu_d(t). \tag{2.18}$$

PROPOSITION 2.9. *Let $g \in L^2(\mathbb{R}^d)$ be a window function. Then, for every function $f \in L^2(\mathbb{R}^d)$, we have*

$$\forall (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \qquad \mathscr{A}(f,g)(x,\omega) = e^{\frac{i}{2}\langle x,\omega\rangle} \mathscr{V}_g(f)(x,\omega). \tag{2.19}$$

In particular,

$$\forall (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \qquad |\mathscr{A}(f,g)(x,\omega)| = |\mathscr{V}_g(f)(x,\omega)|. \tag{2.20}$$

Proof. See [14]. □

DEFINITION 2.10. Let $f, g \in L^2(\mathbb{R}^d)$. The cross Wigner transform of f and g is defined by,

$$\forall (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,\ \mathscr{W}(f,g)(x,\omega) = \int_{\mathbb{R}^d} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{-i\langle t,\omega\rangle}d\mu_d(t). \quad (2.21)$$

PROPOSITION 2.11. *Let $g \in L^2(\mathbb{R}^d)$ be a window function. Then, for every function $f \in L^2(\mathbb{R}^d)$, we have*

$$\forall (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \qquad \mathscr{W}(f,g)(x,\omega) = 2^d e^{2i\langle x,\omega\rangle}\mathscr{V}_{\check{g}}(f)(2x,2\omega). \quad (2.22)$$

In particular,

$$\forall (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \qquad |\mathscr{W}(f,g)(x,\omega)| = 2^d|\mathscr{V}_{\check{g}}(f)(2x,2\omega)|, \quad (2.23)$$

where

$$\forall x \in \mathbb{R}^d, \quad \check{g}(x) = g(-x). \quad (2.24)$$

Proof. See [14]. □

REMARK 2.12. According to Proposition 2.9 and Proposition 2.11, one can see that all the results that will be shown for the short-time Fourier transform, could be naturally and easily deduced for the radar ambiguity function and the cross Wigner transform as well.

3. Uncertainty inequalities for the short time Fourier transform

The purpose of this section is to prove the main results of this paper, more precisely we will establish many uncertainty inequalities related to the short time Fourier transform.

3.1. Nash type inequalities

In the next we recall the well known classical Nash's inequality in the Euclidean case [26].

THEOREM 3.1. *There exists a constant $C(d) \geqslant 0$ such that for every function $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we have*

$$\|f\|_{2,d}^{2+\frac{4}{d}} \leqslant C(d)\|f\|_{1,d}^{\frac{4}{d}}\||\xi|\hat{f}\|_{2,d}^2. \quad (3.1)$$

In the following we shall prove an analogous of Nash's inequality for the STFT.

PROPOSITION 3.2. (Nash type inequality) *Let $g\in L^\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ be a window function. Then,*

i) For every $s>0$ and for every function $f\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, we have

$$\|g\|_{2,d}^{2+\frac{2s}{d}}\|f\|_{2,d}^{2+\frac{2s}{d}} \leqslant C(s,d)\|f\|_{1,d}^{\frac{2s}{d}}\|g\|_{\infty,d}^{\frac{2s}{d}}\||(x,\omega)|^s\mathscr{V}_g(f)\|_{2,2d}^2, \tag{3.2}$$

where

$$C(s,d)=\left(\frac{s+d}{d}\right)^{\frac{s+d}{d}}\left(\frac{1}{2^d s\Gamma(d)}\right)^{\frac{s}{d}},$$

and Γ denotes the Euler gamma function.

2i) For every $s\geqslant 1$ and for every function $f\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, we have

$$\|g\|_{2,d}^{2+\frac{2s}{d}}\|f\|_{2,d}^{2+\frac{2s}{d}} \leqslant C(s,d)\|f\|_{1,d}^{\frac{2s}{d}}\|g\|_{\infty,d}^{\frac{2s}{d}}\||x|^s\mathscr{V}_g(f)\|_{2,2d}\||\omega|^s\mathscr{V}_g(f)\|_{2,2d}, \tag{3.3}$$

where

$$C(s,d)=2^{s+1}\left(\frac{s+d}{d}\right)^{\frac{s+d}{d}}\left(\frac{1}{2^d s\Gamma(d)}\right)^{\frac{s}{d}}.$$

Proof. Let $r>0$, we denote by B_r the open ball given by

$$B_r=\{(x,\omega)\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d;|(x,\omega)|<r\}.$$

By using Plancherel's theorem, we obtain

$$\|f\|_{2,d}^2\|g\|_{2,d}^2=\|\mathscr{V}_g(f)\|_{2,2d}^2=\|\chi_{B_r}\mathscr{V}_g(f)\|_{2,2d}^2+\|\chi_{B_r^c}\mathscr{V}_g(f)\|_{2,2d}^2.$$

By using (2.13), we get

$$\begin{aligned}\|\chi_{B_r}\mathscr{V}_g(f)\|_{2,2d}^2&=\int_{B_r}\left|\mathscr{V}_g(f)(x,\omega)\right|^2 d\mu_{2d}(x,\omega)\\&\leqslant \mu_{2d}(B_r)\left\|\mathscr{V}_g(f)\right\|_{\infty,2d}^2\\&\leqslant \mu_{2d}(B_r)\|f\|_{1,d}^2\|g\|_{\infty,d}^2\\&=\frac{r^{2d}}{2^d d\Gamma(d)}\|f\|_{1,d}^2\|g\|_{\infty,d}^2.\end{aligned}$$

On the other hand

$$\begin{aligned}\|\chi_{B_r^c}\mathscr{V}_g(f)\|_{2,2d}^2&=\int_{B_r^c}\left|\mathscr{V}_g(f)(x,\omega)\right|^2 d\mu_{2d}(x,\omega)\\&\leqslant r^{-2s}\int_{B_r^c}|(x,\omega)|^{2s}\left|\mathscr{V}_g(f)(x,\omega)\right|^2 d\mu_{2d}(x,\omega)\\&\leqslant r^{-2s}\||(x,\omega)|^s\mathscr{V}_g(f)\|_{2,2d}^2.\end{aligned}$$

Hence,

$$\|f\|_{2,d}^2\|g\|_{2,d}^2 \leqslant \frac{1}{2^d d\Gamma(d)}\|f\|_{1,d}^2\|g\|_{\infty,d}^2 r^{2d} + \||(x,\omega)|^s\mathscr{V}_g(f)\|_{2,2d}^2 r^{-2s}. \tag{3.4}$$

By minimizing the right-hand side of inequality (3.4) with respect to the variable $r>0$, we get

$$\|f\|_{2,d}^2\|g\|_{2,d}^2 \leqslant \frac{s+d}{d}\left(\frac{1}{2^d s\Gamma(d)}\right)^{\frac{s}{s+d}} \|f\|_{1,d}^{\frac{2s}{s+d}}\|g\|_{\infty,d}^{\frac{2s}{s+d}}\||(x,\omega)|^s\mathscr{V}_g(f)\|_{2,2d}^{\frac{2d}{s+d}}. \tag{3.5}$$

Then, (3.2) is proved. On the other hand, we have

$$\begin{aligned}\|g\|_{2,d}^{2+\frac{2s}{d}}\|f\|_{2,d}^{2+\frac{2s}{d}} &\leqslant C(s,d)\|f\|_{1,d}^{\frac{2s}{d}}\|g\|_{\infty,d}^{\frac{2s}{d}}\||(x,\omega)|^s\mathscr{V}_g(f)\|_{2,2d}^2\\ &\leqslant 2^s C(s,d)\|f\|_{1,d}^{\frac{2s}{d}}\|g\|_{\infty,d}^{\frac{2s}{d}}\left(\||x|^s\mathscr{V}_g(f)\|_{2,2d}^2 + \||\omega|^s\mathscr{V}_g(f)\|_{2,2d}^2\right).\end{aligned}$$

