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MULTILINEAR OFF–DIAGONAL LIMITED RANGE EXTRAPOLATIONS

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Abstract. We study the Rubio de Francia's extrapolation theorem. We prove an off-diagonal limited range extrapolation theorem. Using this theorem, we obtain a multilinear off-diagonal limited range extrapolation theorem. Our results generalize and refine known results by Duoandikoetxea (2011), Cruz-Uribe and Martel (2018) and Li, Martell and Ombrosi (2020).

Since Rubio de Francia [17, 18] proved the celebrated extrapolation theorem, many studies have been done, The book [6] provides a comprehensive treatment of extrapolation theory; see also [3] for latest results. The classical extrapolation theorem says that if a linear operator T is bounded on weighted L^{p_0} space for some $1 < p_0 < \infty$ uniformly in weights in A_{p_0}:

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x)dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x)dx \quad \text{for all} \quad w \in A_{p_0},$$

then T is bounded on weighted L^p spaces for all $1 < p < \infty$:

$$\int_{\mathbb{R}^n} |Tf(x)|^{p} w(x)dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^{p} w(x)dx \quad \text{for all} \quad w \in A_{p}.$$

For the precise statement of the theorem, in particular, the meaning of "uniformly in weights ", see Section 3.

On the contrary, there are many important operators that are bounded only on L^p where $p_- < p < p_+$. Theories for these operators are called limited range extrapolations [1], and ones for operators that are bounded from L^p to L^q are called off-diagonal extrapolations [12, 2]. There are two theorems for off-diagonal limited range extrapolations by Duoandikoetxea [8] and Cruz-Uribe and Martell [5]. We prove two theorems in a unified manner (Theorem 1) and also refine them.

Since the ceminal work by Lacey and Thiele [14, 15] for Calderón's conjecture about the bilinear Hilbert transform, many studies have been done for bilinear and multilinear operators. A multilinear diagonal limited range extrapolation theorem is obtained by Li, Martell and Ombrosi [16], see also [5, 8, 11]. By using Theorem 1, we prove a multilinear off-diagonal limited range extrapolation theorem. As a corollary we can prove their theorem. Furthermore our theorem generalizes theirs. For the simplicity

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of notation we consider bilinear cases (Theorem 2 in Section 5) . In the final section we shall state a multilinear theorem and show an outline of the proof.

This paper is organized as follows. In Section 2 we define notation and symbols which are used for linear extrapolation theorems. In Section 3 we state Theorem 1 and as corollaries we prove some known results and refine them. In Section 4 we shall prove Theorem 1. In Section 5 we consider bilinear extrapolation theorems and prove Theorem 2. In Section 6 we state a multilinear extrapolation theorem.

1. Preliminaries

For elementary properties about weight functions, see [9, 10], and see [6] about extrapolation theorems. We define some notation. For a nonnegative locally integrable function w,

$$L^p(w) := \left\{ f; \|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \right)^{1/p} < \infty \right\}.$$

When $1 < p < \infty$, $p' = \frac{p}{p-1}$ is the conjugate exponent of p. We use the conventions that $1' = \infty$, $\frac{1}{\infty} = 0$ and $\infty' = 1$. For a ball Q, $|Q|$ denotes the volume of Q.

DEFINITION 1.

$$⨍_Q w(x)\,dx := \frac{1}{|Q|} \int_Q w(x)\,dx \quad \text{and} \quad w(Q) := \int_Q w(x)\,dx.$$

We define some weight classes.

DEFINITION 2. Let $1 < p < \infty$. For a nonnegative locally integrable function w, we say that $w \in A_p$ if

$$[w]_{A_p} := \sup_Q \left(⨍_Q w(x)\,dx \right) \left(⨍_Q w(x)^{-1/(p-1)}\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls. We say that $w \in A_1$ if

$$[w]_{A_1} := \sup_Q \left(⨍_Q w(x)\,dx \right) \operatorname*{esssup}_{x \in Q} w^{-1}(x) < \infty.$$

Let $1 < \alpha < \infty$ and $0 < \beta < \infty$. We say that $w \in A(\alpha;\beta)$ if

$$[w]_{A(\alpha;\beta)} := \sup_Q \left(⨍_Q w(x)^\beta\,dx \right)^{1/\beta} \left(⨍_Q w(x)^{-\alpha'}\,dx \right)^{1/\alpha'} < \infty,$$

and

$$[w]_{A(1;\beta)} := \sup_Q \left(⨍_Q w(x)^\beta\,dx \right)^{1/\beta} \operatorname*{esssup}_{x \in Q} w^{-1}(x).$$

The symbol $A(\alpha;\beta)$ is unfamiliar compared to the standard symbol $A_{p,r}$. However using this symbol, we can state Corollary 1 (a linear case) and Theorem 2 (a bilinear case) in a unified manner.

The next lemma is easily obtained by the definitions.

LEMMA 1. *Let $1<p<\infty$, $1\leqslant\alpha<\infty$ and $0<\beta<\infty$. A weight $w\in A_p$ if and only if $w^{\frac{1}{1-p}}\in A_{p'}$ and*

$$[w^{1/(1-p)}]_{A_{p'}}=[w]_{A_p}^{p'-1}.$$

A weight $w\in A(\alpha;\beta)$ if and only if $w^\beta\in A_{1+\beta/\alpha'}$ and

$$[w]^{\beta}_{A(\alpha;\beta)}=[w^\beta]_{A_{1+\beta/\alpha'}}.$$

The following two lemmas are very important for the extrapolation theorem, see [6] and [8].

LEMMA 2. ([8, Lemma 2.1]) *Let $1\leqslant p<p_0<\infty$. If $u\in A_p$ and $v\in A_1$, then $u\cdot v^{p-p_0}\in A_{p_0}$ and*

$$[u\cdot v^{p-p_0}]_{A_{p_0}}\leqslant[u]_{A_p}[v]_{A_1}^{p_0-p}.$$

Let $1\leqslant p_0<p<\infty$. If $u\in A_p$ and $v\in A_1$, then $\left(u^{p_0-1}\cdot v^{p-p_0}\right)^{1/(p-1)}\in A_{p_0}$ and

$$\left[\left(u^{p_0-1}\cdot v^{p-p_0}\right)^{1/(p-1)}\right]_{A_{p_0}}\leqslant[u]_{A_p}^{(p_0-1)/(p-1)}[v]_{A_1}^{(p-p_0)/(p-1)}.$$

LEMMA 3. ([6, p. 18]) *Let $1<p<\infty$, $H\geqslant 0$ and $H\in L^p(w)$ where $w\in A_p$. By the Rubio de Francia algorithm, we can make the function $\mathscr{R}H(x)$ such that*

$$H(x)\leqslant\mathscr{R}H(x)\,a.e.,\quad \|\mathscr{R}H\|_{L^p(w)}\leqslant 2\|H\|_{L^p(w)}\quad\text{and}\quad[\mathscr{R}H]_{A_1}\leqslant C([w]_{A_p}),$$

where $C([w]_{A_p})$ is a positive constant depending only on $[w]_{A_p}$.

We use the following notation: $C(X,Y)$ is a positive constant depending only on X and Y, and not necessarily same at each occurrence. In this paper we use symbols $C([w]_{A_p})$ and $C([w]_{A(\alpha;\beta)})$ frequently.

2. Linear extrapolations

By the studies in [4, 7], we know that operators do not play a role in extrapolation theorems, see also [6]. Following the notation in [8], we state our theorems.

THEOREM 1. *Let $0<p,q<\infty$, $1\leqslant\alpha<\infty$ and $0<\beta<\infty$. Assume that for some family of pairs of nonnegative functions (f,g), and for all weights w such that $w^\beta\in A_\alpha$,*

$$\left(\int_{\mathbb{R}^n}g^qw^q\,dx\right)^{1/q}\leqslant C([w^\beta]_{A_\alpha})\left(\int_{\mathbb{R}^n}f^pw^p\,dx\right)^{1/p}.$$

Then

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}}\,dx\right)^{1/\widetilde{q}} \leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}})\left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\,dx\right)^{1/\widetilde{p}}$$

for all weights w *such that* $w^{\widetilde{\beta}} \in A_{\widetilde{\alpha}}$, *where*

$$\frac{1}{\widetilde{p}}-\frac{1}{p}=\frac{1}{\widetilde{q}}-\frac{1}{q}=\frac{1}{\widetilde{\beta}}-\frac{1}{\beta},\ \ \widetilde{\alpha}>1,\ \ \widetilde{\beta}>0, \tag{1}$$

$$\frac{\widetilde{\alpha}}{\widetilde{\beta}}=\frac{\alpha}{\beta}, \tag{2}$$

$$\max\left(\frac{1}{p}-\frac{1}{q},\frac{1}{p}-\frac{1}{\beta}\right)<\frac{1}{\widetilde{p}}<\frac{1}{p}-\frac{1}{\beta}+\frac{\alpha}{\beta}. \tag{3}$$

REMARKS. The limited range (3) is necessary for the existence of $0<\widetilde{q}<\infty$, $\widetilde{\alpha}>1$ and $\widetilde{\beta}>0$. When $\max(1/p-1/q,1/p-1/\beta)\leqslant 0$, the first inequality in (3) means $\widetilde{p}<\infty$. When $\alpha=1$, the second inequality in (3) means $\widetilde{p}>p$.

It may be natural to write initial condition by using indices p_0 and q_0. However, when we consider bilinear cases in Section 5, we use many subscripts $p_1,p_2,\ldots$, so we use $\widetilde{p}$ in that section. We use this notation from consistency.

We state some corollaries of this theorem. We shall prove them later.

The next corollary shows the beautiful correspondence between the linear theorem (Corollary 1) and the bilinear theorem (Theorem 2 in Section 5).

COROLLARY 1. *Let* $0<p,q<\infty$, $1\leqslant\alpha<\infty$ *and* $0<\beta<\infty$. *Assume that for some family of pairs of nonnegative functions* (f,g), *and for all weights* $w\in A(\alpha;\beta)$,

$$\left(\int_{\mathbb{R}^n} g^{q} w^{q}\,dx\right)^{1/q} \leqslant C([w]_{A(\alpha;\beta)})\left(\int_{\mathbb{R}^n} f^{p} w^{p}\,dx\right)^{1/p}. \tag{4}$$

Then

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}}\,dx\right)^{1/\widetilde{q}} \leqslant C([w]_{A(\widetilde{\alpha};\widetilde{\beta})})\left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\,dx\right)^{1/\widetilde{p}}$$

for all weights $w\in A(\widetilde{\alpha};\widetilde{\beta})$, *where* $\widetilde{\alpha}>1$, $\widetilde{\beta}>0$ *and*

$$\frac{1}{\widetilde{p}}-\frac{1}{p}=\frac{1}{\widetilde{q}}-\frac{1}{q}=\frac{1}{\widetilde{\alpha}}-\frac{1}{\alpha}=\frac{1}{\widetilde{\beta}}-\frac{1}{\beta}, \tag{5}$$

$$\max\left(\frac{1}{p}-\frac{1}{q},\frac{1}{p}-\frac{1}{\alpha},\frac{1}{p}-\frac{1}{\beta}\right)<\frac{1}{\widetilde{p}}<\frac{1}{p}-\frac{1}{\alpha}+1. \tag{6}$$

REMARK. The condition (6) is necessary for $0<\widetilde{q}<\infty$, $\widetilde{\alpha}>1$ and $\widetilde{\beta}>0$.

