Mathematical
Inequalities
& Applications
Volume 26, Number 2 (2023), 499–510 doi:10.7153/mia-2023-26-32

WEIGHTED DYNAMIC ESTIMATES FOR CONVEX AND SUBHARMONIC FUNCTIONS ON TIME SCALES

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(*Communicated by J. Jakšetić*)

Abstract. This article introduces a new type of weighted square delta integral inequalities involving the delta derivative of a convex function. As an extension, we also establish weighted square delta integral inequalities for subharmonic functions on time scales. Here, we rely on a new definition of the time scales Laplace operator. The significance of this work in the existing literature is provided at the end of the article.

1. Introduction

The evolution of new mathematical inequalities both continuous and discrete often places a rigid support for the interrogative procedures and algorithms practiced in applied sciences. It is useful to ask whether it is plausible to have a scheme which incorporates both discrete and continuous structures simultaneously. One of several attempts to combine discrete and continuous mathematics is the time scale setting, which was founded by Stefan Hilger in 1988 [16]. Since then, thousands of research articles appeared in the theory and its applications to several fields, see e.g., [23,12,3,6,2,5,24]. The treatises of Bohner and Peterson [8,9] and other texts like [1,7] might be consulted by the reader for more detailed explanations of time scale calculus.

In modern calculus, it is notable that the inherent generalization of convex functions to multi-variable functions are subharmonic functions, associated with the well-known Laplace operator. So it is of equal interest to derive analogous inequalities for subharmonic functions. These functions perform a vital role in modern analysis along with classical potential theory and usually serve as a dynamic tool for the inspection of solutions of Dirichlet and classical Poisson problems in the theory of PDEs, some recent examples can be seen in [11,14,15,21].

This manuscript is structured as follows. Section 2 contains some basic concepts needed to prove the main results, while Sections 3 and 4 are devoted to our main results. The weighted square delta integral estimates for the first delta derivative of a twice delta differentiable function are presented in Section 3, and in Section 4, we generalize these estimates for subharmonic as well as for superharmonic functions. Section 5 is devoted to present some examples, and in Section 6, we provide the conclusion.

Mathematics subject classification (2020): 26D20, 39B62, 34N05.
Keywords and phrases: Time scale integrals, subharmonic function, convex function, Poincaré-type inequality.

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Paper MIA-26-32

2. Preliminaries

We assemble some fundamental preliminaries in this section, which we use in the remaining part of the article. We use Δ for time scale Δ-calculus and $\triangle$ for the Laplace operator. A time scale $\mathbb{T}$ is defined to be an arbitrary closed subset of the real numbers $\mathbb{R}$, with the standard inherited topology. We refer the reader to [9, Chapter 1] for basic notions and notations of one-dimensional time scales.

Now let $n \in \mathbb{N}$, and for each $i \in \{1,2,\dots,n\}$, set

$$\begin{aligned}\Lambda^n &= \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n \\ &= \{t = (t_1,t_2,\dots,t_n) :\ t_i \in \mathbb{T}_i,\ i = 1,2,\dots,n\}.\end{aligned}$$

We call Λ^n an n-dimensional time scale. Let σ_i, $i \in \{1,2,\dots,n\}$, be the forward jump operators in $\mathbb{T}_i$. The operator $\sigma : \Lambda^n \to \mathbb{R}^n$ defined by

$$\sigma(t) = (\sigma_1(t_1), \sigma_2(t_2), \dots, \sigma_n(t_n))$$

is said to be the forward jump operator in Λ^n. For the function $\theta : \Lambda^n \to \mathbb{R}$, we put

$$\theta^\sigma(t) = \theta(\sigma_1(t_1), \sigma_2(t_2), \dots, \sigma_n(t_n))$$

and

$$\theta_i^{\sigma_i}(t) = \theta(t_1,\dots,t_{i-1},\sigma_i(t_i),t_{i+1},\dots,t_n).$$

Let us recall the definition of partial derivarives on time scales. For this, we need to introduce the following notations from [7,9], and we put

$$\Lambda^{\kappa n} = \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa \times \ldots \times \mathbb{T}_n^\kappa$$

and

$$\Lambda_i^{\kappa_i n} = \mathbb{T}_1 \times \ldots \times \mathbb{T}_{i-1} \times \mathbb{T}_i^\kappa \times \mathbb{T}_{i+1} \times \ldots \times \mathbb{T}_n.$$