Let $\lambda>0$, replacing f,g by f_λ, g_λ in the last inequality and by using (2.16) and (2.17) we get

$$\|g\|_{2,d}^{2+\frac{2s}{d}}\|f\|_{2,d}^{2+\frac{2s}{d}} \leqslant 2^s C(s,d)\|f\|_{1,d}^{\frac{2s}{d}}\|g\|_{\infty,d}^{\frac{2s}{d}}\left(\frac{1}{\lambda^{2s}}\||x|^s\mathscr{V}_g(f)\|_{2,2d}^2 + \lambda^{2s}\||\omega|^s\mathscr{V}_g(f)\|_{2,2d}^2\right), \tag{3.6}$$

by minimizing the right-hand side of (3.6) with respect to the variable $\lambda>0$, we get

$$\|g\|_{2,d}^{2+\frac{2s}{d}}\|f\|_{2,d}^{2+\frac{2s}{d}} \leqslant 2^{s+1} C(s,d)\|f\|_{1,d}^{\frac{2s}{d}}\|g\|_{\infty,d}^{\frac{2s}{d}}\||x|^s\mathscr{V}_g(f)\|_{2,2d}\||\omega|^s\mathscr{V}_g(f)\|_{2,2d}. \quad \square$$

In the folowing we recall the well known Carlson's inequality that we shall use later, we will also give the proof for sake of completeness.

PROPOSITION 3.3. (Carlson inequality) *Let $s>0$, $0<p<2$ and $0<q<+\infty$ such that $\frac{1}{p}=\frac{1}{2}+\frac{1}{q}$. Then, for every function $f\in L^p(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, we have*

$$\|f\|_{p,d}^{1+\frac{sq}{d}} \leqslant C(s,d,q)\|f\|_{2,d}^{\frac{sq}{d}}\||\xi|^s f\|_{p,d}, \tag{3.7}$$

where

$$C(s,d,q) = \frac{sq}{d}\left(1+\frac{d}{sq}\right)^{1+\frac{sq}{d}}\left(\frac{1}{2^{\frac{d}{2}-1}d\Gamma\left(\frac{d}{2}\right)}\right)^{\frac{s}{d}}.$$

Proof. By using Hölder's inequality we obtain

$$\begin{aligned}\|f\|_{p,d} &= \|\chi_{B_r}f\|_{p,d} + \|\chi_{B_r^c}f\|_{p,d}\\ &\leqslant (\mu_d(B_r))^{\frac{1}{q}}\|f\|_{2,d} + r^{-s}\||\xi|^s f\|_{p,d}\\ &= \left(\frac{1}{2^{\frac{d}{2}-1}d\Gamma(\frac{d}{2})}\right)^{\frac{1}{q}} r^{\frac{d}{q}}\|f\|_{2,d} + r^{-s}\||\xi|^s f\|_{p,d}.\end{aligned} \tag{3.8}$$

By minimizing the right-hand side of inequality (3.8) with respect to the variable r, we get

$$\|f\|_{p,d} \leqslant \left(\frac{sq+d}{sq}\right)\left(\frac{sq}{d}\right)^{\frac{d}{d+sq}}\left(\frac{1}{2^{\frac{d}{2}-1}d\Gamma(\frac{d}{2})}\right)^{\frac{s}{d+sq}}\|f\|_{2,d}^{\frac{sq}{d+sq}}\||\xi|^s f\|_{p,d}^{\frac{d}{d+sq}}. \tag{3.9}$$

Which is the desired result. □

THEOREM 3.4. *Let $g \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ be a window function. Then, for every $s \geqslant 1$ and for every function $f \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, we have*

$$\|g\|_{2,d}^{2+\frac{2s}{d}}\|f\|_{2,d}^2\|f\|_{1,d} \leqslant C(s,d)\|g\|_{\infty,d}^{\frac{2s}{d}}\||\xi|^s f\|_{1,d}\||x|^s\mathscr{V}_g(f)\|_{2,2d}\||\omega|^s\mathscr{V}_g(f)\|_{2,2d}, \tag{3.10}$$

where

$$C(s,d) = \left(\frac{s+d}{d}\right)^{\frac{s+d}{d}}\left(\frac{1}{2^d s\Gamma(d)}\right)^{\frac{s}{d}}\frac{2s}{d}\left(1+\frac{d}{2s}\right)^{1+\frac{2s}{d}}\left(\frac{1}{2^{\frac{d}{2}-1}d\Gamma\left(\frac{d}{2}\right)}\right)^{\frac{s}{d}}.$$

Proof. The proof is a consequence of (3.3) and (3.7) for $p=1$. □

REMARK 3.5. From (2.11) and Carlson's inequality (3.7), we can deduce the following well known Heisenberg's type inequality for the short-time Fourier transform, that is for every window function $g \in S(\mathbb{R}^d)$, for every $1 \leqslant p < 2$, $s > 0$ such that $sp \geqslant 2$ and for all $0 < q < +\infty$ such that $\frac{1}{p} = \frac{1}{2}+\frac{1}{q}$, we have for every $f \in S(\mathbb{R}^d)$

$$\|f\|_{2,d}^2\|g\|_{2,d}^2 \leqslant C(s,d,q,p)\||x|^s\mathscr{V}_g(f)\|_{p,2d}\||\omega|^s\mathscr{V}_g(f)\|_{p,2d}, \tag{3.11}$$

where

$$C(s,d,q,p) = \left(\frac{2}{p}\right)^{\frac{2}{p}(d+\frac{sq}{2})}2^{-\frac{2}{p}(1+\frac{sp}{2})}\left(\frac{sq}{2d}\left(1+\frac{2d}{sq}\right)^{1+\frac{sq}{2d}}\left(\frac{1}{2^d d\Gamma(d)}\right)^{\frac{s}{2d}}\right)^{-2}.$$

PROPOSITION 3.6. *Let $1 < p,q < +\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$, $g \in L^q(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ be a window function. Then, for every $s > 0$ and for every $f \in L^p(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, we have*

$$\|g\|_{2,d}^{2+\frac{2s}{d}}\|f\|_{2,d}^{2+\frac{2s}{d}} \leqslant C(s,d)\|f\|_{p,d}^{\frac{2s}{d}}\|g\|_{q,d}^{\frac{2s}{d}}\||(x,\omega)|^s\mathscr{V}_g(f)\|_{2,2d}^2, \tag{3.12}$$

where

$$C(s,d) = \left(\frac{s+d}{d}\right)^{\frac{s+d}{d}}\left(\frac{1}{2^d s\Gamma(d)}\right)^{\frac{s}{d}}.$$

If $s \geqslant 1$, we have

$$\|g\|_{2,d}^{2+\frac{2s}{d}}\|f\|_{2,d}^{2+\frac{2s}{d}} \leqslant C(s,d)\|f\|_{p,d}^{\frac{2s}{d}}\|g\|_{q,d}^{\frac{2s}{d}}\||x|^s\mathscr{V}_g(f)\|_{2,2d}\||\omega|^s\mathscr{V}_g(f)\|_{2,2d}, \tag{3.13}$$

where

$$C(s,d)=2^{s+1}\left(\frac{s+d}{d}\right)^{\frac{s+d}{d}}\left(\frac{1}{2^d s\Gamma(d)}\right)^{\frac{s}{d}}.$$

Proof. Let $r>0$, by using Plancherel's theorem we obtain

$$\|f\|_{2,d}^2\|g\|_{2,d}^2=\|\mathscr{V}_g(f)\|_{2,2d}^2=\|\chi_{B_r}\mathscr{V}_g(f)\|_{2,2d}^2+\|\chi_{B_r^c}\mathscr{V}_g(f)\|_{2,2d}^2.$$

By using (2.12), we get

$$\begin{aligned}\|\chi_{B_r}\mathscr{V}_g(f)\|_{2,2d}^2&=\int_{B_r}\left|\mathscr{V}_g(f)(x,\omega)\right|^2d\mu_{2d}(x,\omega)\\&\leqslant\mu_{2d}(B_r)\left\|\mathscr{V}_g(f)\right\|_{\infty,2d}^2\\&\leqslant\mu_{2d}(B_r)\|f\|_p^2\|g\|_q^2\\&=\frac{r^{2d}}{2^d d\Gamma(d)}\|f\|_p^2\|g\|_q^2.\end{aligned}$$

