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INEQUALITIES FOR WEIGHTED SPACES WITH VARIABLE EXPONENTS

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Abstract. In this article we obtain an "off-diagonal" version of the Fefferman-Stein vector-valued maximal inequality on weighted Lebesgue spaces with variable exponents. As an application of this result and the atomic decomposition developed in [16] we prove, for certain exponents $q(\cdot)$ in $\mathscr{P}^{\log}(\mathbb{R}^n)$ and certain weights ω, that the Riesz potential I_α, with $0 < \alpha < n$, can be extended to a bounded operator from $H^{p(\cdot)}_{\omega}(\mathbb{R}^n)$ into $L^{q(\cdot)}_{\omega}(\mathbb{R}^n)$, for $\frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n}$.

1. Introduction

Classical Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p < \infty$, were defined by Stein and Weiss [32]. Afterward, Fefferman and Stein [11] introduced real variable methods into this subject and characterized the Hardy spaces by means of maximal functions. One of the most important applications of Hardy spaces is that they are good substitutes of Lebesgue spaces when $p \leqslant 1$. For instance, when $p \leqslant 1$, it is well known that Hilbert transform is not bounded on $L^p(\mathbb{R})$; however, it is bounded on Hardy spaces $H^p(\mathbb{R})$.

The spaces $H^p(\mathbb{R}^n)$ can also be characterized by atomic decompositions [3, 21] and molecular decompositions [33]. These decompositions allow to study the behavior of certain operators on $H^p(\mathbb{R}^n)$ by focusing one's attention on individual atoms. In principle, the continuity of an operator T on $H^p(\mathbb{R}^n)$ can often be proved by estimating Ta when $a(\cdot)$ is an atom. In [4] was observed that, in general, the atoms are not mapped into atoms. However, for many convolution operators – like singular integrals or fractional type operators – $m = Ta$ behaves like an atom. These functions m were called *molecules*, their properties as well as the molecular characterization of $H^p(\mathbb{R}^n)$ were established in [33]. Then, in essence, the continuity of an operator from Hardy spaces into Hardy spaces reduces to showing that it maps atoms into molecules.

With respect to the behavior of Riesz potential I_α $(0 < \alpha < n)$ on classical Hardy spaces, Stein and Weiss [32] used the theory of harmonic functions of several variables to prove that the operator I_α is bounded from $H^1(\mathbb{R}^n)$ into $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$. Taibleson and Weiss [33], by means of the molecular decomposition of Hardy spaces, obtained the boundedness of the Riesz potential I_α from $H^p(\mathbb{R}^n)$ into $H^q(\mathbb{R}^n)$, for $0 < p \leqslant 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$; Krantz independently obtained the same result in [20] by doing use of the atomic decomposition and of the maximal function characterization of $H^p(\mathbb{R}^n)$.

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The weighted Lebesgue spaces $L^p_w(\mathbb{R}^n)$ are a generalization of the classical Lebesgue spaces $L^p(\mathbb{R}^n)$, replacing the Lebesgue measure dx by the measure $w(x)dx$, where w is a non-negative measurable function. Then one can define the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ by generalizing the definition of $H^p(\mathbb{R}^n)$ (see [34]). The atomic characterization of $H^p_w(\mathbb{R}^n)$ has been given in [12] and [34]. The molecular characterization of $H^p_w(\mathbb{R}^n)$ was developed independently by Li and Peng in [23] and by Lee and Lin in [22]. In both works the authors obtained the boundedness of the classical singular integrals on H^p_w for certain weights w. Ding, Lee and Lin in [10] studied homogeneous fractional operators $T_{\Omega,\alpha}$ (where $T_{\Omega,\alpha} = I_\alpha$ if $\Omega \equiv 1$) and gave the $H^p_{w^p}(\mathbb{R}^n) - L^q_{w^q}(\mathbb{R}^n)$ boundedness of $T_{\Omega,\alpha}$ by the weighted atomic decomposition obtained in [12, 22]. They also obtained, applying the weighted atom-molecule theory developed in [12, 22], the $H^p_{w^p}(\mathbb{R}^n) - H^q_{w^q}(\mathbb{R}^n)$ boundedness of $T_{\Omega,\alpha}$. The author in [29], using a weighted molecular decomposition different from those given in [23, 22], obtained the $H^p_{w^p}(\mathbb{R}^n) - H^q_{w^q}(\mathbb{R}^n)$ boundedness of the Riesz potential I_α.