Let $\theta : \Lambda^n \to \mathbb{R}$ be a function. The partial delta derivative of θ with respect to $t_i \in \mathbb{T}_i^\kappa$ at $t \in \Lambda_i^{\kappa_i n}$ is defined as the limit of $s_i \to t_i$, $s_i \neq \sigma_i(t_i)$, of

$$\frac{\theta(t_1,\dots,t_{i-1},\sigma_i(t_i),t_{i+1},\dots,t_n) - \theta(t_1,\dots,t_{i-1},s_i,t_{i+1},\dots,t_n)}{\sigma_i(t_i) - s_i}$$

provided that this limit exists, and it is denoted by

$$\frac{\partial \theta(t)}{\Delta_i t_i} \quad \text{or} \quad \theta^{\Delta_i}(t).$$

Second-order partial delta derivatives of θ are denoted by

$$\frac{\partial^2 \theta(t)}{\Delta_i t_i^2} \quad \text{or} \quad \frac{\partial^2 \theta(t)}{\Delta_i t_i \Delta_j t_j}.$$

Higher-order partial delta derivatives are similarly denoted.

One more important relation of the Δ derivative is given in the next result, see [8, page 13].

REMARK 1. $\theta^{\Delta\sigma}$ and $\theta^{\sigma\Delta}$ are not equal in general even if both exist. This is clear from the relation

$$\theta^{\sigma\Delta} = \left(1+\mu^{\Delta}\right)\theta^{\Delta\sigma}.$$

Note that for time scales with constant graininess such as $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$, we actually have

$$\theta^{\Delta\sigma} = \theta^{\sigma\Delta}. \tag{2.1}$$

A function θ on a convex set $C \subseteq \mathbb{R}$ is called convex provided

$$\theta(\lambda c_0 + (1-\lambda)c_1) \leqslant \lambda\theta(c_0) + (1-\lambda)\theta(c_1) \quad \text{for all} \quad c_0, c_1 \in C \quad \text{and} \quad \lambda \in [0,1]. \tag{2.2}$$

θ is called concave if $-\theta$ is convex. Let us recall some properties of convex functions. The reader may refer to [18, 22] for proofs. A convex function θ defined on C is continuous in the interior of C provided θ is differentiable almost everywhere and the second derivative of θ is always nonnegative, that is, $\theta''(c) \geqslant 0$ for all $c \in C$. For two convex functions θ_1, θ_2 and $k \in \mathbb{R}$, the functions $\theta = \theta_1 + \theta_2$ and $k\theta_1$ are also convex, but the difference of two convex functions may not preserve convexity.

DEFINITION 1. (See [13]) If $\theta_1, \theta_2 : C \to \mathbb{R}$ are convex, then $\theta := \theta_1 - \theta_2$ is called *delta-convex*. Similarly, if $\theta_1, \theta_2 : C \to \mathbb{R}$ are concave, then θ is called *delta-concave*.

3. Energy estimates for convex functions

In order to prove our energy estimates, we use the following lemma.

LEMMA 1. *For $\varphi, \omega \in \mathrm{C}^2_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$, we have*

$$\left(\varphi^{\Delta}\omega - \varphi\omega^{\Delta}\right)^{\Delta} = \varphi^{\Delta\Delta}\omega^{\sigma} - \varphi^{\sigma}\omega^{\Delta\Delta}. \tag{3.1}$$

Proof. By using the product rule for delta derivatives, see [8, page 8, Theorem 1.20], since $\varphi^{\Delta}\omega^{\Delta}$ cancels, we obtain (3.1). □

C. Dinu [10] defined convex functions and their properties on time scales. Assume that θ is an rd-continuous and twice delta differentiable function which is convex on $[a,b]^{\kappa^2}_{\mathbb{T}}$. Then we have $\theta^{\Delta\Delta}(t) \geqslant 0$ for all $t \in [a,b]^{\kappa^2}_{\mathbb{T}}$. Define

$$\mathscr{W} := \left\{\omega \in \mathrm{C}^2_{\mathrm{rd}}([a,b]_{\mathbb{T}}, [0,\infty)) : \ \omega(t) = \omega^{\Delta}(t) = 0,\ t \in \{a,b\}\right\}.$$