On the other hand

$$\begin{aligned}\|\chi_{B_r^c}\mathscr{V}_g(f)\|_{2,2d}^2&=\int_{B_r^c}\left|\mathscr{V}_g(f)(x,\omega)\right|^2d\mu_{2d}(x,\omega)\\&\leqslant r^{-2s}\int_{B_r^c}|(x,\omega)|^{2s}\left|\mathscr{V}_g(f)(x,\omega)\right|^2d\mu_{2d}(x,\omega)\\&\leqslant r^{-2s}\||(x,\omega)|^s\mathscr{V}_g(f)\|_{2,2d}^2.\end{aligned}$$

Hence,

$$\|f\|_{2,d}^2\|g\|_{2,d}^2\leqslant\frac{1}{2^d d\Gamma(d)}\|f\|_p^2\|g\|_q^2r^{2d}+\||(x,\omega)|^s\mathscr{V}_g(f)\|_{2,2d}^2r^{-2s}.\tag{3.14}$$

By minimizing the right-hand side of that inequality over $r>0$, we get

$$\|f\|_{2,d}^2\|g\|_{2,d}^2\leqslant\frac{s+d}{d}\left(\frac{1}{2^d s\Gamma(d)}\right)^{\frac{s}{s+d}}\|f\|_p^{\frac{2s}{s+d}}\|g\|_q^{\frac{2s}{s+d}}\||(x,\omega)|^s\mathscr{V}_g(f)\|_{2,2d}^{\frac{2d}{s+d}}.\tag{3.15}$$

We get the desired result. □

3.2. Sobolev inequality

First we recall the following classical Sobolev inequality in the Euclidean case.

THEOREM 3.7. (Classical Sobolev inequality) *Let $d\geqslant 3$. Then, for every function $f\in\mathscr{C}_c^\infty(\mathbb{R}^d)$, we have*

$$\|f\|_{\frac{2d}{d-2},d}\leqslant C_d\||\xi|\hat{f}\|_{2,d},\tag{3.16}$$

where $\mathscr{C}_c^\infty(\mathbb{R}^d)$ *denotes the vector space of smooth functions with compact support and*

$$C_d = \frac{1}{(2\pi)^{-\frac{d+1}{2}}\pi d(d-2)}\left(\frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)}\right)^{\frac{2}{d}}.$$

Proof. See [29]. □

THEOREM 3.8. (Sobolev's inequality for STFT) *Let* $d \geqslant 3$, $g \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ *be a window function. Then, for every* $f \in \mathscr{C}^\infty(\mathbb{R}^d)$*, we have*

$$\|g\|_{2,d}\|f\|_{\frac{2d}{d-2},d} \leqslant C_d\||\omega|\mathscr{V}_g(f)\|_{2,2d}, \tag{3.17}$$

where $\mathscr{C}^\infty(\mathbb{R}^d)$ *denotes the vector space of smooth functions.*

Proof. Since $f \in \mathscr{C}^\infty(\mathbb{R}^d)$ and $g \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ then for every $x \in \mathbb{R}^d$, $fT_xg \in \mathscr{C}_c^\infty(\mathbb{R}^d)$. Then, by using the classical Sobolev inequality (3.16) we obtain

$$\left(\int_{\mathbb{R}^d}|f(\omega)|^{\frac{2d}{d-2}}|T_xg(\omega)|^{\frac{2d}{d-2}}d\mu_d(\omega)\right)^{\frac{d-2}{d}} \leqslant C_d^2\int_{\mathbb{R}^d}|\omega|^2|\mathscr{V}_g(f)(x,\omega)|^2d\mu_d(\omega). \tag{3.18}$$

Hence, by integrating both sides of equation (3.18) with respect to the variable x and Minkowski's inequality for integrals, we get

$$\begin{aligned}\|g\|_2^2\|f\|_{\frac{2d}{d-2},d}^2 &= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(\omega)|^2|T_xg(\omega)|^2d\mu_d(x)\right)^{\frac{d}{d-2}}d\mu_d(\omega)\right)^{\frac{d-2}{d}}\\ &\leqslant \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(\omega)|^{\frac{2d}{d-2}}|T_xg(\omega)|^{\frac{2d}{d-2}}d\mu_d(\omega)\right)^{\frac{d-2}{d}}d\mu_d(x)\\ &\leqslant C_d^2\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d}|\omega|^2|\mathscr{V}_g(f)(x,\omega)|^2d\mu_{2d}(x,\omega),\end{aligned}$$

which corresponds to the desired result. □

In the following we shall deduce type of Nash's inequality from Sobolev inequality.

COROLLARY 3.9. *Let* $g \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ *be a window function. Then, for every function* $f \in \mathscr{C}^\infty(\mathbb{R}^d)$*, we have*

$$\|g\|_{2,d}^{1+\frac{2}{d}}\|f\|_{2,d}^{1+\frac{2}{d}} \leqslant \|g\|_{2,d}^{\frac{2}{d}}\|f\|_{1,d}^{\frac{2}{d}}C_d\||\omega|\mathscr{V}_g(f)\|_{2,2d}, \tag{3.19}$$

Proof. By interpolation theorem, we get

$$\|f\|_{2,d} \leqslant \|f\|_{1,d}^{1-\theta}\|f\|_{q,d}^{\theta},$$

where $q=\frac{2d}{d-2}$, $\frac{1}{2}=\frac{\theta}{q}+1-\theta$, and $\theta=\frac{d}{d+2}$. Moreover, by (3.17), we obtain

$$\begin{aligned}\|g\|_{2,d}\|f\|_{2,d} &\leqslant \|g\|_{2,d}\|f\|_{1,d}^{\frac{2}{d+2}}\|f\|_{\frac{2d}{d-2},d}^{\frac{d}{d+2}}\\ &= \|g\|_{2,d}^{1-\frac{d}{d+2}}\|f\|_{1,d}^{\frac{2}{d+2}}\left(\|g\|_{2,d}\|f\|_{\frac{2d}{d-2},d}\right)^{\frac{d}{d+2}}\\ &\leqslant C_d^{\frac{d}{d+2}}\|g\|_{2,d}^{\frac{2}{d+2}}\|f\|_{1,d}^{\frac{2}{d+2}}\||\omega|\mathscr{V}_g(f)\|_{2,2d}^{\frac{d}{d+2}},\end{aligned}$$

hence, we get the desired result. □

COROLLARY 3.10. *Let $g\in\mathscr{C}_c^\infty(\mathbb{R}^d)$ be a window function. Then, for every function $f\in\mathscr{C}^\infty(\mathbb{R}^d)$, we have*

$$\|g\|_{2,d}\|f\|_{2+\frac{4}{d},d}^{1+\frac{2}{d}}\leqslant C_d\|f\|_{2,d}^{\frac{2}{d}}\||\omega|\mathscr{V}_g(f)\|_{2,2d}. \tag{3.20}$$

Proof. By interpolation theorem, we have

$$\|f\|_{p,d}\leqslant\|f\|_{2,d}^{1-\theta}\|f\|_{q,d}^{\theta},$$

where $p=2+\frac{4}{d}$, $q=\frac{2d}{d-2}$, $\frac{1}{p}=\frac{\theta}{q}+\frac{1-\theta}{2}$ and $\theta=1-\frac{2}{d+2}$. Therefore, by using Sobolev's inequality (3.17), we get the result for the STFT. □

Now, we will show that Sobolev's inequality (3.17) implies a logarithmic Sobolev inequality.