The Lebesgue spaces with variable exponent on $\mathbb{R}^n$ are a generalization of the classical $L^p(\mathbb{R}^n)$ spaces, via replacing the constant exponent p with an exponent function $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, i.e.: $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions f such that

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx < +\infty.$$

These spaces were first studied by Orlicz [26] in 1931. Sixty years later, it appears the foundational paper of Kováčik and Rákosník [19] of this topic, and then systematically developed in [8, 6]. In this setting, the theory of Hardy type spaces received a considerable impetus.

The theory of variable exponent Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$ was developed independently by Nakai and Sawano in [25] and by Cruz-Uribe and Wang in [7]. Both theories prove equivalent definitions in terms of maximal operators using different approaches. In [25, 7], one of their main goals is the atomic decomposition of elements in $H^{p(\cdot)}$, as an application of the atomic decomposition they proved that singular integrals are bounded on $H^{p(\cdot)}$. Later in [30], the author jointly with Urciuolo proved the $H^{p(\cdot)} - L^{q(\cdot)}$ boundedness of certain generalized Riesz potentials and the $H^{p(\cdot)} - H^{q(\cdot)}$ boundedness of Riesz potential via the infinite atomic and molecular decomposition developed in [25]. In [28], the author gave another proof of the results obtained in [30], but by using the finite atomic decomposition developed in [7].

Others Hardy type spaces are the Orlicz-slice Hardy spaces (see [35]). Kwok-Pun Ho [18] gave, using the extrapolation theory for Hardy type spaces, mapping properties of fractional integral operators on Orlicz-slice Hardy spaces.

Recently, Kwok-Pun Ho [16] developed the weighted theory for variable Hardy spaces on $\mathbb{R}^n$. He established the atomic decompositions for the weighted Hardy spaces with variable exponents $H^{p(\cdot)}_\omega(\mathbb{R}^n)$ and also revealed some intrinsic structures of atomic decomposition for Hardy type spaces. His results generalize the infinite atomic decompositions obtained in [12, 1, 34, 25].

The main results of this work are contained in Theorems 3 and 4 below. The first result concerns to an "off-diagonal" version of the Fefferman-Stein vector-valued maximal inequality on weighted variable Lebesgue spaces (originally due to Harboure, Macías and Segovia [14]). Our second result states, for certain exponent functions $q(\cdot)$ and certain weights ω, the boundedness of Riesz potential I_α from $H^{p(\cdot)}_\omega$ into $L^{p(\cdot)}_\omega$ where $0 < \alpha < n$ and $\frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n}$. This result is achieved via the off-diagonal Fefferman-Stein inequality (Theorem 3), the atomic decomposition established in [16], and the following additional assumption: for every cube $Q \subset \mathbb{R}^n$

$$\|\chi_Q\|_{L^{q(\cdot)}_\omega} \approx |Q|^{-\alpha/n} \|\chi_Q\|_{L^{p(\cdot)}_\omega}. \tag{1}$$

If $\omega \equiv 1$ and $q(\cdot)$ is log-Hölder continuous (locally and at infinite), then (1) holds. This case was proved in [30]. In this article we will give non trivial examples of power weights satisfying (1). So, (1) is an admissible hypothesis.

This paper is organized as follows. Section 2 gives the definition of weighted variable Lebesgue spaces $L^{p(\cdot)}_\omega$ and weighted variable Hardy spaces $H^{p(\cdot)}_\omega$ and some of their preliminary results. Section 3 presents some basics properties of the set of weights $\mathscr{W}_{p(\cdot)}$ used to define $H^{p(\cdot)}_\omega$. The off-diagonal version of the Fefferman-Stein inequality on weighted variable Lebesgue spaces is established in Section 4. The $H^{p(\cdot)}_\omega - L^{q(\cdot)}_\omega$ boundedness of Riesz potential I_α is proved in Section 5. Finally, Section 6 gives non trivial examples of power weights satisfying (1).