We obtain three theorems by Duoandikoetxea [8] and Cruz-Uribe and Martell [5], and also refine their theorems.

THEOREM A. ([8, Theorem 5.1]) *Let* $1 \leqslant p < \infty$ *and* $0 < q, r < \infty$. *Assume that for some family of pairs of nonnegative functions* (f, g), *and for all weights* $w \in A(p;r)$,

$$\left(\int_{\mathbb{R}^n} g^q w^q \, dx\right)^{1/q} \leqslant C([w]_{A(p;r)}) \left(\int_{\mathbb{R}^n} f^p w^p \, dx\right)^{1/p}.$$

Then

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}} \, dx\right)^{1/\widetilde{q}} \leqslant C([w]_{A(\widetilde{p};\widetilde{r})}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}} \, dx\right)^{1/\widetilde{p}}$$

for all weights $w \in A(\widetilde{p};\widetilde{r})$, *where* $0 < \widetilde{q}, \widetilde{r} < \infty$ *and*

$$\frac{1}{\widetilde{p}} - \frac{1}{p} = \frac{1}{\widetilde{q}} - \frac{1}{q} = \frac{1}{\widetilde{r}} - \frac{1}{r}, \tag{7}$$

$$\frac{1}{p} - \frac{1}{r} < \frac{1}{\widetilde{p}} < 1. \tag{8}$$

REMARKS. The condition (8) is not explicitly written in [8], but this condition is necessary for $\widetilde{r} > 0$. Under this formulation, the condition $p \geqslant 1$ is necessary, since $A(p;r)$ is defined for $p \geqslant 1$. Compare this theorem with Corollary 1.

THEOREM B. ([8, Theorem 7.1]) *Let* $0 < p < \infty$, $1 \leqslant q < \infty$ *and* $\beta \geqslant 1$. *Assume that for some family of pairs of nonnegative functions* (f, g), *and for all weights* w *such that* $w^\beta \in A_q$,

$$\left(\int_{\mathbb{R}^n} g^p w \, dx\right)^{1/p} \leqslant C([w^\beta]_{A_q}) \left(\int_{\mathbb{R}^n} f^p w \, dx\right)^{1/p}.$$

Then

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{p}} w \, dx\right)^{1/\widetilde{p}} \leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{q}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w \, dx\right)^{1/\widetilde{p}}$$

for all weights w *such that* $w^{\widetilde{\beta}} \in A_{\widetilde{q}}$ *where* $1 < \widetilde{q} < \infty$,

$$\frac{1}{\widetilde{p}} - \frac{1}{p} = \frac{1}{\widetilde{\beta}\widetilde{p}} - \frac{1}{\beta p} \tag{9}$$

$$\frac{\widetilde{\beta}\widetilde{p}}{\widetilde{q}} = \frac{\beta p}{q},$$

$$\frac{1}{p} - \frac{1}{\beta p} < \frac{1}{\widetilde{p}} < \frac{1}{p} - \frac{1}{\beta p} + \frac{q}{\beta p}. \tag{10}$$

REMARKS. In [8], the condition (9) is written as $q/\widetilde{q} - 1 = \beta(p/\widetilde{p} - 1)$. We can remove the condition $\beta \geqslant 1$. When $\beta < 1$, the left side of (10) means $\widetilde{p} > 0$. See the proof below.

To state the next theorem, we need a new definition.

DEFINITION 3. Let $1<s<\infty$. We say that $w\in RH_s$ if

$$[w]_{RH_s}:=\sup_Q\frac{\left(⨍_Q w^s\,dx\right)^{1/s}}{⨍_Q w\,dx}<\infty.$$

THEOREM C. ([5, Theorem 1.8]) *Given* $0<p_-<p_+\leqslant\infty$. *Let* $0<p,q<\infty$, $p_-\leqslant p<p_+$ *and* $1/q-1/p+1/p_+\geqslant 0$. *Assume that for all weights* w *such that* $w^p\in A_{p/p_-}\cap RH_{(p_+/p)'}$,

$$\left(\int_{\mathbb{R}^n}g^qw^q\,dx\right)^{1/q}\leqslant C([w^p]_{A_{p/p_-}},[w^p]_{RH_{(p_+/p)'}})\left(\int_{\mathbb{R}^n}f^pw^p\,dx\right)^{1/p}.$$

Then

$$\left(\int_{\mathbb{R}^n}g^{\widetilde{q}}w^{\widetilde{q}}\,dx\right)^{1/\widetilde{q}}\leqslant C([w^{\widetilde{p}}]_{A_{\widetilde{p}/p_-}},[w^{\widetilde{p}}]_{RH_{(p_+/\widetilde{p})'}})\left(\int_{\mathbb{R}^n}f^{\widetilde{p}}w^{\widetilde{p}}\,dx\right)^{1/\widetilde{p}}$$

for all weights w *such that* $w^{\widetilde{p}}\in A_{\widetilde{p}/p_-}\cap RH_{(p_+/\widetilde{p})'}$, *where* $p_-<\widetilde{p}<p_+$, $0<\widetilde{q}<\infty$ *and* $1/\widetilde{p}-1/\widetilde{q}=1/p-1/q$.

When $p_+=\infty$, *the condition* $w^p\in A_{p/p_-}\cap RH_{(p_+/p)'}$ *is* $w^p\in A_{p/p_-}$.

REMARKS. We can remove the condition $1/q-1/p+1/p_+\geqslant 0$. In [5], the limited range (p_-,p_+) is defined first, and the proof is long. In our theorem, the limited range is determined by the initial conditions for indices p,q,α and β, see (3).

In [5], the theorem is proved even when $p_-=0$ or $p=p_+$. However, in these cases, they are proved independently. We do not know how to prove them in a uniform way, since the classes A_∞ and RH_∞ are different from A_p and RH_s.

In [1, Theorem 4.9], this theorem is prove where $p=q$.

Now we prove corollaries of Theorem 1.

Proof of Corollary 1. Assume that (4) holds and indices satisfy (5) and (6). By Lemma 1, we have

$$\left(\int_{\mathbb{R}^n}g^qw^q\,dx\right)^{1/q}\leqslant C([w^\beta]_{A_{1+\beta/\alpha'}}\left(\int_{\mathbb{R}^n}f^pw^p\,dx\right)^{1/p}.$$

By (5) and (6)

$$\frac{\widetilde{\beta}}{1+\widetilde{\beta}/\widetilde{\alpha}'}=\frac{\beta}{1+\beta/\alpha'},\tag{11}$$

and

$$\max\left(\frac{1}{p}-\frac{1}{q},\frac{1}{p}-\frac{1}{\beta}\right)<\frac{1}{\widetilde{p}}<\frac{1}{p}-\frac{1}{\beta}+\frac{1+\beta/\alpha'}{\beta}.$$

Applying Theorem 1, we have

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}}\,dx\right)^{1/\widetilde{q}} \leqslant C([w^{\widetilde{\beta}}]_{A_{1+\widetilde{\beta}/\widetilde{\alpha}'}}\left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\,dx\right)^{1/\widetilde{p}},$$

and we obtain

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}}\,dx\right)^{1/\widetilde{q}} \leqslant C([w]_{A(\widetilde{\alpha},\widetilde{\beta})})\left(\int_{\mathbb{R}^n} f^{p} w^{\widetilde{p}}\,dx\right)^{1/\widetilde{p}}. \quad \square$$

In Theorems A, B and C, the condition $0<\widetilde{q}<\infty$ is assumed, that is, $1/\widetilde{p}-1/p+1/q>0$. Therefore we use Theorem 1 under the following conditions:

$$\frac{1}{\widetilde{p}}-\frac{1}{p}=\frac{1}{\widetilde{q}}-\frac{1}{q}=\frac{1}{\widetilde{\beta}}-\frac{1}{\beta},\ \ 0<\widetilde{q}<\infty,\ \ \widetilde{\alpha}>1,\ \ \widetilde{\beta}>0, \tag{1'}$$

$$\frac{\widetilde{\alpha}}{\widetilde{\beta}}=\frac{\alpha}{\beta},$$

$$\frac{1}{p}-\frac{1}{\beta}<\frac{1}{\widetilde{p}}<\frac{1}{p}-\frac{1}{\beta}+\frac{\alpha}{\beta}. \tag{3'}$$

Proof of Theorem A. If $w\in A(p;r)$ then $w^r\in A_{1+r/p'}$ by Lemma 1. When $p=1, w\in A_1$. Let $\alpha=1+r/p'$ and $\beta=r$ in Theorem 1. Then for any w such that $w^{\widetilde{\beta}}\in A_{\widetilde{\alpha}}$,

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}}\,dx\right)^{1/\widetilde{q}} \leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}})\left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\,dx\right)^{1/\widetilde{p}},$$

where

$$\frac{1}{\widetilde{p}}-\frac{1}{p}=\frac{1}{\widetilde{q}}-\frac{1}{q}=\frac{1}{\widetilde{\beta}}-\frac{1}{r},$$

$$\frac{\widetilde{\alpha}}{\widetilde{\beta}}-\frac{1+r/p'}{r},$$

$$\frac{1}{p}-\frac{1}{r}<\frac{1}{\widetilde{p}}<\frac{1}{p}-\frac{1}{r}+\frac{1+r/p'}{r}=1.$$

By (7) we can take $\widetilde{\beta}=\widetilde{r}$. Then

$$\widetilde{\alpha}=\widetilde{\beta}\left(\frac{1}{r}+\frac{1}{p'}\right)=\widetilde{\beta}\left(-\frac{1}{\widetilde{p}}+\frac{1}{p}+\frac{1}{\widetilde{\beta}}+\frac{1}{p'}\right)$$
$$=1+\frac{\widetilde{\beta}}{(\widetilde{p})'}=1+\frac{\widetilde{r}}{(\widetilde{p})'}.$$

Therefore we obtain by Lemma 1

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}}\, dx\right)^{1/\widetilde{q}} \leqslant C([w]_{A(\widetilde{p};\widetilde{r})}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\, dx\right)^{1/\widetilde{p}}$$

for all weights $w \in A(\widetilde{p};\widetilde{r})$. □

Proof of Theorem B. We do not use the condition $\beta \geqslant 1$. The assumption of the theorem is written as follows.

$$\left(\int_{\mathbb{R}^n} g^p (w^{1/p})^p\, dx\right)^{1/p} \leqslant C([w^\beta]_{A_q}) \left(\int_{\mathbb{R}^n} f^p (w^{1/p})^p\, dx\right)^{1/p},$$

and

$$\left(\int_{\mathbb{R}^n} g^p v^p\, dx\right)^{1/p} \leqslant C([v^{\beta p}]_{A_q}) \left(\int_{\mathbb{R}^n} f^p v^p\, dx\right)^{1/p}.$$

By Theorem 1, we have

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{p}} v^{\widetilde{p}}\, dx\right)^{1/\widetilde{p}} \leqslant C([v^{\widetilde{\beta}}]_{A_{\widetilde{q}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} v^{\widetilde{p}}\, dx\right)^{1/\widetilde{p}}, \tag{12}$$

where

$$\frac{1}{\widetilde{p}} - \frac{1}{p} = \frac{1}{\widetilde{q}} - \frac{1}{q} = \frac{1}{\widetilde{\beta}} - \frac{1}{\beta p},$$

$$\frac{\widetilde{q}}{\widetilde{\beta}} = \frac{q}{\beta p}$$

$$\frac{1}{p} - \frac{1}{\beta p} < \frac{1}{\widetilde{p}} < \frac{1}{p} - \frac{1}{\beta p} + \frac{q}{\beta p}.$$

We write (12) as

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{p}} w\, dx\right)^{1/\widetilde{p}} \leqslant C([w^{\widetilde{\beta}/\widetilde{p}}]_{A_{\widetilde{q}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w\, dx\right)^{1/\widetilde{p}} \tag{13}$$

and substituting $\widetilde{\beta}$ for $\frac{\widetilde{\beta}}{\widetilde{p}}$ in (13), we have

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{p}} w\, dx\right)^{1/\widetilde{p}} \leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{q}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w\, dx\right)^{1/\widetilde{p}},$$

where

$$\frac{1}{\widetilde{p}} - \frac{1}{p} = \frac{1}{\widetilde{q}} - \frac{1}{q} = \frac{1}{\widetilde{\beta}\widetilde{p}} - \frac{1}{\beta p},$$

$$\frac{\widetilde{\beta}\widetilde{p}}{\widetilde{q}} = \frac{\beta p}{q}$$

$$\frac{1}{p} - \frac{1}{\beta p} < \frac{1}{\widetilde{p}} < \frac{1}{p} - \frac{1}{\beta p} + \frac{q}{\beta p}. \quad \square$$

To prove Theorem C, we need some lemmas.