COROLLARY 3.11. *Let $d\geqslant 3$ and $g\in\mathscr{C}_c^\infty(\mathbb{R}^d)$ be a window function. Then, for every function $f\in\mathscr{C}^\infty(\mathbb{R}^d)\setminus\{0\}$, we have*

$$\int_{\mathbb{R}^d}\frac{|f(x)|^2}{\|f\|_{2,d}^2}\ln\left(\frac{|f(x)|^2}{\|f\|_{2,d}^2}\right)d\mu_d(x)\leqslant\frac{d}{2}\ln\left(\frac{C_d^2\||\omega|\mathscr{V}_g(f)\|_{2,2d}^2}{\|f\|_{2,d}^2\|g\|_{2,d}^2}\right). \tag{3.21}$$

Proof. Let $g\in\mathscr{C}_c^\infty(\mathbb{R}^d)$ be a window function and $f\in\mathscr{C}^\infty(\mathbb{R}^d)$ such that $\|g\|_{2,d}=\|f\|_{2,d}=1$. Then, by using Jensen's inequality and Sobolev's inequality (3.17) we obtain

$$\begin{aligned}\frac{2}{d-2}\int_{\mathbb{R}^d}|f(x)|^2\ln(|f(x)|^2)d\mu_d(x) &\leqslant \ln\left(\int_{\mathbb{R}^d}|f(x)|^{2+\frac{4}{d-2}}d\mu_d(x)\right)\\ &= \frac{2d}{d-2}\ln\left(\|f\|_{\frac{2d}{d-2},d}\right)\\ &= \frac{2d}{d-2}\ln\left(\|f\|_{\frac{2d}{d-2},d}\|g\|_{2,d}\right)\\ &\leqslant \frac{2d}{d-2}\ln\left(C_d\||\omega|\mathscr{V}_g(f)\|_{2,2d}\right).\end{aligned}$$

Then,

$$\int_{\mathbb{R}^d} |f(x)|^2 \ln(|f(x)|^2) d\mu_d(x) \leqslant \frac{d}{2} \ln\left(C_d^2 \||\omega|\mathscr{V}_g(f)\|_{2,2d}^2\right). \tag{3.22}$$

If f is a nonzero function of $\mathscr{C}^\infty(\mathbb{R}^d)$ then by replacing f, g by $\frac{f}{\|f\|_{2,d}}, \frac{g}{\|g\|_{2,d}}$ in (3.22) we get the result. $\square$

3.3. Pitt's Inequalities

In the following we recall the classical Pitt's inequality in the Euclidean case.

THEOREM 3.12. *For every $f \in S(\mathbb{R}^d)$ and $0 \leqslant \alpha < d$, we have*

$$\int_{\mathbb{R}^d} |y|^{-\alpha} |\hat{f}(y)|^2 d\mu_d(y) \leqslant \frac{1}{2^\alpha} \left(\frac{\Gamma\left(\frac{d-\alpha}{4}\right)}{\Gamma\left(\frac{d+\alpha}{4}\right)} \right)^2 \int_{\mathbb{R}^d} |x|^\alpha |f(x)|^2 d\mu_d(x). \tag{3.23}$$

Proof. See [2]. $\square$

THEOREM 3.13. (Pitt's inequality for STFT) *Let $g \in S(\mathbb{R}^d)$ be a window function. Then, for every function $f \in S(\mathbb{R}^d)$ and $0 \leqslant \alpha < d$, we have*

$$\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |\omega|^{-\alpha} |\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) \leqslant \frac{1}{2^\alpha} \left(\frac{\Gamma\left(\frac{d-\alpha}{4}\right)}{\Gamma\left(\frac{d+\alpha}{4}\right)} \right)^2 \|g\|_{2,d}^2 \int_{\mathbb{R}^d} |t|^\alpha |f(t)|^2 d\mu_d(t). \tag{3.24}$$

Proof. Since $g, f \in S(\mathbb{R}^d)$ this implies that for every $x \in \mathbb{R}^d$, $f\overline{T_x g} \in S(\mathbb{R}^d)$. Using (2.5) and replacing f by $f\overline{T_x g}$ in (3.23), we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} |\omega|^{-\alpha} |\mathscr{V}_g(f)(x,\omega)|^2 d\mu_d(\omega) &= \int_{\mathbb{R}^d} |\omega|^{-\alpha} |\widehat{f\overline{T_x g}}(w)|^2 d\mu_{2d}(\omega) \\ &\leqslant \frac{1}{2^\alpha} \left(\frac{\Gamma\left(\frac{d-\alpha}{4}\right)}{\Gamma\left(\frac{d+\alpha}{4}\right)} \right)^2 \int_{\mathbb{R}^d} |t|^\alpha |f\overline{T_x g}(t)|^2 d\mu_d(t). \end{aligned} \tag{3.25}$$

Hence, by integrating both sides of (3.25) with respect to the variable x, we get

$$\begin{aligned} &\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |\omega|^{-\alpha} |\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) \\ &\leqslant \frac{1}{2^\alpha} \left(\frac{\Gamma\left(\frac{d-\alpha}{4}\right)}{\Gamma\left(\frac{d+\alpha}{4}\right)} \right)^2 \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |t|^\alpha |f\overline{T_x g}(t)|^2 d\mu_{2d}(t,x) \\ &= \frac{1}{2^\alpha} \left(\frac{\Gamma\left(\frac{d-\alpha}{4}\right)}{\Gamma\left(\frac{d+\alpha}{4}\right)} \right)^2 \|g\|_{2,d}^2 \int_{\mathbb{R}^d} |t|^\alpha |f(t)|^2 d\mu_d(t). \quad \square \end{aligned}$$

THEOREM 3.14. *Let $g \in S(\mathbb{R}^d)$ be a window function. Then, for every $f \in S(\mathbb{R}^d)$ and $0 \leqslant \alpha < d$, we have*

$$\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |x|^{-\alpha} |\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) \leqslant \frac{1}{2^\alpha} \left(\frac{\Gamma\left(\frac{d-\alpha}{4}\right)}{\Gamma\left(\frac{d+\alpha}{4}\right)} \right)^2 \|g\|_{2,d}^2 \int_{\mathbb{R}^d} |t|^\alpha |\hat{f}(t)|^2 d\mu_d(t). \tag{3.26}$$

Proof. By using (2.5) and (2.14) we have

$$\begin{aligned} &\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |x|^{-\alpha} |\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) \\ &= \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |x|^{-\alpha} |\mathscr{V}_{\hat{g}}(\hat{f})(\omega,-x)|^2 d\mu_{2d}(x,\omega) \\ &= \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |x|^{-\alpha} |\widehat{\hat{f}\overline{T_\omega \hat{g}}}(-x)|^2 d\mu_{2d}(x,\omega) \\ &= \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |x|^{-\alpha} |\widehat{\hat{f}\overline{T_\omega \hat{g}}}(x)|^2 d\mu_{2d}(x,\omega) \end{aligned}$$

By using (3.23) we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |x|^{-\alpha} |\widehat{\hat{f}\overline{T_\omega \hat{g}}}(x)|^2 d\mu_{2d}(x,\omega) \\ &\leqslant \frac{1}{2^\alpha} \left(\frac{\Gamma\left(\frac{d-\alpha}{4}\right)}{\Gamma\left(\frac{d+\alpha}{4}\right)} \right)^2 \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |y|^\alpha |\hat{f}(y)|^2 |\overline{T_\omega \hat{g}}(y)|^2 d\mu_{2d}(y,\omega) \\ &= \frac{1}{2^\alpha} \left(\frac{\Gamma\left(\frac{d-\alpha}{4}\right)}{\Gamma\left(\frac{d+\alpha}{4}\right)} \right)^2 \|T_\omega \hat{g}\|_{2,d} \int_{\mathbb{R}^d} |y|^\alpha |\hat{f}(y)|^2 d\mu_d(y) \\ &= \frac{1}{2^\alpha} \left(\frac{\Gamma\left(\frac{d-\alpha}{4}\right)}{\Gamma\left(\frac{d+\alpha}{4}\right)} \right)^2 \|g\|_{2,d} \int_{\mathbb{R}^d} |y|^\alpha |\hat{f}(y)|^2 d\mu_d(y), \end{aligned}$$

which is the desired result. □

3.4. Other form of Pitt's inequality

In the following, we give the classical sharp Pitt's inequality that we shall use.