THEOREM 1. *If θ is convex, θ^{σ} is differentiable, and $\omega \in \mathscr{W}$, then*

$$\int_a^b \left((\theta^{\Delta})^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma}\right)(t)\omega(\sigma(t))\Delta t \leqslant \int_a^b (\theta^2 + K\theta)(\sigma(t))\omega^{\Delta\Delta}(t)\Delta t, \tag{3.2}$$

where

$$K = 2 \sup_{\sigma(a) \leqslant t \leqslant \sigma(b)} |\theta(t)| = 2\|\theta\|_\infty.$$

Proof. Define $\varphi := \theta^2 + K\theta$. Then

$$\varphi^\Delta = \theta^\Delta \theta + \theta^\sigma \theta^\Delta + K\theta^\Delta$$

and

$$\begin{aligned} \varphi^{\Delta\Delta} &= \theta^{\Delta\Delta}\theta^\sigma + \theta^\Delta\theta^\Delta + \theta^{\sigma\Delta}\theta^{\Delta\sigma} + \theta^\sigma\theta^{\Delta\Delta} + K\theta^{\Delta\Delta} \\ &= \theta^{\Delta\Delta}(2\theta^\sigma + K) + (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma} \qquad (3.3) \\ &\geqslant \theta^{\Delta\Delta}(-2|\theta^\sigma| + K) + (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma} \\ &\geqslant (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma}. \qquad (3.4) \end{aligned}$$

Now, by integrating (3.1) from a to b, we get

$$\begin{aligned} 0 &\overset{(\omega\in\mathscr{W})}{=} \int_a^b \left(\varphi^\Delta\omega - \varphi\omega^\Delta\right)^\Delta(t)\Delta t \overset{(3.1)}{=} \int_a^b \left(\varphi^{\Delta\Delta}\omega^\sigma - \varphi^\sigma\omega^{\Delta\Delta}\right)(t)\Delta t \\ &\overset{(3.4)}{\geqslant} \int_a^b \left((\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma}\right)(t)\omega^\sigma(t)\Delta t - \int_a^b \left((\theta^\sigma)^2 + K\theta^\sigma\right)(t)\omega^{\Delta\Delta}\Delta t, \end{aligned}$$

which shows (3.2) and completes the proof. □

Our next result shows the variation of estimate (3.2) for a concave function.

THEOREM 2. *If θ is concave, θ^σ is differentiable, and $\omega \in \mathscr{W}$, then*

$$\int_a^b \left((\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma}\right)(t)\omega(\sigma(t))\Delta t \leqslant \int_a^b (\theta^2 - K\theta)(\sigma(t))\omega^{\Delta\Delta}(t)\Delta t, \qquad (3.5)$$

where K is defined as in Theorem 1.

Proof. Define $\varphi := \theta^2 - \theta K$. Then

$$\varphi^\Delta = \theta^\Delta\theta + \theta^\sigma\theta^\Delta - K\theta^\Delta$$

and

$$\begin{aligned} \varphi^{\Delta\Delta} &= \theta^{\Delta\Delta}(2\theta^\sigma - K) + (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma} \\ &\geqslant \theta^{\Delta\Delta}(2|\theta^\sigma| - K) + (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma} \\ &\geqslant (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma}. \qquad (3.6) \end{aligned}$$

Hence, a similar argument as in Theorem 1 completes the proof. □

The next results are associated with delta-convex and delta-concave functions. First we define them on time scales.

DEFINITION 2. A function $\theta \in C^2_{rd}([a,b]_{\mathbb{T}})$ is called *delta-convex* on $[a,b]_{\mathbb{T}}$ if there exists a pair of convex functions $\theta_1, \theta_2 \in C^2_{rd}([a,b]_{\mathbb{T}})$ such that θ is the difference

$$\theta = \theta_1 - \theta_2.$$

Similarly, θ is called *delta-concave* if θ_1, θ_2 are concave.