LEMMA 4. ([13, (P6)], see also [5, Lemma 2.1]) *Let* $1 \leqslant r < \infty$ *and* $1 \leqslant s < \infty$. *Then* $w \in A_r \cap RH_s$ *if and only if* $w^s \in A_{1+s(r-1)}$ *and*

$$[w^s]_{A_{1+s(r-1)}} \leqslant [w]_{A_r}^s [w]_{RH_s}^s, \quad [w]_{A_r} \leqslant [w^s]_{A_{1+s(r-1)}}^{1/s} \quad \textit{and} \quad [w]_{RH_s} \leqslant [w^s]_{A_{1+s(r-1)}}^{1/s}.$$

The next lemma is elementary.

LEMMA 5. *Let* $p_- \leqslant r < p_+$. *Then the indices defined above satisfy the following equalities.*

$$r\left(\frac{p_+}{r}\right)' = \frac{r \cdot p_+}{p_+ - r},$$
$$1 + \left(\frac{p_+}{r}\right)'\left(\frac{r}{p_-} - 1\right) = \frac{r(p_+ - p_-)}{p_-(p_+ - r)}$$

By these lemmas we obtain the following lemma.

LEMMA 6. *Let* $p_- \leqslant r < p_+$. *Then* $w^r \in A_{r/p_-} \cap RH_{(p_+/r)'}$ *if and only if* $w^\beta \in A_\alpha$ *where*

$$\alpha = \frac{r(p_+ - p_-)}{p_-(p_+ - r)} \quad \textit{and} \quad \beta = \frac{r \cdot p_+}{p_+ - r}.$$

Proof of Theorem C. We do not use the condition $1/q - 1/p + 1/p_+ \geqslant 0$.
Case $p_+ < \infty$. Let

$$\alpha = \frac{p(p_+ - p_-)}{p_-(p_+ - p)}, \quad \text{and} \quad \beta = \frac{p \cdot p_+}{p_+ - p}.$$

If $w^p \in A_{p/p_-} \cap RH_{(p_+/p)'}$ then $w^\beta \in A_\alpha$ by Lemma 6. Using Theorem 1, we have for all w such that $w^{\widetilde{\beta}} \in A_{\widetilde{\alpha}}$,

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}}\, dx\right)^{1/\widetilde{q}} \leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\, dx\right)^{1/\widetilde{p}},$$

where

$$\frac{1}{\widetilde{p}} - \frac{1}{p} = \frac{1}{\widetilde{q}} - \frac{1}{q} = \frac{1}{\widetilde{\beta}} - \frac{1}{\beta}, \;\; 0 < \widetilde{q} < \infty, \;\; \widetilde{\alpha} > 1, \;\; \widetilde{\beta} > 0 \tag{14}$$

$$\frac{\widetilde{\alpha}}{\widetilde{\beta}} = \frac{\alpha}{\beta} \tag{15}$$

$$\frac{1}{p} - \frac{1}{\beta} < \frac{1}{\widetilde{p}} < \frac{1}{p} - \frac{1}{\beta} + \frac{\alpha}{\beta} \tag{16}$$

The condition (16) is equivalent to

$$\frac{1}{p_+} < \frac{1}{\widetilde{p}} < \frac{1}{p_-}.$$

By (14),

$$\frac{1}{\widetilde{p}} - \frac{1}{p} = \frac{1}{\widetilde{\beta}} - \frac{p_+ - p}{p \cdot p_+}.$$

We have

$$\widetilde{\beta} = \frac{\widetilde{p} \cdot p_+}{p_+ - \widetilde{p}}.$$

By (15),

$$\widetilde{\alpha} = \widetilde{\beta}\left(\frac{1}{p_-} - \frac{1}{p_+}\right) = \frac{\widetilde{p}(p_+ - p_-)}{p_-(p_+ - \widetilde{p})} > 1.$$

Therefore by Lemma 6, the condition $w^{\widetilde{\beta}} \in A_{\widetilde{\alpha}}$ is equivalent to

$$w^{\widetilde{p}} \in A_{\widetilde{p}/p_-} \cap RH_{(p_+/\widetilde{p})'}.$$

Case $p_+ = \infty$. Since $w^p \in A_{p/p_-}$. We can apply Theorem 1 for $\alpha = p/p_-$, $\beta = p$, $\widetilde{\alpha} = \widetilde{p}/p_-$ and $\widetilde{\beta} = \widetilde{p}$. The limited range is

$$\frac{1}{p} - \frac{1}{p} < \frac{1}{\widetilde{p}} < \frac{1}{p} - \frac{1}{p} + \frac{p/p_-}{p}.$$

We obtain

$$p_- < \widetilde{p} < \infty = p_+.$$

This proves Theorem C. □

3. Proof of Theorem 1

The following lemma is elementary but important for our proof. Assume that all indices satisfy (1), (2) and (3).

LEMMA 7. *Let* $1/s = 1 - \widetilde{q}/q$ *and* $1/t = 1 - p/\widetilde{p}$. *Then the indices defined above satisfy the following equalities.*

$$\left(\frac{\widetilde{p}}{p} - 1\right)\frac{\widetilde{\alpha}\beta}{\widetilde{\alpha} - \alpha} = \widetilde{p}, \tag{17}$$

$$\left(\frac{\widetilde{p}}{p} - \frac{\widetilde{\beta}}{\beta}\right)\frac{\widetilde{\alpha}\beta}{\widetilde{\alpha} - \alpha} + \widetilde{\beta} = \widetilde{p}, \tag{18}$$

$$\frac{(\alpha - \widetilde{\alpha})\widetilde{q}s}{\beta} = \widetilde{\alpha}, \tag{19}$$

$$\left(1-\frac{\widetilde{\beta}}{\beta}\right)\widetilde{q}s=\widetilde{\beta}. \tag{20}$$

$$\frac{\widetilde{\alpha}'(\widetilde{q}-q)}{\widetilde{q}}=\frac{(\widetilde{\alpha}-\alpha)q}{(\widetilde{\alpha}-1)\beta}, \tag{21}$$

$$\left(\frac{\widetilde{\beta}}{1-\widetilde{\alpha}}-\widetilde{q}\right)\frac{\widetilde{q}-q}{\widetilde{q}}+\widetilde{q}=\frac{(\alpha-1)\widetilde{\beta}q}{(\widetilde{\alpha}-1)\beta}, \tag{22}$$

$$\frac{(\widetilde{\alpha}-\alpha)pt}{(\widetilde{\alpha}-1)\beta}=\widetilde{\alpha}', \tag{23}$$

$$\left(\frac{(\alpha-1)\widetilde{\beta}p}{(\widetilde{\alpha}-1)\beta}-p\right)t=\frac{\widetilde{\beta}}{1-\widetilde{\alpha}}. \tag{24}$$

Proof. We shall prove only (18), (22) and (24). The proofs for the others are easy. The equality (18) is equivalent to the following:

$$\frac{\widetilde{p}}{p}-\frac{\widetilde{\beta}}{\beta}+\frac{(\widetilde{\alpha}-\alpha)\widetilde{\beta}}{\widetilde{\alpha}\beta}=\frac{\widetilde{p}(\widetilde{\alpha}-\alpha)}{\widetilde{\alpha}\beta},$$
$$\frac{\widetilde{p}}{p}-\frac{\widetilde{\beta}}{\beta}+\left(1-\frac{\beta}{\widetilde{\beta}}\right)\frac{\widetilde{\beta}}{\beta}=\frac{\widetilde{p}}{\beta}\left(1-\frac{\beta}{\widetilde{\beta}}\right),$$

and the last equality is easily proved.

The equality (22) is written as follows.

$$\frac{\widetilde{\beta}}{1-\widetilde{\alpha}}\frac{\widetilde{q}-q}{\widetilde{q}}+q=\frac{(\alpha-1)\widetilde{\beta}q}{(\widetilde{\alpha}-1)\beta},$$
$$\widetilde{\beta}\left(\frac{1}{\widetilde{q}}-\frac{1}{q}\right)+\widetilde{\alpha}-1=\frac{\widetilde{\beta}}{\beta}(\alpha-1),$$
$$\widetilde{\beta}\left(\frac{1}{\widetilde{\beta}}-\frac{1}{\beta}\right)+\widetilde{\alpha}-1=\frac{\widetilde{\beta}}{\beta}(\alpha-1).$$

We can prove the last equality easily.

We write (24) as

$$\frac{(\alpha-1)\widetilde{\beta}}{\beta}-\widetilde{\alpha}+1=-\widetilde{\beta}\left(\frac{1}{p}-\frac{1}{\widetilde{p}}\right),$$
$$\frac{(\alpha-1)\widetilde{\beta}}{\beta}-\widetilde{\alpha}+1=-\widetilde{\beta}\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}}\right).$$

This is easily proved. □

Proof of Theorem 1. We follow the argument in [8, p. 1894].