THEOREM 3.15. *Let $1 \leqslant p < 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for every $f \in S(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} |\xi|^{d(1-\frac{2}{p})} |\hat{f}(\xi)|^2 d\mu_d(\xi) \leqslant K_p \|f\|_{p,d}^2, \tag{3.27}$$

where

$$K_p = 2^{\frac{d}{2}-\frac{d}{p}} \frac{\Gamma\left(\frac{d}{p'}\right)}{\Gamma\left(\frac{d}{p}\right)} \left(\frac{\Gamma(d)}{\Gamma\left(\frac{d}{p}\right)} \right)^{\frac{2}{p}-1}.$$

Proof. See [2]. □

REMARK 3.16. (Pitt's inequality for STFT) From (2.5) and (3.27), one can easily deduce the following well known Pitt's inequality for STFT that is for a window function $g \in S(\mathbb{R}^d)$ and for every $1 \leqslant p < 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$, we have for every $f \in S(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |\omega|^{d(1-\frac{2}{p})} |\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) \leqslant K_p \|g\|_{2,d}^2 \|f\|_{p,d}^2. \tag{3.28}$$

3.5. Generalized Pitt's inequality

In the following we first recall the generalized Pitt's inequality in the Euclidean case which will allow us to prove a generalized Pitt's inequality type for the STFT.

THEOREM 3.17. *Let $1 < p \leqslant q < \infty$, $0 < \alpha < \frac{d}{q}$, $0 < \beta < \frac{d}{p'}$ and $d \geqslant 2$. Then, there is a nonnegative constant A such that for every $f \in S(\mathbb{R}^d)$, we have*

$$\left(\int_{\mathbb{R}^d} ||x|^{-\alpha} \hat{f}(x)|^q d\mu_d(x) \right)^{\frac{1}{q}} \leqslant A \left(\int_{\mathbb{R}^d} ||x|^{\beta} f(x)|^p d\mu_d(x) \right)^{\frac{1}{p}}, \tag{3.29}$$

with the index constraint

$$\frac{d}{p} + \frac{d}{q} + \beta - \alpha = d, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Proof. See [3]. □

THEOREM 3.18. (Generalized Pitt's inequality for STFT) *Let $g \in S(\mathbb{R}^d)$ be a window function , $1 < p \leqslant q < +\infty$, $0 < \alpha < \frac{d}{q}$, $0 < \beta < \frac{d}{p'}$ and $d \geqslant 2$. Then, there is a nonnegative constant A such that for every $f \in S(\mathbb{R}^d)$, we have*

$$\| |\omega|^{-\alpha} \mathscr{V}_g(f) \|_{q,2d} \leqslant A \|g\|_{q,d} \| |x|^{\beta} f \|_{p,d}, \tag{3.30}$$

with the index constraint

$$\frac{d}{p} + \frac{d}{q} + \beta - \alpha = d, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Proof. Since $g, f \in S(\mathbb{R}^d)$ this implies that for every $x \in \mathbb{R}^d$, $f\overline{T_x g} \in S(\mathbb{R}^d)$.
We use (2.5) and we replace f by $f\overline{T_x g}$ in (3.29) we obtain

$$\int_{\mathbb{R}^d} ||w|^{-\alpha} \mathscr{V}_g(f)(x,\omega)|^q d\mu_d(\omega) \leqslant A^q \left(\int_{\mathbb{R}^d} ||\omega|^{\beta} f(\omega) T_x g(\omega)|^p d\mu_d(\omega) \right)^{\frac{q}{p}}. \tag{3.31}$$

Hence, by integrating both sides of equation (3.31) with respect to the variable x and using Minkowski's inequality for integrals, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} ||w|^{-\alpha}\mathscr{V}_g(f)(x,\omega)|^q \mu_{2d}(x,\omega)\\
&\leqslant A^q \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} ||\omega|^\beta f(\omega)T_x g(\omega)|^p d\mu_d(\omega)\right)^{\frac{q}{p}} d\mu_d(x)\\
&\leqslant A^q \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} ||\omega|^\beta f(\omega)T_x g(\omega)|^q d\mu_d(x)\right)^{\frac{p}{q}} d\mu_d(\omega)\right)^{\frac{q}{p}}\\
&= A^q \left(\int_{\mathbb{R}^d} |f(\omega)|^p|\omega|^{\beta p}\|g\|_q^p d\mu_d(\omega)\right)^{\frac{q}{p}}\\
&= A^q\|g\|_q^q \left(\int_{\mathbb{R}^d} |f(\omega)|^p|\omega|^{\beta p} d\mu_d(\omega)\right)^{\frac{q}{p}},
\end{aligned}$$

which corresponds to the desired result. □

3.6. Logarithmic uncertainty principle

In this subsection, we use the Beckner's logarithmic uncertainty principle for the classical Fourier transform in the Euclidean case to obtain the Logarithmic uncertainty principle for the STFT.

THEOREM 3.19. (Beckner's logarithmic uncertainty principle) *For every $f\in S(\mathbb{R}^d)$ we have*

$$\begin{aligned}
&\int_{\mathbb{R}^d} \ln|x||f(x)|^2 d\mu_d(x) + \int_{\mathbb{R}^d}\ln|y||\hat{f}(y)|^2 d\mu_d(y)\\
&\geqslant \left(\psi\left(\frac{d}{4}\right)+\ln(2)\right)\int_{\mathbb{R}^d}|f(x)|^2 d\mu_d(x),
\end{aligned} \tag{3.32}$$

where ψ is the digamma function given by

$$\psi(t) = \frac{d}{dt}\left[\ln\Gamma(t)\right].$$

Proof. See [2]. □

THEOREM 3.20. (Logarithmic uncertainty for STFT) *Let $g\in S(\mathbb{R}^d)$ be a window function. Then, for every $f\in S(\mathbb{R}^d)$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \ln|\omega||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) + \|g\|_{2,d}^2\int_{\mathbb{R}^d}\ln|t||f(t)|^2 d\mu_d(t)\\
&\geqslant \left(\psi\left(\frac{d}{4}\right)+\ln(2)\right)\|f\|_{2,d}^2\|g\|_{2,d}^2.
\end{aligned} \tag{3.33}$$

Proof. Since $g, f \in S(\mathbb{R}^d)$, this implies that for every $x \in \mathbb{R}^d$, $f\overline{T_x g} \in S(\mathbb{R}^d)$. Using (2.5) and replacing f by $f\overline{T_x g}$ in Theorem 3.19, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \ln|\omega||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_d(\omega) + \int_{\mathbb{R}^d} \ln|t||f(t)\overline{T_x g}(t)|^2 d\mu_d(t) \\
&= \int_{\mathbb{R}^d} \ln|\omega||\widehat{f\overline{T_x g}}(\omega)|^2 d\mu_d(\omega) + \int_{\mathbb{R}^d} \ln|t||f(t)\overline{T_x g}(t)|^2 d\mu_d(t) \\
&\geqslant \left(\psi\left(\frac{d}{4}\right) + \ln(2)\right) \int_{\mathbb{R}^d} |f(t)\overline{T_x g}(t)|^2 d\mu_d(t), \qquad (3.34)
\end{aligned}$$

then, by integrating both sides of equation (3.34) with respect to the variable x, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \ln|\omega||\widehat{f\overline{T_x g}}(\omega)|^2 d\mu_{2d}(x,\omega) + \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \ln|t||f(t)\overline{T_x g}(t)|^2 d\mu_{2d}(t,x) \\
&\geqslant \left(\psi\left(\frac{d}{4}\right) + \ln(2)\right) \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |f(t)\overline{T_x g}(t)|^2 d\mu_{2d}(t,x),
\end{aligned}$$

hence

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \ln|\omega||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) + \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \ln|t||f(t)|^2 |\overline{g(t-x)}|^2 d\mu_{2d}(t,x) \\
&\geqslant \left(\psi\left(\frac{d}{4}\right) + \ln(2)\right) \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |f(t)|^2 |\overline{g(t-x)}|^2 d\mu_{2d}(t,x),
\end{aligned}$$

we get

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \ln|\omega||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) + \|g\|_{2,d}^2 \int_{\mathbb{R}^d} \ln|t||f(t)|^2 d\mu_d(t) \\
&\geqslant \left(\psi\left(\frac{d}{4}\right) + \ln(2)\right) \|f\|_{2,d}^2 \|g\|_{2,d}^2,
\end{aligned}$$

which corresponds to the desired result. □

Now, we give another proof of Beckner's Logarithmic uncertainty principle by using Pitt's inequality for STFT.