THEOREM 3. *If $\theta = \theta_1 - \theta_2$ is δ-convex, θ^σ is differentiable, and $\omega \in \mathscr{W}$, then*

$$\int_a^b \left((\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma}\right)(t)\omega(\sigma(t))\Delta t \leqslant \int_a^b (\theta^2 + K(\theta_1+\theta_2))(\sigma(t))\omega^{\Delta\Delta}(t)\Delta t, \quad (3.7)$$

where K is defined as in Theorem 1.

Proof. Define $\varphi := \theta^2 + K(\theta_1+\theta_2)$. Then

$$\begin{aligned}
\varphi^{\Delta\Delta} &= 2\theta^{\Delta\Delta}\theta^\sigma + K\left(\theta_1^{\Delta\Delta} + \theta_2^{\Delta\Delta}\right) + (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma} \\
&\geqslant -2\left|\theta^{\Delta\Delta}\right||\theta^\sigma| + K\left(\theta_1^{\Delta\Delta} + \theta_2^{\Delta\Delta}\right) + (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma} \\
&\geqslant -2\left|\theta_1^{\Delta\Delta} + \theta_2^{\Delta\Delta}\right||\theta^\sigma| + K\left(\theta_1^{\Delta\Delta} + \theta_2^{\Delta\Delta}\right) + (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma} \\
&= \left(\theta_1^{\Delta\Delta} + \theta_2^{\Delta\Delta}\right)(K - 2|\theta^\sigma|) + (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma} \\
&\geqslant (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma}.
\end{aligned}$$

Thus, by applying Lemma 1, we get the desired result. □

Now we can also give an estimate for δ-concave functions.

THEOREM 4. *If $\theta = \theta_1 - \theta_2$ is δ-concave, θ^σ is differentiable, and $\omega \in \mathscr{W}$, then*

$$\int_a^b \left((\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma}\right)(t)\omega(\sigma(t))\Delta t \leqslant \int_a^b (\theta^2 - K(\theta_1+\theta_2))(\sigma(t))\omega^{\Delta\Delta}(t)\Delta t,$$

where K is defined as in Theorem 1.

Proof. Define $\varphi := \theta^2 - K(\theta_1+\theta_2)$. Then

$$\begin{aligned}
\varphi^{\Delta\Delta} &= 2\theta^{\Delta\Delta}\theta^\sigma - K\left(\theta_1^{\Delta\Delta} + \theta_2^{\Delta\Delta}\right) + (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma} \\
&\geqslant 2\left|\theta^{\Delta\Delta}\right||\theta^\sigma| - K\left(\theta_1^{\Delta\Delta} + \theta_2^{\Delta\Delta}\right) + (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma} \\
&= \left(\theta_1^{\Delta\Delta} + \theta_2^{\Delta\Delta}\right)(2|\theta^\sigma| - K) + (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma} \\
&\geqslant (\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma}.
\end{aligned}$$

We may now achieve the desired outcome by using Lemma 1. □

Applying Hölder's inequality to estimate (3.7), we get the following result.

COROLLARY 1. *If $\theta = \theta_1 - \theta_2$ is δ-convex, θ^σ is differentiable, and $\omega \in \mathscr{W}$, then*

$$\int_a^b \left((\theta^\Delta)^2 + \theta^{\sigma\Delta}\theta^{\Delta\sigma}\right)(t)\omega(\sigma(t))\Delta t \leqslant \left\|(\theta^\sigma)^2 + K(\theta_1^\sigma + \theta_2^\sigma)\right\|_p \left\|\omega^{\Delta\Delta}\right\|_q,$$

where K is defined as in Theorem 1, p and q are conjugate exponents, $1 \leqslant p < \infty$, and

$$\|\varphi\|_p = \left(\int_a^b |\varphi(t)|^p \Delta t\right)^{\frac{1}{p}}.$$

Furthermore, we can also apply Hölder's inequality to the right-hand side of other estimates such as (3.2) and (3.5) to get similar results.