NOTATION. The symbol $A \lesssim B$ stands for the inequality $A \leqslant cB$ for some positive constant c. The symbol $A \approx B$ stands for $B \lesssim A \lesssim B$. We denote by $Q(z,r)$ the cube centered at $z \in \mathbb{R}^n$ with side lenght r. Given a cube $Q = Q(z,r)$, we set $kQ = Q(z,kr)$ and $\ell(Q) = r$. For a measurable subset $E \subseteq \mathbb{R}^n$ we denote by $|E|$ and χ_E the Lebesgue measure of E and the characteristic function of E respectively. Given a real number $s \geqslant 0$, we write $\lfloor s \rfloor$ for the integer part of s. As usual we denote with $\mathscr{S}(\mathbb{R}^n)$ the space of smooth and rapidly decreasing functions and with $\mathscr{S}'(\mathbb{R}^n)$ the dual space. If β is the multiindex $\beta = (\beta_1, \ldots, \beta_n)$, then $|\beta| = \beta_1 + \ldots + \beta_n$.

2. Preliminaries

Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a measurable function. Given a measurable set E, let

$$p_-(E) = \operatorname*{ess\,inf}_{x\in E} p(x), \quad \text{and} \quad p_+(E) = \operatorname*{ess\,sup}_{x\in E} p(x).$$

When $E = \mathbb{R}^n$, we will simply write $p_- := p_-(\mathbb{R}^n)$ and $p_+ := p_+(\mathbb{R}^n)$.

Given a measurable function f on $\mathbb{R}^n$, define the modular ρ associated with $p(\cdot)$ by

$$\rho(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx.$$

We define the variable Lebesgue space $L^{p(\cdot)} = L^{p(\cdot)}(\mathbb{R}^n)$ to be the set of all measurable functions f such that, for some $\lambda > 0$, $\rho(f/\lambda) < \infty$. This becomes a quasi normed space when equipped with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : \rho(f/\lambda) \leqslant 1\}.$$

Given a weight ω, i.e.: a locally integrable function on $\mathbb{R}^n$ such that $0 < \omega(x) < \infty$ almost everywhere, we define the weighted variable Lebesgue space $L^{p(\cdot)}_{\omega}$ as the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\|f\omega\|_{L^{p(\cdot)}} < \infty$. If $f \in L^{p(\cdot)}_{\omega}$, we define its quasi-norm by

$$\|f\|_{L^{p(\cdot)}_{\omega}} := \|f\omega\|_{L^{p(\cdot)}}.$$

LEMMA 1. *Given a measurable function* $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ *with* $0 < p_- \leqslant p_+ < \infty$ *and a weight* ω*, then for every* $s > 0$*,*

$$\|f\|^s_{L^{p(\cdot)}_{\omega}} = \||f|^s\|_{L^{p(\cdot)/s}_{\omega^s}}.$$

Proof. The condition $p_+ < \infty$ implies that $|\{x : p(x) = \infty\}| = 0$. Then, for every $s > 0$,

$$\begin{aligned} \||f|^s\|_{L^{p(\cdot)/s}_{\omega^s}} &= \inf\left\{\lambda > 0 : \int \left(\frac{|f(x)|\,\omega(x)}{\lambda^{1/s}}\right)^{p(x)} dx \leqslant 1\right\} \\ &= \inf\left\{\mu^s > 0 : \int \left(\frac{|f(x)|\,\omega(x)}{\mu}\right)^{p(x)} dx \leqslant 1\right\} = \|f\|^s_{L^{p(\cdot)}_{\omega}}. \quad \square \end{aligned}$$

For a measurable function $p(\cdot) : \mathbb{R}^n \to [1,\infty)$, its conjugate function $p'(\cdot)$ is defined by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. We have the following generalization of Hölder's inequality and an equivalent expression for the $L^{p(\cdot)}_{\omega}$-norm.