Proof. For $0 \leqslant \alpha < d$, let $c(\alpha) = \frac{1}{2^\alpha}\left(\frac{\Gamma\left(\frac{d-\alpha}{4}\right)}{\Gamma\left(\frac{d+\alpha}{4}\right)}\right)^2$ and

$$h(\alpha) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |\omega|^{-\alpha} |\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) - c(\alpha)\|g\|_{2,d}^2 \int_{\mathbb{R}^d} |t|^\alpha |f(t)|^2 d\mu_d(t).$$

Then,

$$\begin{aligned}
h'(\alpha) = &- \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \ln|\omega||\omega|^{-\alpha} |\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) \\
&- c(\alpha)\|g\|_{2,d}^2 \int_{\mathbb{R}^d} \ln|t||t|^\alpha |f(t)|^2 d\mu_d(t) \\
&- c'(\alpha)\|g\|_{2,d}^2 \int_{\mathbb{R}^d} |t|^\alpha |f(t)|^2 d\mu_d(t) \qquad (3.35)
\end{aligned}$$

By (3.24), $h(\alpha) \leqslant 0$ and $h(0) = 0$ which implies that $h'(0^+) \leqslant 0$. Hence, by (3.35) we get

$$\begin{aligned}
&-c'(0)\|g\|_{2,d}^2\|f\|_{2,d}^2 \\
&\leqslant \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \ln|\omega||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) + \|g\|_{2,d}^2 \int_{\mathbb{R}^d} \ln|t||f(t)|^2 d\mu_d(t),
\end{aligned}$$

hence we obtain (3.33). □

THEOREM 3.21. *Let $g \in S(\mathbb{R}^d)$ be a window function. Then, for every function $f \in S(\mathbb{R}^d)$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \ln|x||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) + \|g\|_{2,d}^2 \int_{\mathbb{R}^d} \ln|t||\hat{f}(t)|^2 d\mu_d(t) \\
&\geqslant \left(\psi\left(\frac{d}{4}\right) + \ln(2)\right)\|f\|_{2,d}^2\|g\|_{2,d}^2. \qquad (3.36)
\end{aligned}$$

Proof. By using (2.5) and (2.14) we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \ln|x||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_d(x) &= \int_{\mathbb{R}^d} \ln|x||\widehat{\hat{f}\overline{T_\omega\hat{g}}}(-x)|^2 d\mu_d(x) \\
&= \int_{\mathbb{R}^d} \ln|x||\widehat{\hat{f}\overline{T_\omega\hat{g}}}(x)|^2 d\mu_d(x).
\end{aligned}$$

Since $f, g \in S(\mathbb{R}^d)$ then for every $\omega \in \mathbb{R}^d$, we have $\hat{f}\overline{T_\omega\hat{g}} \in S(\mathbb{R}^d)$, we apply (2.5) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \ln|x||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_d(x) + \int_{\mathbb{R}^d} \ln|x||\hat{f}\overline{T_\omega\hat{g}}(x)|^2 d\mu_d(x) \\
&\geqslant \left(\psi\left(\frac{d}{4}\right) + \ln(2)\right)\int_{\mathbb{R}^d} |\hat{f}\overline{T_\omega\hat{g}}(x)|^2 d\mu_d(x). \qquad (3.37)
\end{aligned}$$

Then, by integrating both sides of equation (3.37) with respect to the variable ω, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \ln|x||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) + \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \ln|x||\hat{f}\overline{T_\omega\hat{g}}(x)|^2 d\mu_{2d}(x,\omega) \\
&= \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \ln|x||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) \\
&\quad + \int_{\mathbb{R}^d} \ln|x||\hat{f}(x)|^2 \int_{\mathbb{R}^d} |\overline{T_\omega\hat{g}}(x)|^2 d\mu_d(\omega) d\mu_d(x) \\
&= \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \ln|x||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) + \|\hat{g}\|_{2,d}^2 \int_{\mathbb{R}^d} \ln|x||\hat{f}(x)|^2 d\mu_d(x) \\
&= \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \ln|x||\mathscr{V}_g(f)(x,\omega)|^2 d\mu_{2d}(x,\omega) + \|g\|_{2,d}^2 \int_{\mathbb{R}^d} \ln|x||\hat{f}(x)|^2 d\mu_d(x) \\
&\geqslant \left(\psi\left(\frac{d}{4}\right) + \ln(2)\right)\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\hat{f}\overline{T_\omega\hat{g}}(x)|^2 d\mu_{2d}(x,\omega)
\end{aligned}$$

$$= \left(\psi\left(\frac{d}{4}\right) + \ln(2)\right) \int_{\mathbb{R}^d} |\hat{f}(x)|^2 \int_{\mathbb{R}^d} |\overline{T_\omega \hat{g}}(x)|^2 d\mu_d(\omega) d\mu_d(x)$$
$$= \left(\psi\left(\frac{d}{4}\right) + \ln(2)\right) \|\hat{g}\|_{2,d}^2 \|\hat{f}\|_{2,d}^2$$
$$= \left(\psi\left(\frac{d}{4}\right) + \ln(2)\right) \|g\|_{2,d}^2 \|f\|_{2,d}^2,$$

which corresponds to the desired result. □

3.7. Gross's inequality

We denote by $H^1(\mathbb{R}^d)$ the Sobolev space defined by

$$H^1(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) | \nabla f \in L^2(\mathbb{R}^d)\},$$

where ∇f is the standard gradiant function.

THEOREM 3.22. *For every $d \geqslant 2$ and $f \in H^1(\mathbb{R}^d) \setminus \{0\}$, we have*

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{\|f\|_{2,d}^2} \ln\left(\frac{|f(x)|^2}{\|f\|_{2,d}^2}\right) d\mu_d(x) \leqslant \frac{d}{2} \ln\left(\frac{1}{d\pi^2 e} \frac{\||\xi|\hat{f}\|_{2,d}^2}{\|f\|_{2,d}^2}\right). \tag{3.38}$$

Proof. See [4]. □

Let $g \in S(\mathbb{R}^d)$ be a window function, for $\beta > 0$ we define the Gabor modulation space with respect to the frequency polynomial weight on $\mathbb{R}^d$, by

$$M_2^\beta(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) : \left(1 + |\omega|^2\right)^{\frac{\beta}{2}} \mathscr{V}_g(f) \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \right\}.$$

For every nonzero function $f \in M_2^\beta(\mathbb{R}^d)$, we define the measure $d\eta_{2d}$ by

$$d\eta_{2d}(x,\omega) = \frac{|\mathscr{V}_g(f)(x,\omega)|^2}{\|f\|_{2,d}^2 \|g\|_{2,d}^2} d\mu_{2d}(x,\omega).$$

By Jensen's inequality and Plancherel's formula, we get

$$\int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |\mathscr{V}_g(f)(x,\omega)|^2 \ln|\omega| d\mu_{2d}(x,\omega)$$
$$= \frac{1}{2\beta} \|f\|_{2,d}^2 \|g\|_{2,d}^2 \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \ln(|\omega|^{2\beta}) d\eta_{2d}(x,\omega)$$
$$\leqslant \frac{1}{2\beta} \|f\|_{2,d}^2 \|g\|_{2,d}^2 \ln\left(\frac{\||\omega|^\beta \mathscr{V}_g(f)\|_{2,2d}^2}{\|f\|_{2,d}^2 \|g\|_{2,d}^2}\right) < +\infty. \tag{3.39}$$