Case $\widetilde{p} < p$. Assume that $w^{\widetilde{\beta}} \in A_{\widetilde{\alpha}}$ and $f \in L^{\widetilde{p}}(w^{\widetilde{p}})$. Let

$$H(x) := \left(f(x)^{\frac{\widetilde{p}}{p}-1} w(x)^{\frac{\widetilde{p}}{p}-\frac{\widetilde{\beta}}{\beta}} \right)^{\beta/(\widetilde{\alpha}-\alpha)}.$$

By (17) and (18),

$$H(x)^{\widetilde{\alpha}} w(x)^{\widetilde{\beta}} = f(x)^{\widetilde{p}} w(x)^{\widetilde{p}} \quad \text{and} \quad H \in L^{\widetilde{\alpha}}(w^{\widetilde{\beta}}).$$

Using Lemma 3 we can make the function $\mathscr{R}H(x)$ such that

$$H(x) \leqslant \mathscr{R}H(x), \quad \|\mathscr{R}H\|_{L^{\widetilde{\alpha}}(w^{\widetilde{\beta}})} \leqslant 2\|H\|_{L^{\widetilde{\alpha}}(w^{\widetilde{\beta}})} \quad \text{and} \quad [\mathscr{R}H]_{A_1} \leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}).$$

Let

$$V(x) := w(x)^{\frac{\widetilde{\beta}}{\beta}} \mathscr{R}H(x)^{\frac{\widetilde{\alpha}-\alpha}{\beta}}.$$

By Lemma 2, we have $[V^{\beta}]_{A_\alpha} \leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}})$. Since $\widetilde{q} < q$, we write

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}} \, dx \right)^{1/\widetilde{q}} &= \left(\int_{\mathbb{R}^n} g^{\widetilde{q}} V^{\widetilde{q}} \cdot V^{-\widetilde{q}} w^{\widetilde{q}} \, dx \right)^{1/\widetilde{q}} \\
&\leqslant \left(\int_{\mathbb{R}^n} g^q V^q \, dx \right)^{1/q} \left(\int_{\mathbb{R}^n} \left(V^{-\widetilde{q}} w^{\widetilde{q}} \right)^s dx \right)^{1/s\widetilde{q}} \\
&=: I^{1/q} \cdot II^{1/s\widetilde{q}},
\end{aligned}$$

where $1/s = 1 - \widetilde{q}/q$. By the assumption

$$\begin{aligned}
I^{1/q} &\leqslant C([V^{\beta}]_{A_\alpha}) \left(\int_{\mathbb{R}^n} f^p V^p \, dx \right)^{1/p} \\
&\leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}) \left(\int_{\mathbb{R}^n} f^p V^p \, dx \right)^{1/p}.
\end{aligned}$$

Since $\widetilde{\alpha} < \alpha$, we have

$$f(x)^p V(x)^p = f(x)^p w(x)^{\frac{\widetilde{\beta}p}{\beta}} (\mathscr{R}H)^{\frac{(\widetilde{\alpha}-\alpha)p}{\beta}} \leqslant f(x)^p w(x)^{\frac{\widetilde{\beta}p}{\beta}} H(x)^{\frac{(\widetilde{\alpha}-\alpha)p}{\beta}} = f(x)^{\widetilde{p}} w(x)^{\widetilde{p}}.$$

Therefore we obtain

$$I^{1/q} \leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}} \, dx \right)^{1/p}. \tag{25}$$

By (19) and (20)

$$(V(x)^{-\widetilde{q}} w(x)^{\widetilde{q}})^s = \left(w(x)^{-\frac{\widetilde{\beta}}{\beta}} (\mathscr{R}H(x))^{\frac{\alpha-\widetilde{\alpha}}{\beta}} \right)^{\widetilde{q}s} = (\mathscr{R}H(x))^{\widetilde{\alpha}} w(x)^{\widetilde{\beta}},$$

and we have

$$II = \int_{\mathbb{R}^n} (\mathscr{R}H)^{\widetilde{\alpha}} w^{\widetilde{\beta}}\, dx \leqslant 2^{\widetilde{\alpha}} \int_{\mathbb{R}^n} H^{\widetilde{\alpha}} w^{\widetilde{\beta}}\, dx = 2^{\widetilde{\alpha}} \int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\, dx. \tag{26}$$

By (25) and (26), we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}}\, dx \right)^{1/\widetilde{q}} &\leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\, dx \right)^{1/p+1/s\widetilde{q}} \\ &= C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\, dx \right)^{1/\widetilde{p}}. \end{aligned}$$

Case $p < \widetilde{p}$. Assume that $w^{\widetilde{\beta}} \in A_{\widetilde{\alpha}}$. By duality, we can take a nonnegative function h such that

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}}\, dx \right)^{\frac{1}{\widetilde{q}}} = \left(\int_{\mathbb{R}^n} g^{q} h w^{\widetilde{q}}\, dx \right)^{1/q} \tag{27}$$

and

$$\int_{\mathbb{R}^n} h^{\frac{\widetilde{q}}{\widetilde{q}-q}} w^{\widetilde{q}}\, dx = 1.$$

Let

$$H(x) := \left(h(x)^{\frac{\widetilde{q}}{\widetilde{q}-q}} w(x)^{\widetilde{q} - \frac{\widetilde{\beta}}{1-\widetilde{\alpha}}} \right)^{1/\widetilde{\alpha}'}.$$

Then

$$H(x)^{\widetilde{\alpha}'} w(x)^{\frac{\widetilde{\beta}}{1-\widetilde{\alpha}}} = h(x)^{\frac{\widetilde{q}}{\widetilde{q}-q}} w(x)^{\widetilde{q}}$$

and

$$\int_{\mathbb{R}^n} H^{\widetilde{\alpha}'} w^{\frac{\widetilde{\beta}}{1-\widetilde{\alpha}}}\, dx = 1. \tag{28}$$

By Lemma 1 we have $w(x)^{\widetilde{\beta}/(1-\widetilde{\alpha})} \in A_{\widetilde{\alpha}'}$. Using Lemma 3, we can make the function $\mathscr{R}H(x)$ such that

$$H(x) \leqslant \mathscr{R}H(x), \quad \|\mathscr{R}H\|_{L^{\widetilde{\alpha}'}(w^{\widetilde{\beta}/(1-\widetilde{\alpha})})} \leqslant 2\|H\|_{L^{\widetilde{\alpha}'}(w^{\widetilde{\beta}/(1-\widetilde{\alpha})})} = 2$$

and

$$[\mathscr{R}H]_{A_1} \leqslant C([w^{\widetilde{\beta}/(1-\widetilde{\alpha})}]_{A_{\widetilde{\alpha}'}}) \leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}).$$

Since $\widetilde{q} > q$, by the definition of H we have

$$h(x) = \left(H(x)^{\widetilde{\alpha}'} w(x)^{\frac{\widetilde{\beta}}{1-\widetilde{\alpha}} - \widetilde{q}} \right)^{(\widetilde{q}-q)/\widetilde{q}} \leqslant \left(\mathscr{R}H(x)^{\widetilde{\alpha}'} w(x)^{\frac{\widetilde{\beta}}{1-\widetilde{\alpha}} - \widetilde{q}} \right)^{(\widetilde{q}-q)/\widetilde{q}}.$$

By (27), (21) and (22), we have

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}}\,dx\right)^{\frac{1}{\widetilde{q}}} \leqslant \left(\int_{\mathbb{R}^n} g^q \left((\mathscr{R}H)^{\widetilde{\alpha}'} w^{\frac{\widetilde{\beta}}{1-\widetilde{\alpha}}-\widetilde{q}}\right)^{(\widetilde{q}-q)/\widetilde{q}} w^{\widetilde{q}}\,dx\right)^{1/q}$$

$$= \left(\int_{\mathbb{R}^n} g^q \left(w^{\widetilde{\beta}(\alpha-1)}(\mathscr{R}H)^{\widetilde{\alpha}-\alpha}\right)^{q/(\widetilde{\alpha}-1)\beta} dx\right)^{1/q}.$$

Let

$$V(x) := \left(w(x)^{\widetilde{\beta}(\alpha-1)}\mathscr{R}H(x)^{\widetilde{\alpha}-\alpha}\right)^{1/(\widetilde{\alpha}-1)\beta}.$$

Since $\widetilde{\alpha} > \alpha$, we have $V^\beta \in A_\alpha$ and $[V^\beta]_{A_\alpha} \leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}})$ by Lemma 2. Let $1/t = 1 - p/\widetilde{p}$. By the assumption and (23), (24) and (28), we have

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}}\,dx\right)^{1/\widetilde{q}} \leqslant \left(\int_{\mathbb{R}^n} g^q V^q\,dx\right)^{1/q} \leqslant C([V^\beta]_{A_\alpha}) \left(\int_{\mathbb{R}^n} f^p V^p\,dx\right)^{1/p}$$

$$\leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}) \left(\int_{\mathbb{R}^n} f^p \left(w^{\widetilde{\beta}(\alpha-1)}(\mathscr{R}H)^{\widetilde{\alpha}-\alpha}\right)^{p/(\widetilde{\alpha}-1)\beta} dx\right)^{1/p}$$

$$= C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}) \left(\int_{\mathbb{R}^n} f^p w^p \cdot (\mathscr{R}H)^{\frac{(\widetilde{\alpha}-\alpha)p}{(\widetilde{\alpha}-1)\beta}} w^{\frac{\widetilde{\beta}(\alpha-1)p}{\beta(\widetilde{\alpha}-1)}-p}\,dx\right)^{1/p}$$

$$\leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\,dx\right)^{1/\widetilde{p}} \left(\int_{\mathbb{R}^n} (\mathscr{R}H)^{\frac{(\widetilde{\alpha}-\alpha)pt}{(\widetilde{\alpha}-1)\beta}} w^{\left(\frac{\widetilde{\beta}(\alpha-1)p}{\beta(\widetilde{\alpha}-1)}-p\right)t}\,dx\right)^{1/pt}$$

$$= C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\,dx\right)^{1/\widetilde{p}} \left(\int_{\mathbb{R}^n} (\mathscr{R}H)^{\widetilde{\alpha}'} w^{\frac{\widetilde{\beta}}{1-\widetilde{\alpha}}}\,dx\right)^{1/pt}$$

$$\leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\,dx\right)^{1/\widetilde{p}} \left(\int_{\mathbb{R}^n} H^{\widetilde{\alpha}'} w^{\frac{\widetilde{\beta}}{1-\widetilde{\alpha}}}\,dx\right)^{1/pt}$$

$$= C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}}\,dx\right)^{1/\widetilde{p}}. \quad \square$$

4. Bilinear extrapolations

We defne a new weight class.

DEFINITION 4. Let $\alpha_1, \alpha_2 \geqslant 1$ and $\beta > 0$. For a pair of weights w_1 and w_2, we say $(w_1, w_2) \in A(\alpha_1, \alpha_2; \beta)$ if

$$[(w_1, w_2)]_{A(\alpha_1,\alpha_2;\beta)} := \sup_Q \left(⨍_Q \left(w_1(x)w_2(x)\right)^\beta dx\right)^{1/\beta} \prod_{i=1}^2 \left(⨍_Q w_i(x)^{-\alpha_i'}\,dx\right)^{1/\alpha_i'} < \infty.$$

When $\alpha_i = 1$, $\left(⨍_Q w_i(x)^{-\alpha_i'}\,dx\right)^{1/\alpha_i'} := \operatorname{esssup}_{x\in Q} w_i(x)^{-1}$.

Statements of bilinear extrapolations are a little complicated, so we use the following notation for short.

DEFINITION 5. If some family of 3-tuple of nonnegative functions (f_1, f_2, g) satisfies the next inequalities:

$$\left(\int_{\mathbb{R}^n} g^q (w_1 w_2)^q \, dx\right)^{1/q} \leqslant C([(w_1, w_2)]_{A(\alpha_1,\alpha_2;\beta)}) \left(\int_{\mathbb{R}^n} f_1^{p_1} w_1^{p_1} \, dx\right)^{1/p_1} \left(\int_{\mathbb{R}^n} f_2^{p_2} w_2^{p_2} \, dx\right)^{1/p_2}$$

for all $(w_1, w_2) \in A(\alpha_1, \alpha_2; \beta)$, then we say

$$(f_1, f_2, g) \quad \text{satisfies weighted} \quad L^{p_1} \times L^{p_2} \to L^q \quad \text{for} \quad A(\alpha_1, \alpha_2; \beta).$$

Our result is the following.

