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ON THE CONSTANT IN THE HARDY INEQUALITY FOR FINITE SEQUENCES

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Abstract. We investigate the behaviour of the smallest possible constant d_n in the Hardy's inequality

$$\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}a_j\right)^2 \leqslant d_n\sum_{k=1}^{n}a_k^2, \qquad (a_1,\dots,a_n)\in\mathbb{R}^n.$$

A new proof of the Hardy's inequality is given which allows us to give another much simpler proof of the upper estimation of d_n

$$d_n < 4-\frac{c}{\ln^2 n}, \qquad c>0.$$

1. Introduction

In series of papers Hardy [4, 5, 6] proved for $p>1$ the inequality

$$\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}a_j\right)^p \leqslant C\sum_{k=1}^{n}a_k^p, \quad a_k\geqslant 0, \quad k=1,2,\dots,n \tag{1}$$

where the constant C is an absolute constant in a sense it does not depend on the sequence $\{a_k\}$ and n. Initially Hardy proved the inequality (1) with the constant $\frac{p^2}{p-1}$. Later Landau [9] proved that the constant $\left(\frac{p}{p-1}\right)^p$ is the smallest possible one, for which (1) holds for every n.

For p-even integer the assumption for nonnegativity of $\{a_k\}$ can be dropped, and for $p=2$ the inequality (1) becomes

$$\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}a_j\right)^2 \leqslant 4\sum_{k=1}^{n}a_k^2. \tag{2}$$

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There are many papers investigating different generalizations and applications of Hardy's inequality – see for instance [8] and the bibliography of the book [7].

Let allow the constant C in (1) to depend on n and let us denote it by d_n. Then we can write (1) for $p=2$ as

$$\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}a_j\right)^2 \leqslant d_n\sum_{k=1}^{n}a_k^2, \qquad (a_1,a_2,\dots,a_n)\in\mathbb{R}^n \tag{3}$$

The behavior of the constant d_n as a function of n was studied in many papers – see, for instance, [1], [2], [10], [11], [12]. In [12] Herbert S. Wilf established the exact rate of convergence of the constant d_n

$$d_n = 4-\frac{16\pi^2}{\ln^2 n}+O\left(\frac{\ln\ln n}{\ln^3 n}\right).$$

In [3] we also studied the asymptotic behavior of d_n and proved that the next inequalities are true

$$4\left(1-\frac{4}{\ln n+4}\right)\leqslant d_n\leqslant 4\left(1-\frac{8}{(\ln n+4)^2}\right), \qquad n\geqslant 3,$$

i.e.

$$4-\frac{c_1}{\ln n}\leqslant d_n\leqslant 4-\frac{c_2}{\ln^2 n}, \qquad n\geqslant 2 \tag{4}$$

where the constants c_1 and c_2 do not depend on n. By considering the sequence

$$a_k=\sqrt{k}-\sqrt{k-1}, \qquad k=1,\dots,n$$

establishing the left inequality is not difficult. But the proof of the right inequality was very complicated and we used the special properties of the space ℓ_+^2 where ℓ_+^p is the class of nonnegative sequences $\{a_k\}$.

In this paper we give a much simpler proof of the upper estimation of d_n in (4) which could be used (with some modifications) in order to prove a similar result for $p\neq 2$. Our main result read as follows:

THEOREM 1. *The next estimation of d_n is true*

$$d_n\leqslant 4-\frac{d}{\ln^2(n+1)}, \tag{5}$$

where $d=1/4$ for $n\geqslant 16$, $d=1/6$ for $n\geqslant 7$, $d=1/8$ for $n\geqslant 5$ and $d=1/16$ for $n\geqslant 2$.

REMARK 1. The constants $1/4, 1/6, 1/8$ and $1/16$ in the above estimation (5) are by no means the best ones. They could be significantly improved in a lot of ways but that would have made the proof longer and much more complicated. Our goal was to keep the proof as simple as possible.