THEOREM 3.23. *Let $g \in S(\mathbb{R}^d)$ be a window function. Then, for every $f \in S(\mathbb{R}^d)$ such that $\|f\|_{2,d} = 1$, we have*

$$\frac{d}{2}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\mathscr{V}_g(f)(x,\omega)|^2 \ln|\omega| d\mu_{2d}(x,\omega) \geqslant A^2\|g\|_{2,d}^2 \int_{\mathbb{R}^d} |f(x)|^2 \ln|f(x)| d\mu_d(x). \tag{3.40}$$

Proof. Let $f \in S(\mathbb{R}^d)$ such that $\|f\|_{2,d} = 1$ and h is the function defined by

$$\forall p \in]1,2[, \quad h(p) = \||\omega|^{\frac{d}{2}-\frac{d}{p}}\mathscr{V}_g(f)\|_{2,2d}^2 - A^2\|g\|_{2,d}^2\|f\|_{p,d}^2,$$

$$\begin{aligned} h'(p) = {} & \frac{2d}{p^2}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} |\omega|^{d(1-\frac{2}{p})}|\mathscr{V}_g(f)(x,\omega)|^2 \ln|\omega| d\mu_{2d}(x,\omega) \\ & + \frac{2}{p}A^2\|g\|_{2,d}^2\|f\|_{p,d}^2\left(\ln(\|f\|_{p,d}) - \|f\|_{p,d}^{-p}\int_{\mathbb{R}^d}|f(x)|^p \ln|f(x)| d\mu_d(x)\right). \end{aligned} \tag{3.41}$$

By (3.30), $h(p) \leqslant 0$ and $h(2) = 0$ which implies that $h'(2^-) \geqslant 0$.

Hence, by (3.42) we get

$$A^2\|g\|_{2,d}^2\int_{\mathbb{R}^d}|f(x)|^2\ln|f(x)| d\mu_d(x) \leqslant \frac{d}{2}\int_{\mathbb{R}^d}|\mathscr{V}_g(f)(x,\omega)|^2 \ln|\omega| d\mu_{2d}(x,\omega). \quad \square$$

COROLLARY 3.24. *Let $g \in L^2(\mathbb{R}^d)$ be a window function and $\beta > 0$. Then, for every $f \in M_2^1(\mathbb{R}^d)\setminus\{0\}$, we have*

$$\int_{\mathbb{R}^d}\frac{|f(x)|^2}{\|f\|_{2,d}^2}\ln\left(\frac{|f(x)|^2}{\|f\|_{2,d}^2}\right) d\mu_d(x) \leqslant \frac{d}{2A^2}\ln\left(\frac{\||\omega|\mathscr{V}_g(f)\|_{2,2d}^2}{\|f\|_{2,d}^2\|g\|_{2,d}^2}\right). \tag{3.42}$$

Proof. By replacing f by $\dfrac{f}{\|f\|_{2,d}}$ in (3.40), we obtain the desired result. $\square$

3.8. Logarithmic Sobolev inequalities

THEOREM 3.25. *For every $f \in S(\mathbb{R}^d)$ with $\|f\|_{2,d} = 1$, we have*

$$\frac{d}{2}\int_{\mathbb{R}^d}|\hat{f}(\xi)|^2\ln|\xi| d\mu_d(\xi) \geqslant \int_{\mathbb{R}^d}|f(x)|^2\ln|f(x)| d\mu_d(x) + B_d, \tag{3.43}$$

where

$$B_d = (2\pi)^{-\frac{d}{2}}\left(\frac{d}{2}\psi(\frac{d}{2}) + \frac{d}{2}\ln 2 + \frac{d}{4}\ln\pi - \frac{1}{2}\ln\left(\frac{\Gamma(d)}{\Gamma(\frac{d}{2})}\right)\right)$$

up to conformal automorphism, extremal functions are of the form $A(1+|x|^2)^{-\frac{d}{2}}$.

Proof. See [2]. $\square$

THEOREM 3.26. (Logarithmic Sobolev's inequality for STFT) *Let $g \in S(\mathbb{R}^d)$ be a window function. Then, for every $f \in S(\mathbb{R}^d)$ such that $\|f\|_{2,d} = 1$, we have*

$$\frac{d}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} |\mathscr{V}_g(f)(x,\omega)|^2 \ln|\omega| d\mu_d(x,\omega) \geqslant \|g\|_{2,d}^2 \int_{\mathbb{R}^d} |f(x)|^2 \ln|f(x)| d\mu_d(x) + B_d\|g\|_{2,d}^2, \tag{3.44}$$

where $B_d = \frac{d}{4}\ln(2) - \frac{\ln\Gamma(d)}{2} + \frac{\Gamma(d)}{2} + \frac{d}{2}\psi\left(\frac{d}{2}\right)$.

Proof. Let $f \in S(\mathbb{R}^d)$ such that $\|f\|_{2,d} = 1$. Let $p \in]1,2[$ and h is the function defined by

$$\begin{aligned} h(p) &= \||\omega|^{\frac{d}{2}}\mathscr{V}_g(f)\|_{2,2d}^2 - c(p)\|g\|_{2,d}^2\|f\|_{p,d}^2,\\ h'(p) &= \frac{2d}{p^2}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} |\omega|^{d(1-\frac{2}{p})}|\mathscr{V}_g(f)(x,\omega)|^2 \ln|\omega| d\mu_{2d}(x,\omega)\\ &\quad + \frac{2}{p}c(p)\|g\|_{2,d}^2\|f\|_{p,d}^2\left(\ln(\|f\|_{p,d}) - \|f\|_{p,d}^{-p}\int_{\mathbb{R}^d}|f(x)|^p\ln|f(x)|d\mu_d(x)\right)\\ &\quad - c'(p)\|g\|_{2,d}^2\|f\|_{p,d}^2. \end{aligned} \tag{3.45}$$

By (3.28), $h(p) \leqslant 0$ and $h(2) = 0$ which implies that $h'(2^-) \geqslant 0$.
Hence, by (3.45) we get

$$\begin{aligned} &\|g\|_{2,d}^2\int_{\mathbb{R}^d}|f(x)|^2\ln|f(x)|d\mu_d(x)\\ &\leqslant \frac{d}{2}\int_{\mathbb{R}^d}|\mathscr{V}_g(f)(x,\omega)|^2\ln|\omega|d\mu_{2d}(x,\omega) - h'(2)\|g\|_{2,d}^2. \quad \square \end{aligned}$$

THEOREM 3.27. *For every $f \in H^1(\mathbb{R}^d)\backslash\{0\}$, we have*

$$\int_{\mathbb{R}^d}|f(x)|^2\ln\left(\frac{|f(x)|^2}{\|f\|^2}\right)d\mu_d(x) \leqslant \frac{d}{2}\int_{\mathbb{R}^d}|\hat{f}(\xi)|^2\ln\left(S_d|\xi|^2\right)d\mu_d(\xi) - d\psi\left(\frac{d}{2}\right)\|f\|_{2,d}^2, \tag{3.46}$$

where $S_d = \frac{1}{4\pi}\left(\frac{\Gamma(d)}{\Gamma(\frac{d}{2})}\right)^{\frac{2}{d}}$ *is the best possible. This constant is attained by up to conformal automorphism* $f(x) = \left(1+|x|^2\right)^{-\frac{d}{2}}$.