THEOREM 2. *Let* $p_1, p_2, q > 0$, $\alpha_1, \alpha_2 \geqslant 1$ *and* $\beta > 0$. *If*

$$(f_1, f_2, g) \quad \textit{satisfies weighted} \quad L^{p_1} \times L^{p_2} \to L^q \quad \textit{for} \quad A(\alpha_1, \alpha_2; \beta),$$

then

$$(f_1, f_2, g) \quad \textit{satisfies weighted} \quad L^{\widetilde{p}_1} \times L^{\widetilde{p}_2} \to L^{\widetilde{q}} \quad \textit{for} \quad A(\widetilde{\alpha}_1, \widetilde{\alpha}_2; \widetilde{\beta}),$$

where $\widetilde{p}_1, \widetilde{p}_2, \widetilde{q} > 0$, $\widetilde{\alpha}_1, \widetilde{\alpha}_2 > 1$, $\widetilde{\beta} > 0$ *and*

$$\frac{1}{\widetilde{p}_i} - \frac{1}{p_i} = \frac{1}{\widetilde{\alpha}_i} - \frac{1}{\alpha_i} \quad i = 1, 2, \tag{29}$$

$$\frac{1}{\widetilde{q}} - \frac{1}{q} = \sum_{i=1}^{2} \left(\frac{1}{\widetilde{p}_i} - \frac{1}{p_i}\right) = \frac{1}{\widetilde{\beta}} - \frac{1}{\beta}, \tag{30}$$

$$\frac{1}{p_i} - \frac{1}{\alpha_i} < \frac{1}{\widetilde{p}_i} < \frac{1}{p_i} - \frac{1}{\alpha_i} + 1 \quad i = 1, 2, \tag{31}$$

$$\max\left(\sum_{i=1}^{2} \frac{1}{p_i} - \frac{1}{q}, \sum_{i=1}^{2} \frac{1}{p_i} - \frac{1}{\beta}\right) < \sum_{i=1}^{2} \frac{1}{\widetilde{p}_i}. \tag{32}$$

REMARKS. When $1/p_i - 1/\alpha_i \leqslant 0$ in (31), this means $\widetilde{p}_i < \infty$. The condition (31) is necessary for $1 < \widetilde{\alpha}_i < \infty$ and (32) is necessary for $\widetilde{q} > 0$ and $\widetilde{\beta} > 0$.

As a corollary we can prove the diagonal extrapolation theorem by Li, Martell and Ombrosi [16]. Note that they prove a multilinear extrapolation theorem. We restate their theorem for the bilinear case. Let $1 \leqslant r_1 \leqslant p_1 < \infty$, $1 \leqslant r_2 \leqslant p_2 < \infty$, $1 \leqslant r_3 < \infty$ and $r_3' > p_1 p_2/(p_1 + p_2)$ be fixed, and let $1/p := 1/p_1 + 1/p_2$. We define a new weight class.

DEFINITION 6.

$$[(w_1,w_2)]_{A(p_1,p_2;r_1,r_2,r_3)} := \sup_Q \left(\fint_Q \left(w_1^{1/p_1} w_2^{1/p_2}\right)^{\frac{pr_3'}{r_3'-p}} \right)^{1/p-1/r_3'} \prod_{i=1}^{2} \left(\fint_Q (w_i^{1/p_i})^{\frac{-p_i r_i}{p_i - r_i}} \right)^{1/r_i - 1/p_i}.$$

THEOREM D. (Theorem 1.1 in [5]) *If*

$$\left(\int_{\mathbb{R}^n} g^p \left(w_1^{1/p_1} w_2^{1/p_2}\right)^p dx \right)^{1/p} \leqslant C([(w_1,w_2)]_{A(p_1,p_2;r_1,r_2,r_3)}) \left(\int_{\mathbb{R}^n} f_1^{p_1} w_1\, dx \right)^{1/p_1} \left(\int_{\mathbb{R}^n} f_2^{p_2} w_2\, dx \right)^{1/p_2} \quad (33)$$

for all weights w such that $[(w_1,w_2)]_{A(p_1,p_2;r_1,r_2,r_3)} < \infty$, then

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{p}} \left(w_1^{1/\widetilde{p}_1} w_2^{1/\widetilde{p}_2}\right)^{\widetilde{p}} dx \right)^{1/q} \leqslant C([(w_1,w_2)]_{A(\widetilde{p}_1,\widetilde{p}_2;r_1,r_2,r_3)}) \left(\int_{\mathbb{R}^n} f_1^{\widetilde{p}_1} w_1\, dx \right)^{1/\widetilde{p}_1} \left(\int_{\mathbb{R}^n} f_2^{\widetilde{p}_2} w_2\, dx \right)^{1/\widetilde{p}_2} \quad (34)$$

for all weights w such that $[(w_1,w_2)]_{A(\widetilde{p}_1,\widetilde{p}_2;r_1,r_2,r_3)} < \infty$, where $1 \leqslant \widetilde{p}_1, \widetilde{p}_2 < \infty$, $1/\widetilde{p} = 1/\widetilde{p}_1 + 1/\widetilde{p}_2$ and

$$r_1 < \widetilde{p}_1, \quad r_2 < \widetilde{p}_2, \quad r_3' > \frac{\widetilde{p}_1 \widetilde{p}_2}{\widetilde{p}_1 + \widetilde{p}_2}. \quad (35)$$

REMARK. This theorem assumes that $1 \leqslant r_i \leqslant p_i$ and $1 \leqslant \widetilde{p}_i$ $(i = 1,2)$. We can weaken this condition as follows.

$$0 \leqslant \frac{1}{r_i} - \frac{1}{p_i} < 1 \quad \text{and} \quad 0 \leqslant \frac{1}{r_i} - \frac{1}{\widetilde{p}_i} < 1, \quad \text{see the proof below.}$$

Proof of Theorem D. Assume that indices satisfy $1/\widetilde{p} = 1/\widetilde{p}_1 + 1/\widetilde{p}_2$ and (35). As in the proof of Theorem B, we rewrite (33) as follows.

$$\left(\int_{\mathbb{R}^n} g^p (v_1 v_2)^p dx \right)^{1/p} \leqslant C([(v_1^{p_1}, v_2^{p_2})]_{A(p_1,p_2;r_1,r_2,r_3)}) \left(\int_{\mathbb{R}^n} f_1^{p_1} v_1^{p_1}\, dx \right)^{1/p_1} \left(\int_{\mathbb{R}^n} f_2^{p_2} v_2^{p_2}\, dx \right)^{1/p_2}.$$

We define α_i and β as

$$\alpha_i' = \frac{p_i r_i}{p_i - r_i} \quad \text{and} \quad \beta = \frac{pr_3'}{r_3' - p}.$$

Note that if $r_3 = 1$ we define $\beta = p$. Then

$$[(v_1^{p_1}, v_2^{p_2})]_{A(p_1,p_2;r_1,r_2,r_3)} = [(v_1,v_2)]_{A(\alpha_1,\alpha_2;\beta)}$$

and we have

$$\left(\int_{\mathbb{R}^n} g^p (v_1 v_2)^p dx\right)^{1/p} \leqslant C([(v_1,v_2)]_{A(\alpha_1,\alpha_2;\beta)}) \left(\int_{\mathbb{R}^n} f_1^{p_1} v_1^{p_1}\, dx\right)^{1/p_1} \left(\int_{\mathbb{R}^n} f_2^{p_2} v_2^{p_2}\, dx\right)^{1/p_2}.$$

Let

$$\frac{1}{\widetilde{\alpha}_i} = \frac{1}{\widetilde{p}_i} - \frac{1}{p_i} + \frac{1}{\alpha_i}, \quad \frac{1}{\widetilde{\beta}} = \frac{1}{\widetilde{p}} - \frac{1}{p} + \frac{1}{\beta}, \quad q = p \quad \text{and} \quad \widetilde{q} = \widetilde{p}.$$

Then all indices satisfy the assumption of Theorem 2 and we have

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{p}} (v_1 v_2)^{\widetilde{p}} dx\right)^{1/\widetilde{p}} \leqslant C([(v_1,v_2)]_{A(\widetilde{\alpha}_1,\widetilde{\alpha}_2;\widetilde{\beta})}) \left(\int_{\mathbb{R}^n} f_1^{\widetilde{p}_1} v_1^{\widetilde{p}_1}\, dx\right)^{1/\widetilde{p}_1} \left(\int_{\mathbb{R}^n} f_2^{\widetilde{p}_2} v_2^{\widetilde{p}_2}\, dx\right)^{1/\widetilde{p}_2}.$$

Since

$$\widetilde{\alpha}_i' = \frac{\widetilde{p}_i r_i}{\widetilde{p}_i - r_i} \quad \text{and} \quad \widetilde{\beta} = \frac{\widetilde{p} r_3'}{r_3' - \widetilde{p}},$$

we obtain (34) by the same argument above. □

4.1. Proof of Theorem 2

By the method of freezing one parameter, we can reduce the proof of Theorem 2 to the following theorem, see [16] Section 4.

THEOREM 3. *If*

$$(f_1, f_2, g) \quad \textit{satisfies weighted} \quad L^{p_1} \times L^{p_2} \to L^q \quad \textit{for} \quad A(a,b;c),$$

then

$$(i) \quad (f_1, f_2, g) \quad \textit{satisfies weighted} \quad L^{p_1} \times L^{\widetilde{p}_2} \to L^{\widetilde{q}_2} \quad \textit{for} \quad A(a,\widetilde{b};\widetilde{c}_2)$$

where $\widetilde{b} > 1$ *and*

$$\frac{1}{\widetilde{p}_2} - \frac{1}{p_2} = \frac{1}{\widetilde{q}_2} - \frac{1}{q} = \frac{1}{\widetilde{b}} - \frac{1}{b} = \frac{1}{\widetilde{c}_2} - \frac{1}{c},$$

$$\max\left(\frac{1}{p_2} - \frac{1}{q}, \frac{1}{p_2} - \frac{1}{b}, \frac{1}{p_2} - \frac{1}{c},\right) < \frac{1}{\widetilde{p}_2} < \frac{1}{p_2} - \frac{1}{b} + 1,$$

and

$$(ii)\quad (f_1,f_2,g)\quad \textit{satisfies weighted}\quad L^{\widetilde{p}_1}\times L^{p_2}\to L^{\widetilde{q}_1}\quad \textit{for}\quad A(\widetilde{a},b;\widetilde{c}_1)$$

where $\widetilde{a}>1$ *and*

$$\frac{1}{\widetilde{p}_1}-\frac{1}{p_1}=\frac{1}{\widetilde{q}_1}-\frac{1}{q}=\frac{1}{\widetilde{a}}-\frac{1}{a}=\frac{1}{\widetilde{c}_1}-\frac{1}{c},$$
$$\max\left(\frac{1}{p_1}-\frac{1}{q},\frac{1}{p_1}-\frac{1}{a},\frac{1}{p_1}-\frac{1}{c},\right)<\frac{1}{\widetilde{p}_1}<\frac{1}{p_1}-\frac{1}{a}+1.$$

REMARK. We prove Theorem 2 by using Theorem 3. We shall prove Thereom 3 in the next subsection. The notation $A(a,b;c)$ is not standard, but we substitute several indices into a,b and c in the proof of Theorem 2.

To prove Theorem 2, we need a lemma. Assume that all indices satisfy (29)–(32).