2. Proof of the Theorem 1

From Cauchy's inequality we have for every two sequences μ_i and η_i, $i = 1, \ldots n$

$$\left(\sum_{i=1}^{k} \mu_i \eta_i\right)^2 \leqslant \left(\sum_{i=1}^{k} \mu_i^2\right)\left(\sum_{i=1}^{k} \eta_i^2\right).$$

Let us denote $a_i = \mu_i \eta_i$. Then

$$\left(\frac{1}{k}\sum_{i=1}^{k} a_i\right)^2 \leqslant \frac{1}{k^2}\left(\sum_{i=1}^{k} \mu_i^2\right)\left(\sum_{i=1}^{k} \frac{a_i^2}{\mu_i^2}\right)$$

and after changing the order of summation

$$\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{i=1}^{k} a_i\right)^2 \leqslant \sum_{i=1}^{n} M_i a_i^2 \leqslant \left(\max_{1\leqslant i\leqslant n} M_i\right)\sum_{i=1}^{n} a_i^2,$$

where

$$M_i = \frac{1}{\mu_i^2} M_i^*, \quad M_i^* = \sum_{k=i}^{n} \frac{1}{k^2}\sum_{j=1}^{k} \mu_j^2.$$

Obviously

$$d_n \leqslant \max_{1\leqslant i\leqslant n} M_i, \quad \text{so we want to minimize} \quad \max_{1\leqslant i\leqslant n} M_i$$

over all sequences $\mu = \{\mu_i\}$, $i = 1, 2, \ldots, n$, i.e. to find

$$\min_{\mu} \max_{1\leqslant i\leqslant n} M_i$$

or, at least, to make it as small as possible.

REMARK 2. By choosing, for instance,

$$\mu_k = k^{-1/4}, \quad k = 1, 2, \ldots n$$

it is not very difficult to prove that $\max_{1\leqslant i\leqslant n} M_i < 4$, i.e. $d_n < 4$. In fact, by taking the sequence

$$\mu_k^2 = \frac{k\sqrt{k}}{k+1} - \frac{(k-1)\sqrt{k-1}}{k}, \quad k = 1, 2, \ldots n$$

the next upper estimation of d_n could be proved

$$d_n < 4 - \frac{4}{\sqrt{n+1}}.$$

It is similar to the result in [2] where the authors proved the estimation

$$d_n \leqslant n^{-1}\left(\sum_{k=1}^{n} k^{-1/2}\right)^2.$$

Although the results of this type give better estimations for some n, asymptotically they are worse.

In order to prove the estimation (5) we need to make a more complicated choice of the sequence μ_k.

Let

$$\mu_k^2 = c\int_{k-1}^{k}\frac{dx}{\sqrt{x}} - \frac{1}{\ln^2(n+1)}\int_k^{k+1}\frac{\ln^2 x}{\sqrt{x}}dx$$

where $c \geqslant 1$. It is obvious that μ_k is well defined. Then

$$\begin{aligned}
\mu_i^2 &= c\int_{i-1}^{i}\frac{dx}{\sqrt{x}} - \frac{1}{\ln^2(n+1)}\int_i^{i+1}\frac{\ln^2 x}{\sqrt{x}}dx \\
&> c\int_{i-1}^{i}\frac{dx}{\sqrt{x}} - \frac{\ln^2(i+1)}{\sqrt{i}\ln^2(n+1)}.
\end{aligned} \tag{6}$$

For M_i^* we have

$$\begin{aligned}
\sum_{j=1}^{k}\mu_j^2 &= c\int_0^k\frac{dx}{\sqrt{x}} - \frac{1}{\ln^2(n+1)}\int_1^{k+1}\frac{\ln^2 x}{\sqrt{x}}dx \\
&\leqslant c\int_0^k\frac{dx}{\sqrt{x}} - \frac{1}{\ln^2(n+1)}\int_1^{k}\frac{\ln^2 x}{\sqrt{x}}dx \\
&= 2c\sqrt{k} - \frac{2\sqrt{k}}{\ln^2(n+1)}\left[\ln^2 k - 4\ln k + 8 - \frac{8}{\sqrt{k}}\right]
\end{aligned}$$

and

$$M_i^* \leqslant 2c\sum_{k=i}^{n}\frac{1}{k^{3/2}} - \frac{2}{\ln^2(n+1)}\sum_{k=i}^{n}\left[\frac{\ln^2 k - 4\ln k + 8}{k^{3/2}} - \frac{8}{k^2}\right].$$

For the first term in RHS we have (for $i \geqslant 1$)

$$\sum_{k=i}^{n}\frac{1}{k^{3/2}} \leqslant 2\int_{i-1}^{i}\frac{dx}{\sqrt{x}} - \frac{2}{\sqrt{n+\frac{1}{2}}}.$$

Indeed it follows from easily verifiable inequalities

$$\frac{1}{k^{3/2}} \leqslant \frac{2}{\sqrt{k-\frac{1}{2}}} - \frac{2}{\sqrt{k+\frac{1}{2}}} \quad \text{and} \quad \frac{1}{\sqrt{i-\frac{1}{2}}} \leqslant \int_{i-1}^{i}\frac{dx}{\sqrt{x}}.$$