4. Energy estimates for subharmonic functions

As mentioned earlier, subharmonic functions are a natural generalization of univariable convex functions to the case of several variables. In a similar fashion, superharmonic functions are a generalization of univariable concave functions. It is of great interest to establish the preceding estimates for subharmonic and superharmonic functions. In order to achieve this, we first define subharmonic functions and superharmonic functions in the multivariable time scales case.

Let $a_i, b_i \in \mathbb{T}_i$ and $R := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. Define

$$R^\sigma := [\sigma(a_1), \sigma(b_1)] \times [\sigma(a_2), \sigma(b_2)] \times \cdots \times [\sigma(a_n), \sigma(b_n)].$$

We introduce the class of nonnegative bounded n variable weight functions

$$\mathscr{W} = \left\{\omega \in \mathrm{C}^2_{\mathrm{rd}}(R, [0,\infty)) : \ \omega(t) = \omega^{\Delta_i}(t) = 0,\ t \in \partial B,\ i \in \{1, \dots, n\}\right\}. \tag{4.1}$$

Let us introduce the limited forward jump operator

$$\theta^{\bar{\sigma}_i}(t) = \theta(\sigma_1(t_1), \dots, \sigma_{i-1}(t_{i-1}), t_i, \sigma_{i+1}(t_{i+1}), \dots, \sigma_n(t_n)). \tag{4.2}$$

If we apply σ_i to $\theta^{\bar{\sigma}_i}$, then we get θ^σ back, i.e.,

$$\theta^{\bar{\sigma}_i\sigma_i}(t) = \theta(\sigma_1(t_1), \dots, \sigma_{i-1}(t_{i-1}), \sigma_i(t_i), \sigma_{i+1}(t_{i+1}), \dots, \sigma_n(t_n)) = \theta^\sigma(t).$$

The following definition of the Laplace operator on time scales is new, and it will serve the purpose we are looking for in this study.

DEFINITION 3. We define the Laplace operator $\triangle$ on time scales as

$$\triangle\theta = \sum_{i=1}^n \frac{\partial^2 \theta^{\bar{\sigma}_i}}{\Delta_i t_i^2} = \sum_{i=1}^n \theta^{\bar{\sigma}_i\Delta_i\Delta_i}.$$

DEFINITION 4. The n-dimensional Laplace equation on time scales is defined as

$$\triangle\theta = 0, \quad \text{i.e.,} \quad \sum_{i=1}^{n} \theta^{\bar{\sigma}_i \Delta_i \Delta_i} = 0. \tag{4.3}$$

DEFINITION 5. Any solution of (4.3) is called *harmonic*. A function θ is called a *subsolution* of (4.3) or *subharmonic* if $\triangle\theta \geqslant 0$. A function θ is called a *supersolution* of (4.3) or *superharmonic* if $\triangle\theta \leqslant 0$.

The following theorem generalizes Theorem 1 to the multivariable case, and it is the main result of this paper.

THEOREM 5. (Reverse Poincaré-type inequality) *If θ is subharmonic, θ^σ is differentiable on R^σ, and $\omega \in \mathscr{W}$, then*

$$\int_R L_\theta(t)\omega(\sigma(t))\Delta t \leqslant \int_R (\theta^2 + K\theta)(\sigma(t)) \triangle \omega(t)\Delta t, \tag{4.4}$$

where

$$K = 2 \sup_{t \in R^\sigma} |\theta(t)| = 2\|\theta\|_\infty$$

and

$$L_\theta = \sum_{i=1}^{n} (\theta^{\bar{\sigma}_i \Delta_i})^2 + \sum_{i=1}^{n} \theta^{\sigma \Delta_i} \theta^{\bar{\sigma}_i \Delta_i \sigma_i}.$$