LEMMA 8. *When* $\beta<q$*,*

$$\frac{1}{p_1}-\frac{1}{q}<\frac{1}{\widetilde{p}_1}\quad or\quad \frac{1}{p_2}-\frac{1}{q}<\frac{1}{\widetilde{p}_2}. \tag{36}$$

When $\beta\geqslant q$*,*

$$\frac{1}{p_1}-\frac{1}{\beta}<\frac{1}{\widetilde{p}_1}\quad or\quad \frac{1}{p_2}-\frac{1}{\beta}<\frac{1}{\widetilde{p}_2}. \tag{37}$$

Proof. When $\beta<q$, assume that both inequalities in (36) do not hold, then

$$\frac{1}{q}-\frac{1}{\widetilde{q}}=\frac{1}{p_1}-\frac{1}{\widetilde{p}_1}+\frac{1}{p_2}-\frac{1}{\widetilde{p}_2}\geqslant\frac{2}{q}.$$

Therefore $-1/\widetilde{q}\geqslant 1/q$. This is a contradiction. The proof for the case $\beta\geqslant q$ is the same. □

Proof of Theorem 2 by using Theorem 3. Let $\beta<q$. Taking account of Lemma 8, we assume that $1/p_1-1/q<1/\widetilde{p}_1$. We define auxiliary indices $\widetilde{q}_1$ and $\widetilde{\beta}_1$ as follows.

$$\frac{1}{\widetilde{q}_1}:=\frac{1}{\widetilde{p}_1}-\frac{1}{p_1}+\frac{1}{q}>0,\quad \frac{1}{\widetilde{\beta}_1}:=\frac{1}{\widetilde{\alpha}_1}-\frac{1}{\alpha_1}+\frac{1}{\beta}>0.$$

Then

$$\frac{1}{\widetilde{p}_1}-\frac{1}{p_1}=\frac{1}{\widetilde{q}_1}-\frac{1}{q}=\frac{1}{\widetilde{\alpha}_1}-\frac{1}{\alpha_1}=\frac{1}{\widetilde{\beta}_1}-\frac{1}{\beta}$$

and

$$\max\left(\frac{1}{p_1}-\frac{1}{q},\frac{1}{p_1}-\frac{1}{\alpha_1},\frac{1}{p_1}-\frac{1}{\beta}\right)<\frac{1}{\widetilde{p}_1}<\frac{1}{p_1}-\frac{1}{\alpha_1}+1.$$

Applying Theorem 3 (ii) for $a=\alpha_1$, $\widetilde{a}=\widetilde{\alpha}_1$, $b=\alpha_2$, $c=\beta$ and $\widetilde{c}_1=\widetilde{\beta}_1$,

$$(f_1,f_2,g) \quad \text{satisfies weighted} \quad L^{\widetilde{p}_1}\times L^{p_2}\to L^{\widetilde{q}_1} \quad \text{for} \quad A(\widetilde{\alpha}_1,\alpha_2;\widetilde{\beta}_1).$$

By the definitions of indices, we have

$$\frac{1}{\widetilde{p}_2}-\frac{1}{p_2}=\frac{1}{\widetilde{q}}-\frac{1}{\widetilde{q}_1}=\frac{1}{\widetilde{\alpha}_2}-\frac{1}{\alpha_2}=\frac{1}{\widetilde{\beta}}-\frac{1}{\widetilde{\beta}_1}$$

and

$$\max\left(\frac{1}{p_2}-\frac{1}{\widetilde{q}_1},\frac{1}{p_2}-\frac{1}{\alpha_2},\frac{1}{p_2}-\frac{1}{\widetilde{\beta}_1}\right)<\frac{1}{\widetilde{p}_2}<\frac{1}{p_2}-\frac{1}{\alpha_2}+1.$$

Therefore we can apply Theorem 3 (i) for $a=\widetilde{\alpha}_1$, $b=\alpha_2$, $\widetilde{b}=\widetilde{\alpha}_2$, $c=\widetilde{\beta}_1$, $\widetilde{c}_2=\widetilde{\beta}$, $q=\widetilde{q}_1$ and $\widetilde{q}_2=\widetilde{q}$, and have that

$$(f_1,f_2,g) \quad \text{satisfies weighted} \quad L^{\widetilde{p}_1}\times L^{\widetilde{p}_2}\to L^{\widetilde{q}} \quad \text{for} \quad A(\widetilde{\alpha}_1,\widetilde{\alpha}_2;\widetilde{\beta}).$$

The proofs for the cases $1/p_2-1/q<1/\widetilde{p}_2$ and $\beta\geqslant q$ are similar. □

4.2. Proof of Theorem 3

Now return back to the standard notation and we shall prove (i) for the following formula. The proof of (ii) is the same.

THEOREM 3'. *If*

$$(f_1,f_2,g) \quad \textit{satisfies weighted} \quad L^{p_1}\times L^{p_2}\to L^{q} \quad \textit{for} \quad A(\alpha_1,\alpha_2;\beta),$$

then

$$(i) \quad (f_1,f_2,g) \quad \textit{satisfies weighted} \quad L^{p_1}\times L^{\widetilde{p}_2}\to L^{\widetilde{q}_2} \quad \textit{for} \quad A(\alpha_1,\widetilde{\alpha}_2;\widetilde{\beta})$$

where $\widetilde{\alpha}_2>1$ *and*

$$\frac{1}{\widetilde{p}_2}-\frac{1}{p_2}=\frac{1}{\widetilde{q}}-\frac{1}{q}=\frac{1}{\widetilde{\alpha}_2}-\frac{1}{\alpha_2}=\frac{1}{\widetilde{\beta}}-\frac{1}{\beta}, \tag{38}$$

$$\max\left(\frac{1}{p_2}-\frac{1}{q},\frac{1}{p_2}-\frac{1}{\alpha_2},\frac{1}{p_2}-\frac{1}{\beta},\right)<\frac{1}{\widetilde{p}_2}<\frac{1}{p_2}-\frac{1}{\alpha_2}+1.$$

For the proof we define a new weight class.

DEFINITION 7. Let μ be a measure on $\mathbb{R}^n$. When $p>1$, we say that $w\in A_p(d\mu)$ if

$$[w]_{A_p(d\mu)}:=\sup_Q\left(\frac{1}{\mu(Q)}\int_Q w\,d\mu\right)\left(\frac{1}{\mu(Q)}\int_Q w^{-1/(p-1)}\,d\mu\right)^{p-1}<\infty,$$

and

$$[w]_{A_1(d\mu)}:=\sup_Q\left(\frac{1}{\mu(Q)}\int_Q w\,d\mu\right)\operatorname*{esssup}_{x\in Q} w(x)^{-1}<\infty.$$

We use a variant of Theorem 1 for measure spaces. Since its proof is same as the one of Theorem 1, we omit the proof, see [16] Theorem 3.1.

DEFINITION 8. We say that a measure μ is a doubling measure if

$$\sup_{Q} \frac{\mu(2Q)}{\mu(Q)} < \infty,$$

where $2Q$ is the ball with the same center as Q whose radius is twice as large.

The following lemma is well-known, see, for example, [9].

LEMMA 9. *If $w \in \cup_{p \geqslant 1} A_p$, then the measure $w(x)dx$ is a doubling measure.*

THEOREM 1'. *Let μ be a doubling measure and $0 < p, q < \infty$, $1 \leqslant \alpha < \infty$ and $0 < \beta < \infty$. Assume that for some family of pairs of nonnegative functions (f,g), and for all weights w such that $w^{\beta} \in A_{\alpha}(d\mu)$,*

$$\left(\int_{\mathbb{R}^n} g^q w^q \, d\mu\right)^{1/q} \leqslant C([w^{\beta}]_{A_{\alpha}(d\mu)}) \left(\int_{\mathbb{R}^n} f^p w^p \, d\mu\right)^{1/p}.$$

Then

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} w^{\widetilde{q}} \, d\mu\right)^{1/\widetilde{q}} \leqslant C([w^{\widetilde{\beta}}]_{A_{\widetilde{\alpha}}(d\mu)}) \left(\int_{\mathbb{R}^n} f^{\widetilde{p}} w^{\widetilde{p}} \, d\mu\right)^{1/\widetilde{p}}$$

for all weights w such that $w^{\widetilde{\beta}} \in A_{\widetilde{\alpha}}(d\mu)$, where indices satisfy (1)–(3).

We need some lemmas. Assume that all indices satisfy (38). Let

$$\frac{1}{s} := \frac{1}{\alpha_2'} + \frac{1}{\beta}, \quad \frac{1}{\widetilde{s}} := \frac{1}{\widetilde{\alpha}_2'} + \frac{1}{\widetilde{\beta}}. \tag{39}$$

The next lemma is easily obtained from the definitions.

LEMMA 10.

$$\widetilde{s} = s,$$

$$\frac{\widetilde{\beta}}{\beta} = \frac{1 + \widetilde{\beta}/\widetilde{\alpha}_2'}{1 + \beta/\alpha_2'},$$

$$s\widetilde{q}\left(\frac{1}{\beta} - \frac{1}{q}\right) + \frac{s\widetilde{q}}{\widetilde{\alpha}_2'} + s = \widetilde{q},$$

$$\widetilde{p}_2\left(-1 + \frac{s}{\beta} - \frac{s}{p_2}\right) + \frac{s\widetilde{p}_2}{\widetilde{\alpha}_2'} + s = 0.$$

The next lemma is obtained from Hölder's inequality.

LEMMA 11. *For any $a > 0, b > 0$, cubes Q and weights w,*

$$\left(\fint_Q w^a\,dx\right)^{1/a}\left(\fint_Q w^{-b}\,dx\right)^{1/b} \geqslant 1.$$

Proof. Let $1 = w^{ab/(a+b)} \cdot w^{-ab/(a+b)}$ and use Hölder's inequality. $\square$

The next three lemmas are important for the proof of Theorem 3'.

LEMMA 12. *If $(w_1, w_2) \in A(\alpha_1, \alpha_2; \beta)$, then $w_1^s \in A_{1+s/\alpha_1'}$ and*

$$[w_1^s]_{A_{1+s/\alpha_1'}} \leqslant [(w_1, w_2)]^s_{A(\alpha_1,\alpha_2;\beta)} \quad \textit{where} \quad 1/s = 1/\alpha_2' + 1/\beta.$$

Note that if $\alpha_1 = 1$ then $w_1^s \in A_1$.
In the proof of Theorem 3' we use the following formula.

LEMMA $\widetilde{12}$. *If $(w_1, v_2) \in A(\alpha_1, \widetilde{\alpha}_2; \widetilde{\beta})$, then $w_1^{\widetilde{s}} \in A_{1+\widetilde{s}/\alpha_1'}$ and*

$$[w_1^{\widetilde{s}}]_{A_{1+\widetilde{s}/\alpha_1'}} \leqslant [(w_1, v_2)]^{\widetilde{s}}_{A(\alpha_1,\widetilde{\alpha}_2;\widetilde{\beta})}.$$

REMARK. Since $s = \widetilde{s}$,

$$[w_1^s]_{A_{1+s/\alpha_1'}} = [w_1^{\widetilde{s}}]_{A_{1+\widetilde{s}/\alpha_1'}}.$$

This equality is used in the estimate (43) below.

Proof of Lemma 12. We prove the case that $\alpha_i > 1$. When $\alpha_i = 1$ for some i, the proof is similar.