Then

$$\begin{aligned}
M_i^* &\leqslant 4c\int_{i-1}^{i}\frac{dx}{\sqrt{x}} - \frac{4c}{\sqrt{n+\frac{1}{2}}} - \frac{2}{\ln^2(n+1)}\sum_{k=i}^{n}\left[\frac{\ln^2 k - 4\ln k + 8}{k^{3/2}} - \frac{8}{k^2}\right] \\
&= 4c\int_{i-1}^{i}\frac{dx}{\sqrt{x}} - \frac{4c}{\sqrt{n+\frac{1}{2}}} - \frac{2}{\ln^2(n+1)}\sum_{k=i}^{n}\left[f(k) - \frac{8}{k^2}\right]
\end{aligned}$$

where for brevity we denoted by

$$f(x) = x^{-3/2}\left[\ln^2 x - 4\ln x + 8\right].$$

We have $f(x) > 0$ and $f(x)$ is decreasing since

$$f'(x) = \frac{-3\ln^2 x + 16\ln x - 32}{2x^{5/2}} < 0.$$

Then

$$\sum_{k=i}^{n} f(k) > \int_i^n f(x)\,dx = \frac{2\ln^2 i + 16}{\sqrt{i}} - \frac{2\ln^2 n + 16}{\sqrt{n}}.$$

Consequently

$$\begin{aligned}
M_i^* &\leqslant 4c\int_{i-1}^{i}\frac{dx}{\sqrt{x}} - \frac{4c}{\sqrt{n+\frac{1}{2}}} + \frac{16}{\ln^2(n+1)}\sum_{k=i}^{n}\frac{1}{k^2} \\
&\quad - \frac{2}{\ln^2(n+1)}\left[\frac{2\ln^2 i + 16}{\sqrt{i}} - \frac{2\ln^2 n + 16}{\sqrt{n}}\right] \\
&= 4\left[c\int_{i-1}^{i}\frac{dx}{\sqrt{x}} - \frac{\ln^2(i+1)}{\sqrt{i}\ln^2(n+1)}\right] + \frac{4\left(\ln^2(i+1) - \ln^2 i\right)}{\sqrt{i}\ln^2(n+1)} \\
&\quad - \frac{4c}{\sqrt{n+\frac{1}{2}}} - \frac{32}{\sqrt{i}\ln^2(n+1)} + \frac{4\ln^2 n + 32}{\sqrt{n}\ln^2(n+1)} + \frac{16}{\ln^2(n+1)}\sum_{k=i}^{n}\frac{1}{k^2}.
\end{aligned}$$

Now

$$\ln^2(i+1) - \ln^2 i = \ln\frac{i+1}{i}\ln i(i+1) < \frac{\ln i(i+1)}{i} < 1, \tag{7}$$

and

$$\frac{32}{\sqrt{i}\ln^2(n+1)} - \frac{16}{\ln^2(n+1)}\sum_{k=i}^{n}\frac{1}{k^2} > \frac{5}{\sqrt{i}\ln^2(n+1)}. \tag{8}$$

By taking $c = 2$ for $n \geqslant 16$, $c = 3$ for $n \geqslant 7$, $c = 4$ for $n \geqslant 5$ and $c = 8$ for $n \geqslant 2$ we have also the estimation

$$\frac{4c}{\sqrt{n+\frac{1}{2}}} - \frac{4\ln^2 n + 32}{\sqrt{n}\ln^2(n+1)} > 0. \tag{9}$$

Then from all of the above estimations (7), (8) and (9) it follows that

$$M_i^* \leqslant 4\left[c\int_{i-1}^{i}\frac{dx}{\sqrt{x}} - \frac{\ln^2(i+1)}{\sqrt{i}\ln^2(n+1)}\right] - \frac{1}{\sqrt{i}\ln^2(n+1)}. \tag{10}$$

Since

$$c\int_{i-1}^{i}\frac{dx}{\sqrt{x}} = \frac{2c}{\sqrt{i-1}+\sqrt{i}} < \frac{2c}{\sqrt{i}}$$

we have from (6) and (10)

$$M_i \leqslant 4 - \frac{\frac{1}{\sqrt{i}\ln^2(n+1)}}{c\int_{i-1}^{i}\frac{dx}{\sqrt{x}} - \frac{\ln^2(i+1)}{\sqrt{i}\ln^2(n+1)}} < 4 - \frac{1}{2c\ln^2(n+1)} = 4 - \frac{d}{\ln^2(n+1)}$$

and consequently

$$d_n \leqslant 4 - \frac{d}{\ln^2(n+1)}.$$

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