Proof. Define $\varphi_i := (\theta^{\bar{\sigma}_i})^2 + K\theta^{\bar{\sigma}_i}$ for $i \in \{1, 2, \ldots, n\}$. Then, by (3.3), we get

$$\begin{aligned}
\varphi_i^{\Delta_i \Delta_i} &= \theta^{\bar{\sigma}_i \Delta_i \Delta_i} \left(2\theta^{\bar{\sigma}_i \sigma_i} + K\right) + (\theta^{\bar{\sigma}_i \Delta_i})^2 + \theta^{\bar{\sigma}_i \sigma_i \Delta_i} \theta^{\bar{\sigma}_i \Delta_i \sigma_i} \\
&= \theta^{\bar{\sigma}_i \Delta_i \Delta_i} (2\theta^\sigma + K) + (\theta^{\bar{\sigma}_i \Delta_i})^2 + \theta^{\sigma \Delta_i} \theta^{\bar{\sigma}_i \Delta_i \sigma_i},
\end{aligned}$$

and by applying summation, we get

$$\sum_{i=1}^{n} \varphi_i^{\Delta_i \Delta_i} = (2\theta^\sigma + K) \triangle \theta + L_\theta \geqslant L_\theta. \tag{4.5}$$

Now fix again $i \in \{1, 2, \ldots, n\}$. We apply Lemma 1 with φ replaced by φ_i and ω replaced by $\omega^{\bar{\sigma}_i}$ to obtain for $t = (t_1, \ldots, t_n)$ with $t_j \in [a_j, b_j]$ for $j \in \{1, 2, \ldots, n\} \setminus \{i\}$

$$\begin{aligned}
0 &\overset{(\omega \in \mathscr{W})}{=} \int_{a_i}^{b_i} \left(\varphi_i^{\Delta_i} \omega^{\bar{\sigma}_i} - \varphi_i \omega^{\bar{\sigma}_i \Delta_i}\right)^{\Delta_i} (t)\Delta t_i \\
&= \int_{a_i}^{b_i} \left(\varphi_i^{\Delta_i \Delta_i} \omega^{\bar{\sigma}_i \sigma_i} - \varphi_i^{\sigma_i} \omega^{\bar{\sigma}_i \Delta_i \Delta_i}\right)(t)\Delta t_i,
\end{aligned}$$

so

$$0 = \int_R \left(\varphi_i^{\Delta_i \Delta_i} \omega^\sigma - \left((\theta^\sigma)^2 + K\theta^\sigma\right) \omega^{\bar{\sigma}_i \Delta_i \Delta_i}\right)(t)\Delta t.$$

Adding all those up, we get

$$\begin{aligned} 0 &= \int_R \left(\sum_{i=1}^n \varphi_i^{\Delta_i \Delta_i} \omega^\sigma - \left((\theta^\sigma)^2 + K\theta^\sigma \right) \sum_{i=1}^n \omega^{\bar{\sigma}_i \Delta_i \Delta_i} \right)(t)\Delta t \\ &\overset{(4.5)}{\geqslant} \int_R \left(L_\theta \omega^\sigma - \left((\theta^\sigma)^2 + K\theta^\sigma \right) \triangle \omega \right)(t)\Delta t, \end{aligned}$$

showing (4.4). □

THEOREM 6. (Reverse Poincaré-type inequality) *If θ is superharmonic, θ^σ is differentiable on R^σ, and $\omega \in \mathscr{W}$, then*

$$\int_R L_\theta(t)\omega(\sigma(t))\Delta t \leqslant \int_R (\theta^2 - K\theta)(\sigma(t)) \triangle \omega(t)\Delta t, \tag{4.6}$$

where K and L_θ are defined as in Theorem 5.

5. Examples

In this section, we state our results for the special examples $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = q^{\mathbb{N}_0}$, i.e., we give corresponding inequalities in the continuous, the discrete, and the quantum cases. While the results in the continuous case are well known (but indeed our proofs presented here also are often simpler than the ones available in the literature), the inequalities in the discrete case and in the quantum case are new.

5.1. Continuous case

EXAMPLE 1. If $\mathbb{T} = \mathbb{R}$ in (3.2), then

$$2\int_a^b (\theta')^2(t)\omega(t)\mathrm{d}t \leqslant \int_a^b \left(\theta^2 + K\theta \right)(t) w''(t)\mathrm{d}t,$$

where

$$K = 2 \sup_{a \leqslant t \leqslant b} |\theta(t)|.$$

This inequality represents the weighted square integral inequality for the first derivative of a real-valued function, given in [17, Theorem 2.1].