We write $w_1^s = (w_1 w_2)^s \cdot w_2^{-s}$. Using Hölder's inequality, we have

$$\begin{aligned}\fint_Q w_1^s\,dx &\leqslant \left(\fint_Q (w_1 w_2)^\beta\,dx\right)^{s/\beta}\left(\fint_Q w_2^{-\beta s/(\beta - s)}\,dx\right)^{(\beta-s)/\beta} \\ &= \left(\fint_Q (w_1 w_2)^\beta\,dx\right)^{s/\beta}\left(\fint_Q w_2^{-\alpha_2'}\,dx\right)^{s/\alpha_2'},\end{aligned}$$

and obtain

$$[w_1^s]_{A_{1+s/\alpha_1'}} = \sup_Q\left(\fint_Q w_1^s\,dx\right)\left(\fint_Q w_1^{-\alpha_1'}\,dx\right)^{s/\alpha_1'} \leqslant [(w_1, w_2)]^s_{A(\alpha_1,\alpha_2;\beta)}. \quad \square$$

LEMMA 13. *Let $w_1^s \in A_{1+s/\alpha_1'}$. If a weight W satisfies $W^\beta \in A_{1+\beta/\alpha_2'}(w_1^s dx)$, then $(w_1, W \cdot w_1^{-1+s/\beta}) \in A(\alpha_1, \alpha_2; \beta)$ and*

$$\left[\left(w_1,\, W \cdot w_1^{-1+s/\beta}\right)\right]_{A(\alpha_1,\alpha_2;\beta)} \leqslant [W^\beta]^{1/\beta}_{A_{1+\beta/\alpha_2'}(w_1^s dx)} \cdot [w_1^s]^{1/s}_{A_{1+s/\alpha_1'}}.$$

Proof. We prove the case that $\alpha_i > 1$.

$$
\begin{aligned}
&\left(\fint_Q \left(w_1 \cdot W w_1^{-1+s/\beta}\right)^{\beta} dx\right)^{1/\beta} \left(\fint_Q w_1^{-\alpha_1'} dx\right)^{1/\alpha_1'} \left(\fint_Q \left(W w_1^{-1+s/\beta}\right)^{-\alpha_2'} dx\right)^{1/\alpha_2'} \\
=&\left(\fint_Q W^{\beta} w_1^s \, dx\right)^{1/\beta} \left(\fint_Q w_1^{-\alpha_1'} dx\right)^{1/\alpha_1'} \left(\fint_Q W^{-\alpha_2'} w_1^s \, dx\right)^{1/\alpha_2'} \\
=&\left(\frac{1}{w_1^s(Q)} \int_Q W^{\beta} w_1^s \, dx\right)^{1/\beta} \left(\frac{1}{w_1^s(Q)} \int_Q W^{-\alpha_2'} w_1^s \, dx\right)^{1/\alpha_2'} \\
&\times \left(\fint_Q w_1^s \, dx\right)^{1/\beta+1/\alpha_2'} \left(\fint_Q w_1^{-\alpha_1'} dx\right)^{1/\alpha_1'} \\
\leqslant&[W^{\beta}]^{1/\beta}_{A_{1+\beta/\alpha_2'}(w_1^s\,dx)} \left(\fint_Q w_1^s \, dx\right)^{1/s} \left(\fint_Q w_1^{-\alpha_1'} dx\right)^{1/\alpha_1'} \\
\leqslant&[W^{\beta}]^{1/\beta}_{A_{1+\beta/\alpha_2'}(w_1^s\,dx)} [w_1^s]^{1/s}_{A_{1+s/\alpha_1'}}. \quad \square
\end{aligned}
$$

LEMMA 14. *Assume that* $(w_1, v_2) \in A(\alpha_1, \widetilde{\alpha}_2; \widetilde{\beta})$, *and let* $W := w_1^{s/\widetilde{\alpha}_2'} \cdot v_2$. *Then* $W^{\widetilde{\beta}} \in A_{1+\widetilde{\beta}/\widetilde{\alpha}_2'}(w_1^s dx)$ *and*

$$
[W^{\widetilde{\beta}}]_{A_{1+\widetilde{\beta}/\widetilde{\alpha}_2'}(w_1^s dx)} \leqslant [(w_1, v_2)]^{\widetilde{\beta}}_{A(\alpha, \widetilde{\alpha}_2; \widetilde{\beta})}.
$$

Proof. We prove the case that $\alpha_i > 1$. Let

$$
I := \left(\frac{1}{w_1^s(Q)} \int_Q W^{\widetilde{\beta}} w_1^s \, dx\right) \left(\frac{1}{w_1^s(Q)} \int_Q W^{-\widetilde{\alpha}_2'} w_1^s \, dx\right)^{\widetilde{\beta}/\alpha_2'}.
$$

Note that

$$
W^{\widetilde{\beta}} w_1^s = (w_1 v_2)^{\widetilde{\beta}} \quad \text{and} \quad W^{-\widetilde{\alpha}_2'} w_1^s = v_2^{-\widetilde{\alpha}_2'}.
$$

We have

$$
\begin{aligned}
I^{1/\widetilde{\beta}} =& \left(\fint_Q (w_1 v_2)^{\widetilde{\beta}} dx\right)^{1/\widetilde{\beta}} \left(\fint_Q w_1^{-\alpha_1'} dx\right)^{1/\alpha_1'} \left(\fint_Q v_2^{-\widetilde{\alpha}_2'} dx\right)^{1/\widetilde{\alpha}_2'} \\
&\times \left(\fint_Q w_1^{-\alpha_1'} dx\right)^{-1/\alpha_1'} \left(\fint_Q w_1^s \, dx\right)^{-1/\widetilde{\beta}-1/\widetilde{\alpha}_2'} \\
&\leqslant [(w_1, v_2)]_{A(\alpha_1, \widetilde{\alpha}_2; \widetilde{\beta})} \left(\fint_Q w_1^{-\alpha_1'} dx\right)^{-1/\alpha_1'} \left(\fint_Q w_1^s \, dx\right)^{-1/s} \leqslant [(w_1, v_2)]_{A(\alpha_1, \widetilde{\alpha}_2; \widetilde{\beta})}
\end{aligned}
$$

by Lemma 11. $\square$

Proof of Theorem 3' (i). We shall prove when $\alpha_1, \alpha_2 > 1$. The proofs for the cases $\alpha_i = 1$ are similar. We follow the argument in [16] Section 4. Let

$$\frac{1}{s} := \frac{1}{\alpha_2'} + \frac{1}{\beta}, \quad \frac{1}{\widetilde{s}} := \frac{1}{\widetilde{\alpha}_2'} + \frac{1}{\widetilde{\beta}}.$$

Note that $s = \widetilde{s}$ by Lemma 10. Let f_1 be fixed and take w_1 such that

$$w_1^s \in A_{1+s/\alpha_1'}$$

and fix. For any W such that $W^\beta \in A_{1+\beta/\alpha_2'}(w_1^s dx)$, we have by Lemma 13

$$(w_1, W \cdot w_1^{-1+s/\beta}) \in A(\alpha_1, \alpha_2; \beta).$$

By the assumption of Theorem 3',

$$\left(\int_{\mathbb{R}^n} g^q (w_1 \cdot W w_1^{-1+s/\beta})^q \, dx\right)^{1/q} \leqslant C([W^\beta]_{A_{1+\beta/\alpha_2'}(w_1^s dx)}, [w_1^s]_{A_{1+s/\alpha_1'}}) \times \left(\int_{\mathbb{R}^n} f_1^{p_1} w_1^{p_1} \, dx\right)^{1/p_1} \left(\int_{\mathbb{R}^n} f_2^{p_2} (W w_1^{-1+s/\beta})^{p_2} \, dx\right)^{1/p_2},$$

and we have

$$\left(\int_{\mathbb{R}^n} (g \cdot w_1^{s(1/\beta - 1/q)})^q W^q \cdot w_1^s \, dx\right)^{1/q} \leqslant C([W^\beta]_{A_{1+\beta/\alpha_2'}(w_1^s dx)}, [w_1^s]_{A_{1+s/\alpha_1'}}) \times \left(\int_{\mathbb{R}^n} f_1^{p_1} w_1^{p_1} \, dx\right)^{1/p_1} \left(\int_{\mathbb{R}^n} (f_2 \cdot w_1^{-1+s/\beta - s/p_2})^{p_2} W^{p_2} \cdot w_1^s \, dx\right)^{1/p_2}.$$

Since f_1 and w_1 are fixed, we can apply Theorem 1' to the pair

$$\left(f_2 \cdot w_1^{-1+s/\beta - s/p_2}, g \cdot w_1^{s(1/\beta - 1/q)}\right)$$

and the weight W with respect to the measure $d\mu = w_1^s \, dx$ and obtain

$$\left(\int_{\mathbb{R}^n} (g \cdot w_1^{s(1/\beta - 1/q)})^{\widetilde{q}} W^{\widetilde{q}} \cdot w_1^s \, dx\right)^{1/\widetilde{q}} \leqslant C([W^{\widetilde{\beta}}]_{A_{\widetilde{\gamma}}(w_1^s dx)}, [w_1^s]_{A_{1+s/\alpha_1'}}) \times \left(\int_{\mathbb{R}^n} f_1^{p_1} w_1^{p_1} \, dx\right)^{1/p_1} \left(\int_{\mathbb{R}^n} (f_2 \cdot w_1^{-1+s/\beta - s/p_2})^{\widetilde{p}_2} W^{\widetilde{p}_2} \cdot w_1^s \, dx\right)^{1/\widetilde{p}_2},$$

where $W^{\widetilde{\beta}} \in A_{\widetilde{\gamma}}(w_1^s dx)$, $\widetilde{\beta} > 0$, $\widetilde{\gamma} > 1$ and

$$\frac{1}{\widetilde{p}_2} - \frac{1}{p_2} = \frac{1}{\widetilde{q}} - \frac{1}{q} = \frac{1}{\widetilde{\beta}} - \frac{1}{\beta}, \tag{40}$$

$$\frac{\widetilde{\beta}}{\widetilde{\gamma}} = \frac{\beta}{1 + \beta/\alpha_2'}, \tag{41}$$

$$\max\left(\frac{1}{p_2} - \frac{1}{q}, \frac{1}{p_2} - \frac{1}{\beta}\right) < \frac{1}{\widetilde{p}_2} < \frac{1}{p_2} - \frac{1}{\beta} + \frac{1 + \beta/\alpha_2'}{\beta}. \tag{42}$$

The conditions (40) and (42) are satisfied by the assumptions of theorem. By Lemma 10 we have $\widetilde{\gamma} = 1 + \widetilde{\beta}/\widetilde{\alpha}_2' > 1$. We can write

$$\left(\int_{\mathbb{R}^n} (g \cdot w_1^{s(1/\beta - 1/q)})^{\widetilde{q}} W^{\widetilde{q}} \cdot w_1^s \, dx\right)^{1/\widetilde{q}} \leqslant C([W^{\widetilde{\beta}}]_{A_{1+\widetilde{\beta}/\widetilde{\alpha}_2'}(w_1^s dx)}, [w_1^s]_{A_{1+s/\alpha_1'}}) \times \left(\int_{\mathbb{R}^n} f_1^{p_1} w_1^{p_1} \, dx\right)^{1/p_1} \left(\int_{\mathbb{R}^n} (f_2 \cdot w_1^{-1+s/\beta - s/p_2})^{\widetilde{p}_2} W^{\widetilde{p}_2} \cdot w_1^s \, dx\right)^{1/\widetilde{p}_2} \qquad (\star)$$

where $W^{\widetilde{\beta}} \in A_{1+\widetilde{\beta}/\widetilde{\alpha}_2'}(w_1^s dx)$.