Proof. See [19]. $\square$

COROLLARY 3.28. *Let $g \in L^2(\mathbb{R}^d)$ be a window function and $f \in M_1(\mathbb{R}^d)\setminus\{0\}$. Then, we have*

$$\begin{aligned} &\|g\|_{2,d}^2\int_{\mathbb{R}^d}|f(x)|^2\ln\left(\frac{|f(x)|^2}{\|f\|_{2,d}^2}\right)d\mu_d(x)\\ &\leqslant \frac{d}{2}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}|\mathscr{V}_g(f)(x,\omega)|^2\ln|\omega|d\mu_{2d}(x,\omega) - B_d\|g\|_{2,d}^2\|f\|_{2,d}^2. \end{aligned} \tag{3.47}$$

Proof. By replacing f by $\dfrac{f}{\|f\|_{2,d}}$ in (3.44), we obtain the desired result. $\square$

COROLLARY 3.29. *Let $\beta > 0$ and $g \in L^2(\mathbb{R}^d)$ be a window function. Then, there exists a constant $D(\beta,d)$ such that for every $f \in M_1^{\beta}(\mathbb{R}^d)\setminus\{0\}$ we have*

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{\|f\|_{2,d}^2} \ln\left(\frac{|f(x)|^2}{\|f\|_{2,d}^2}\right) d\mu_d(x) \leqslant \frac{d}{4\beta} \ln\left(D(\beta,d)\frac{\||\omega|^{\beta}\mathscr{V}_g(f)\|_{2,2d}^2}{\|f\|_{2,d}^2\|g\|_{2,d}^2}\right). \tag{3.48}$$

Proof. By using (3.47) and (3.39) we get the desired result with $D(\beta,d) = e^{-\frac{4\beta}{d}B_d}$. $\square$

We denote by

$$L_b^p(\mathbb{R}^d) = \{f \in L_{Loc}^p(\mathbb{R}^d) |\, \left(1+|x|^2\right)^{\frac{b}{2}} f \in L^p(\mathbb{R}^d)\}.$$

THEOREM 3.30. *Let $1 < b < +\infty$. Then, for every $f \in L_b^1(\mathbb{R}^d)\setminus\{0\}$, we have*

$$-\int_{\mathbb{R}^d} |f(x)| \ln\left(\frac{|f(x)|}{\|f\|_{1,d}}\right) d\mu_d(x) \leqslant d\int_{\mathbb{R}^d} |f(x)| \ln(C_{d,b}(1+|x|^b))d\mu_d(x), \tag{3.49}$$

where

$$C_{d,b} = \left(\frac{2\pi^{\frac{d}{2}}\Gamma(\frac{d}{b})\Gamma(\frac{d}{b'})}{b\Gamma(d)\Gamma(\frac{d}{2})}\right)^{\frac{1}{d}}$$

is the best possible and $\frac{1}{b}+\frac{1}{b'} = 1$. Moreover, it is attained up to conformal automorphism by $f(x) = (1+|x|^b)^{-d}$.

Proof. See [19]. $\square$

If $f \in L_b^2(\mathbb{R}^d)\setminus\{0\}$, then $|f|^2 \in L_{2b}^1(\mathbb{R}^d)\setminus\{0\}$ and by (3.49) we get

$$-\int_{\mathbb{R}^d} |f(x)|^2 \ln\left(\frac{|f(x)|^2}{\|f\|_{2,d}^2}\right) d\mu_d(x) \leqslant d\int_{\mathbb{R}^d} |f(x)|^2 \ln(C_{d,2b}(1+|x|^{2b})d\mu_d(x). \tag{3.50}$$

COROLLARY 3.31. *Let $g \in L^2(\mathbb{R}^d)$ be a window function. Then, there exists a constant $C(d)$ such that for every $f \in L_1^2(\mathbb{R}^d)\cap M_2^1(\mathbb{R}^d)\setminus\{0\}$, we have*

$$\int_{\mathbb{R}^d} |f(x)|^2 \ln(1+|x|^2)d\mu_d(x) + \frac{1}{2\|g\|_{2,d}^2}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\mathscr{V}_g(f)(x,\omega)|^2 \ln|\omega| d\mu_{2d}(x,\omega)$$
$$\geqslant C(d)\|f\|_{2,d}^2. \tag{3.51}$$

Proof. Let $f \in L^2_1(\mathbb{R}^d)\cap M^1_2(\mathbb{R}^d)\setminus\{0\}$. By using (3.50) and (3.47) we get

$$\begin{aligned}
&-d\int_{\mathbb{R}^d}|f(x)|^2\ln\left(C_{d,2}(1+|x|^2)\right)d\mu_d(x)\\
&\leqslant \int_{\mathbb{R}^d}|f(x)|^2\ln\left(\frac{|f(x)|^2}{\|f\|^2_{2,d}}\right)d\mu_d(x)\\
&\leqslant \frac{d}{2\|g\|^2_{2,d}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}\ln|\omega||\mathscr{V}_g(f)(x,\omega)|^2d\mu_{2d}(x,\omega)-B_d\|f\|^2_{2,d},
\end{aligned}$$

hence,

$$\begin{aligned}
&\left(\frac{B_d}{d}-\ln C_{d,2}\right)\|f\|^2_{2,d}\\
&\leqslant \int_{\mathbb{R}^d}|f(x)|^2\ln(1+|x|^2)d\mu_d(x)+\frac{1}{2\|g\|^2_{2,d}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}\ln|\omega||\mathscr{V}_g(f)(x,\omega)|^2d\mu_{2d}(x,\omega).
\end{aligned}$$

Which corresponds to the desired result. □

COROLLARY 3.32. (Shannon's inequality) *Let $1<b<+\infty$. Then, for every $f\in L^1_b(\mathbb{R}^d)\setminus\{0\}$, we have*

$$-\int_{\mathbb{R}^d}|f(x)|\ln\left(\frac{|f(x)|}{\|f\|_{1,d}}\right)d\mu_d(x)\leqslant\frac{d}{b}\|f\|_{1,d}\ln\left(\frac{C'_{d,b}}{\|f\|_{1,d}}\int_{\mathbb{R}^d}|x|^b|f(x)|d\mu_d(x)\right),\tag{3.52}$$

where

$$C'_{d,b}=b^b(b-1)^{1-b}\left(\frac{2\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{b}\right)\Gamma\left(\frac{d}{b'}\right)}{b\Gamma(d)\Gamma\left(\frac{d}{2}\right)}\right)^{\frac{b}{d}}.$$

Proof. See [19]. □

THEOREM 3.33. *Let $\beta>0$ and $\frac{1}{2}<b<+\infty$ and $g\in L^2(\mathbb{R}^d)$ be a window function. Then, for every $f\in L^2_b(\mathbb{R}^d)\cap M^\beta_1(\mathbb{R}^d)\setminus\{0\}$ we have*

$$\||x|^bf\|^\beta_{2,d}\||\omega|^\beta\mathscr{V}_g(f)\|^{\frac{b}{2}}_{2,2d}\geqslant C(d,\beta,b)\|f\|^{\frac{b}{2}+\beta}_{2,d}\|g\|^{\frac{\beta}{2}}_{2,d}.\tag{3.53}$$

Proof. By replacing b by $2b$ and f by $|f|^2$ in (3.52), we get

$$-\int_{\mathbb{R}^d}|f(x)|^2\ln\left(\frac{|f(x)|^2}{\|f\|^2_{2,d}}\right)d\mu_d(x)\leqslant\frac{d}{2b}\|f\|^2_{2,d}\ln\left(\frac{C'_{d,2b}}{\|f\|^2_{2,d}}\||x|^bf\|^2_{2,d}\right),\tag{3.54}$$

and by (3.48) we get

$$\ln\left(\left(\frac{C'_{d,2b}}{\|f\|^2_{2,d}}\||x|^bf\|^2_{2,d}\right)^{\frac{1}{2b}}\left(D(\beta,d)\frac{\||\omega|^\beta\mathscr{V}_g(f)\|^2_{2,2d}}{\|f\|^2_{2,d}\|g\|^2_{2,d}}\right)^{\frac{1}{4\beta}}\right)\geqslant 0.\tag{3.55}$$

Which implies the desired result with $C(d,\beta,b)=D(\beta,d)^{\frac{-1}{4\beta}}C'^{\frac{-1}{2b}}_{d,2b}$. □

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