EXAMPLE 2. If $\mathbb{T} = \mathbb{R}$ in (3.7), then

$$2\int_a^b (\theta')^2(t)\omega(t)\mathrm{d}t \leqslant \int_a^b \left(\theta^2 + K(\theta_1 + \theta_2) \right)(t) w''(t)\mathrm{d}t,$$

where K is defined as in Example 1.

EXAMPLE 3. If $\mathbb{T}=\mathbb{R}$ in (4.4), then

$$\int_R L_\theta(t)\omega(t)\mathrm{d}t \leqslant \int_R \left(\theta^2+K\theta\right)(t) \triangle \omega(t)\mathrm{d}t,$$

where

$$K = 2\sup_{t\in R}|\theta(t)| = 2\|\theta\|_\infty \quad \text{and} \quad L_\theta = 2\sum_{i=1}^{n}\left(\frac{\partial\theta}{\partial t_i}\right)^2.$$

5.2. Discrete case

EXAMPLE 4. If $\mathbb{T}=\mathbb{Z}$ in (3.2), then

$$\sum_{t=a}^{b-1}\left((\Delta\theta(t))^2+(\Delta\theta(t+1))^2\right)\omega(t+1) \leqslant \sum_{t=a}^{b-1}\left(\theta^2+K\theta\right)(t+1)\Delta^2\omega(t),$$

where

$$K = 2\max_{a+1\leqslant t\leqslant b+1}|\theta(t)|.$$

EXAMPLE 5. If $\mathbb{T}=\mathbb{Z}$ in (3.7), then

$$\sum_{t=a}^{b-1}\left((\Delta\theta(t))^2+(\Delta\theta(t+1))^2\right)\omega(t+1) \leqslant \sum_{t=a}^{b-1}\left(\theta^2+K(\theta_1+\theta_2)\right)(t+1)\Delta^2\omega(t),$$

where K is defined as in Example 4.

EXAMPLE 6. If $\mathbb{T}=\mathbb{Z}$ in (4.4), then

$$\sum_{t_1=a_1}^{b_1-1}\cdots\sum_{t_n=a_n}^{b_n-1} L_\theta(t_1,\dots,t_n)\omega(t_1+1,\dots,t_n+1)$$
$$\leqslant \sum_{t_1=a_1}^{b_1-1}\cdots\sum_{t_n=a_n}^{b_n-1}\left(\theta^2+K\theta\right)(t_1+1,\dots,t_n+1)\triangle\omega(t_1,\dots,t_n),$$

where

$$K = 2\max_{\substack{a_i+1\leqslant t_i\leqslant b_i+1\\ 1\leqslant i\leqslant n}}|\theta(t_1,\dots,t_n)| = 2\|\theta\|_\infty,$$

$$L_\theta(t_1,\dots,t_n) = \sum_{i=1}^{n}\left(\theta(t_1+1,\dots,t_n+1)-\theta(t_1+1,\dots,t_i,\dots,t_n+1)\right)^2$$
$$+\sum_{i=1}^{n}\left(\theta(t_1+1,\dots,t_i+2,\dots,t_n+1)-\theta(t_1+1,\dots,t_n+1)\right)^2,$$

and

$$\triangle\omega(t_1,\dots,t_n) = \omega(t_1+1,\dots,t_i+2,\dots,t_n+1)$$
$$-2\omega(t_1+1,\dots,t_n+1)+\omega(t_1+1,\dots,t_i,\dots,t_n+1).$$

5.3. Quantum calculus case

EXAMPLE 7. If $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$ and $a=q^{\alpha}$, $b=q^{\beta}$ in (3.2), then

$$\sum_{k=\alpha}^{\beta-1} \frac{q\left(\theta(q^{k+1})-\theta(q^k)\right)^2+\left(\theta(q^{k+2})-\theta(q^{k+1})\right)^2}{q^{k+1}}\omega(q^{k+1})$$
$$\leqslant \sum_{k=\alpha}^{\beta-1}\left(\theta^2+K\theta\right)(q^{k+1})\frac{\omega(q^{k+2})-(q+1)\omega(q^{k+1})+q\omega(q^k)}{q^{k+1}},$$

where

$$K=\max_{\alpha+1\leqslant k\leqslant \beta+1}\left|\theta(q^k)\right|.$$

EXAMPLE 8. If $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$ and $a=q^{\alpha}$, $b=q^{\beta}$ in (3.7), then