For any $(w_1, v_2) \in A(\alpha_1, \widetilde{\alpha}_2; \widetilde{\beta})$, we have by Lemma $\widetilde{12}$

$$w_1^{\widetilde{s}} \in A_{1+\widetilde{s}/\alpha_1'},$$

and

$$[w_1^s]_{A_{1+s/\alpha_1'}} = [w_1^{\widetilde{s}}]_{A_{1+\widetilde{s}/\alpha_1'}} \leqslant [(w_1, v_2)]_{A(\alpha_1, \widetilde{\alpha}_2; \widetilde{\beta})}^{\widetilde{s}}. \qquad (43)$$

Let $W := w_1^{s/\widetilde{\alpha}_2'} \cdot v_2$. By Lemma 14 we have

$$W^{\widetilde{\beta}} \in A_{1+\widetilde{\beta}/\widetilde{\alpha}_2'}(w_1^s dx).$$

Therefore we can substitute W into $(\star)$. By Lemma 10 we have

$$w_1^{s(1/\beta - 1/q)\widetilde{q}} W^{\widetilde{q}} w_1^s = (w_1 v_2)^{\widetilde{q}},$$
$$w_1^{(-1+s/\beta - s/p_2)\widetilde{p}_2} W^{\widetilde{p}_2} w_1^s = v_2^{\widetilde{p}_2}.$$

By (43) and Lemma 14, we obtain

$$\left(\int_{\mathbb{R}^n} g^{\widetilde{q}} (w_1 v_2)^{\widetilde{q}} \, dx\right)^{1/\widetilde{q}} \leqslant C([(w_1, v_2)_{A(\alpha_1, \widetilde{\alpha}_2; \widetilde{\beta})}]) \left(\int_{\mathbb{R}^n} f_1^{p_1} w_1^{p_1} \, dx\right)^{1/p_1} \left(\int_{\mathbb{R}^n} f_2^{\widetilde{p}_2} v_2^{\widetilde{p}_2} \, dx\right)^{1/\widetilde{p}_2}. \quad \square$$

5. Multilinear extrapolations

Finally we consider multilinear extrapolations.

DEFINITION 9. We say that m-tuple of weights $(w_1, w_2, \ldots, w_m) \in A(\alpha_1, \alpha_2, \ldots, \alpha_m; \beta)$ if

$$[(w_1, w_2, \ldots, w_m)]_{A(\alpha_1, \alpha_2, \ldots, \alpha_m; \beta)} := \sup_Q \left(\fint_Q (w_1 w_2 \cdots w_m)^\beta \, dx\right)^{1/\beta} \prod_{i=1}^m \left(\fint_Q w_i^{-\alpha_i'} \, dx\right)^{1/\alpha_i'} < \infty.$$

We obtain the following.

THEOREM 4. *Let* $p_i, q > 0$, $\alpha_i \geqslant 1$ *and* $\beta > 0$. *If* $(f_1, f_2, \dots f_m, g)$ *satisfies*

$$\textit{weighted} \quad L^{p_1} \times \cdots \times L^{p_m} \to L^q \quad \textit{for} \quad A(\alpha_1, \alpha_2, \dots, \alpha_m; \beta),$$

then $(f_1, f_2, \dots f_m, g)$ *satisfies*

$$\textit{weighted} \quad L^{\widetilde{p}_1} \times \cdots \times L^{\widetilde{p}_m} \to L^{\widetilde{q}} \quad \textit{for} \quad A(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \dots, \widetilde{\alpha}_m; \widetilde{\beta}),$$

where $\widetilde{p}_i, \widetilde{q} > 0$, $\widetilde{\alpha}_i > 1$, $\widetilde{\beta} > 0$ *and*

$$\frac{1}{\widetilde{p}_i} - \frac{1}{p_i} = \frac{1}{\widetilde{\alpha}_i} - \frac{1}{\widetilde{\alpha}_i} \qquad i = 1, 2, \dots, m,$$
$$\frac{1}{\widetilde{q}} - \frac{1}{q} = \sum_{i=1}^{m} \left(\frac{1}{\widetilde{p}_i} - \frac{1}{p_i} \right) = \frac{1}{\widetilde{\beta}} - \frac{1}{\beta},$$
$$\frac{1}{p_i} - \frac{1}{\alpha_i} < \frac{1}{\widetilde{p}_i} < \frac{1}{p_i} - \frac{1}{\alpha_i} + 1 \qquad i = 1, 2, \dots, m,$$
$$\max \left(\sum_{i=1}^{m} \frac{1}{p_i} - \frac{1}{q}, \sum_{i=1}^{m} \frac{1}{p_i} - \frac{1}{\beta} \right) < \sum_{i=1}^{m} \frac{1}{\widetilde{p}_i}.$$

We show an outline of the proof.

Important lemmas. Corresponding to three important Lemmas 12–14, we have the following lemmas. Let

$$\frac{1}{s_i} := \sum_{j \neq i} \frac{1}{\alpha_j} \quad \text{and} \quad \frac{1}{s} := \frac{1}{\alpha'_m} + \frac{1}{\beta},$$

where $\sum_{j \neq i}$ means the summation over $\{1, 2, \dots, m\} \setminus \{i\}$.

LEMMA 12'. *If* $(w_1, w_2, \dots, w_m) \in A(\alpha_1, \alpha_2, \dots, \alpha_m; \beta)$, *then*

$$w_i^{s_i} \in A_{1 + s_i / \widetilde{\alpha}'_i} \qquad i = 1, 2, \dots, m,$$
$$(w_1 w_2 \cdots w_{m-1})^s \in A_{1 + s(1/\alpha'_1 + 1/\alpha'_2 + \cdots + 1/\alpha'_{m-1})}.$$

REMARK. In fact we need to consider $(w_2 \cdots w_m)^s$ where $1/s = 1/\alpha'_1 + 1/\beta$, and so on.

LEMMA 13'. *Assume that*

$$w_i^{s_i} \in A_{1 + s_i / \widetilde{\alpha}'_i} \qquad i = 1, 2, \dots, m-1,$$
$$(w_1 w_2 \cdots w_{m-1})^s \in A_{1 + s(1/\alpha'_1 + 1/\alpha'_2 + \cdots + 1/\alpha'_{m-1})}.$$

Then for any W *such that* $W^\beta \in A_{1 + \beta / \alpha'_m}((w_1 w_2 \cdots w_{m-1})^s \, dx)$,

$$(w_1, w_2, \dots, w_{m-1}, W \cdot (w_1 w_2 \cdots w_{m-1})^{-s/\alpha'_m}) \in A(\alpha_1, \alpha_2, \dots, \alpha_m; \beta).$$

LEMMA 14'. *Assume that* $(w_1,\dots,w_{m-1},v_m)\in A(\alpha_1,\dots,\alpha_{m-1},\widetilde{\alpha}_m;\widetilde{\beta})$, *and let*

$$W := (w_1\cdots w_{m-1})^{s/\widetilde{\alpha}'_m}\cdot v_m.$$

Then

$$W^{\widetilde{\beta}}\in A_{1+\widetilde{\beta}/\widetilde{\alpha}'_m}((w_1\cdots w_{m-1})^s\,dx).$$

Reduction to linear cases. Corresponding to Thorem 3 we prove the following.

THEOREM 5. *Let* $p_i,q>0$, $\alpha_i\geqslant 1$ *and* $\beta>0$. *If* $(f_1,f_2,\dots f_m,g)$ *satisfies*

$$\textit{weighted}\quad L^{p_1}\times\cdots\times L^{p_m}\to L^q\quad \textit{for}\quad A(\alpha_1,\alpha_2,\dots,\alpha_m;\beta),$$

then $(f_1,f_2,\dots f_m,g)$ *satisfies*

$$\textit{weighted}\quad L^{p_1}\times\cdots\times L^{p_{m-1}}\times L^{\widetilde{p}_m}\to L^{\widetilde{q}}\quad \textit{for}\quad A(\alpha_1,\dots\alpha_{m-1},\widetilde{\alpha}_m;\widetilde{\beta}),$$

where $\widetilde{p}_m,\widetilde{q}>0$, $\widetilde{\alpha}_m>1$, $\widetilde{\beta}>0$ *and*

$$\frac{1}{\widetilde{p}_m}-\frac{1}{p_m}=\frac{1}{\widetilde{q}}-\frac{1}{q}=\frac{1}{\widetilde{\alpha}_m}-\frac{1}{\widetilde{\alpha}_m},$$
$$\max\left(\frac{1}{p_m}-\frac{1}{\alpha_m},\frac{1}{p_m}-\frac{1}{\beta}\right)<\frac{1}{\widetilde{p}_m}<\frac{1}{p_m}-\frac{1}{\alpha_m}+1.$$

REMARK. In fact we need to freeze several indices, for example, $L^{\widetilde{p}_1}\times L^{p_2}\times\cdots\times L^{p_m}\to L^{\widetilde{q}}$, and so on.

Proof of Theorem 4 by using Theorem 5. Different from the bilinear cases, it is important what indices to be frozen.

We consider the case that $\beta<q$. By the definitions of indices, there exists $1\leqslant i\leqslant m$ such that $1/p_i-1/q<1/\widetilde{p}_i$. Assume that

$$1/p_m-1/q<1/\widetilde{p}_m.$$

We define auxiliary indices $\widetilde{q}_m$ and $\widetilde{\beta}_m$ as follows.

$$\frac{1}{\widetilde{q}_m}:=\frac{1}{\widetilde{p}_m}-\frac{1}{p_m}+\frac{1}{q},$$
$$\frac{1}{\widetilde{\beta}_m}:=\frac{1}{\widetilde{p}_m}-\frac{1}{p_m}+\frac{1}{\beta}.$$

By Theorem 4 we have that $(f_1,f_2,\dots f_m,g)$ satsifies

$$\text{weighted}\quad L^{p_1}\times\cdots\times L^{p_{m-1}}\times L^{\widetilde{p}_m}\to L^{\widetilde{q}_m}\quad\text{for}\quad A(\alpha_1,\dots\alpha_{m-1},\widetilde{\alpha}_m;\widetilde{\beta}_m).$$

Similarly, there exists $1\leqslant i\leqslant m-1$ such that $1/p_i-1/\widetilde{q}_m<1/\widetilde{p}_i$. Assume that

$$1/p_{m-1}-1/\widetilde{q}_m<1/\widetilde{p}_{m-1}.$$

Let

$$\frac{1}{\widetilde{q}_{m-1}} := \frac{1}{\widetilde{p}_{m-1}} - \frac{1}{p_{m-1}} + \frac{1}{\widetilde{q}_m},$$
$$\frac{1}{\widetilde{\beta}_{m-1}} := \frac{1}{\widetilde{p}_{m-1}} - \frac{1}{p_{m-1}} + \frac{1}{\widetilde{\beta}_m},$$

and we have that $(f_1, f_2, \dots f_m, g)$ satisfies

$$\text{weighted} \quad L^{p_1} \times \cdots \times L^{p_{m-2}} \times L^{\widetilde{p}_{m-1}} \times L^{\widetilde{p}_m} \to L^{\widetilde{q}_{m-1}} \quad \text{for} \quad A(\alpha_1, \dots \widetilde{\alpha}_{m-1}, \widetilde{\alpha}_m; \widetilde{\beta}_{m-1}).$$

Continuing this procedure we obtain the desired result.

The proof for the case $\beta \geqslant q$ is similar. □

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