$$\sum_{k=\alpha}^{\beta-1} \frac{q\left(\theta(q^{k+1})-\theta(q^k)\right)^2+\left(\theta(q^{k+2})-\theta(q^{k+1})\right)^2}{q^{k+1}}\omega(q^{k+1})$$
$$\leqslant \sum_{k=\alpha}^{\beta-1}\left(\theta^2+K(\theta_1+\theta_2)\right)(q^{k+1})\frac{\omega(q^{k+2})-(q+1)\omega(q^{k+1})+q\omega(q^k)}{q^{k+1}},$$

where K is defined as in Example 7.

EXAMPLE 9. If $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$ and $a_i=q^{\alpha_i}$, $b_i=q^{\beta_i}$, $1\leqslant i\leqslant n$, in (4.4), then

$$\sum_{k_1=\alpha_1}^{\beta_1-1}\cdots\sum_{k_n=\alpha_n}^{\beta_n-1} q^{\sum_{i=1}^n k_i} L_\theta(q^{k_1},\ldots,q^{k_n})\omega(q^{k_1+1},\ldots,q^{k_n+1})$$
$$\leqslant \sum_{k_1=\alpha_1}^{\beta_1-1}\cdots\sum_{k_n=\alpha_n}^{\beta_n-1} q^{\sum_{i=1}^n k_i}\left(\theta^2+K\theta\right)(q^{k_1+1},\ldots,q^{k_n+1})\,\triangle\,\omega(q^{k_1},\ldots,q^{k_n}),$$

where

$$K=2\max_{\substack{\alpha_i+1\leqslant k_i\leqslant \beta_i+1\\ 1\leqslant i\leqslant n}}\left|\theta(q^{k_1},\ldots,q^{k_n})\right|=2\|\theta\|_\infty,$$

$$L_\theta(q^{k_1},\ldots,q^{k_n})=\sum_{i=1}^n\left\{\frac{\left(\theta(q^{k_1+1},\ldots,q^{k_n+1})-\theta(q^{k_1+1},\ldots q^{k_i},\ldots,q^{k_n+1})\right)^2}{q^{2k_i}(q-1)^2}\right.$$
$$\left.+\frac{\left(\theta(q^{k_1+1},\ldots,q^{k_i+2},\ldots,q^{k_n+1})-\theta(q^{k_1+1},\ldots,q^{k_n+1})\right)^2}{q^{2k_i+1}(q-1)^2}\right\},$$

and

$$\triangle\,\omega(q^{k_1},\ldots,q^{k_n})=\frac{1}{q^{2k_i+1}(q-1)^2}\left\{\omega(q^{k_1+1},\ldots,q^{k_i+2},\ldots,q^{k_n+1})\right.$$
$$\left.-(1+q)\omega(q^{k_1+1},\ldots q^{k_n+1})+q\omega(q^{k_1+1},\ldots,q^{k_i},\ldots,q^{k_n+1})\right\}.$$

6. Conclusion

The time scales Laplacian was introduced in this paper. Subharmonic and superharmonic functions on time scales were defined. Weighted energy estimates have been developed for subharmonic and superharmonic functions on time scales. To accomplish this task, first we established estimates (3.2) and (3.5) for convex and concave functions, respectively, on an arbitrary time scale. Similar energy estimates have been developed in [4] for 4-convex functions on time scales having constant graininess functions. Moreover, we have discussed particular cases of the inequalities presented in Sections 3 and 4 by considering different time scales. When we take $\mathbb{T} = \mathbb{R}$, then estimate (3.2) in Theorem 1 reduces to [17, estimate (2.3)]. The continuous case of superharmonic function resembles the reverse Poincaré-type inequality presented in [19,20] for the difference of superharmonic functions. The other estimates for the continuous, discrete, quantum, and general time scales cases are new in the literature.

Competing interest. It is declared that no competing interest exists regarding this manuscript.

Authors contribution. All authors have contributed equally to this manuscript.

Acknowledgement. All authors have checked and approved the final version.

Funding Information. There is no funding source that supported this manuscript financially.

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(Received November 24, 2022)

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