LEMMA 2. (Hölder's inequality) *Let* $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ *be a measurable function and* ω *be a locally integrable function such that* $0 < \omega(x) < \infty$ *almost everywhere. Then, there exists a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant C\|f\|_{L^{p(\cdot)}_{\omega}} \|g\|_{L^{p'(\cdot)}_{\omega^{-1}}}.$$

Proof. The lemma follows from [8, Lemma 3.2.20]. $\square$

PROPOSITION 1. *Let* $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ *be a measurable function and* ω *be a locally integrable function such that* $0 < \omega(x) < \infty$ *almost everywhere. Then*

$$\|f\|_{L^{p(\cdot)}_{\omega}} \approx \sup\left\{\int_{\mathbb{R}^n} |f(x)g(x)|dx : \|g\|_{L^{p'(\cdot)}_{\omega^{-1}}} \leqslant 1\right\}.$$

Proof. The proposition follows from [8, Corollary 3.2.14]. $\square$

Now, we briefly present the basics of weighted variable Hardy spaces theory and recall the atomic decomposition developed by K.-P. Ho in [16].

We introduce the weights used in [16] to define weighted Hardy spaces with variable exponents.

DEFINITION 1. Let $p(\cdot):\mathbb{R}^n \to (0,\infty)$ be a measurable function with $0<p_- \leqslant p_+ <\infty$. We define $\mathscr{W}_{p(\cdot)}$ as the set of all weights ω such that

(i) there exists $0<p_*<\min\{1,p_-\}$ such that $\|\chi_Q\|_{L^{p(\cdot)/p_*}_{\omega^{p_*}}}<\infty$, and

$$\|\chi_Q\|_{L^{(p(\cdot)/p_*)'}_{\omega^{-p_*}}}<\infty, \text{ for all cube } Q;$$

(ii) there exist $\kappa>1$ and $s>1$ such that Hardy-Littlewood maximal

operator M is bounded on $L^{(sp(\cdot))'/\kappa}_{\omega^{-\kappa/s}}$.

REMARK 1. In [16, Definition 2.3], the author considers $p_*=\min\{1,p_-\}$. We observe that the whole theory is still valid if we take $0<p_*<\min\{1,p_-\}$ instead of $p_*=\min\{1,p_-\}$. This slight modification allows us to compare the sets $\mathscr{W}_{q(\cdot)}$ and $\mathscr{W}_{p(\cdot)}$ related by $\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}=\frac{\alpha}{n}$ with $0<\alpha<n$ (see Proposition 3 below).

For a measurable function $p(\cdot):\mathbb{R}^n\to(0,\infty)$ such that $0<p_-\leqslant p_+<\infty$ and $\omega\in\mathscr{W}_{p(\cdot)}$, in [16] the author give a variety of distinct approaches, based on differing definitions, all lead to the same notion of weighted variable Hardy space $H^{p(\cdot)}_{\omega}$.

We recall some terminologies and notations from the study of maximal functions. Given $N\in\mathbb{N}$, define

$$\mathscr{F}_N=\left\{\varphi\in\mathscr{S}(\mathbb{R}^n):\sum_{|\beta|\leqslant N}\sup_{x\in\mathbb{R}^n}(1+|x|)^N\left|\partial^\beta\varphi(x)\right|\leqslant 1\right\}.$$

For any $f\in\mathscr{S}'(\mathbb{R}^n)$, the grand maximal function of f is given by

$$\mathscr{M}f(x)=\sup_{t>0}\sup_{\varphi\in\mathscr{F}_N}|(\varphi_t*f)(x)|,$$

where $\varphi_t(x)=t^{-n}\varphi(t^{-1}x)$.

DEFINITION 2. Let $p(\cdot):\mathbb{R}^n\to(0,\infty)$, $0<p_-\leqslant p_+<\infty$, and $\omega\in\mathscr{W}_{p(\cdot)}$. The weighted variable Hardy space $H^{p(\cdot)}_{\omega}(\mathbb{R}^n)$ is the set of all $f\in\mathscr{S}'(\mathbb{R}^n)$ for which $\|\mathscr{M}f\|_{L^{p(\cdot)}_{\omega}}<\infty$. In this case we define $\|f\|_{H^{p(\cdot)}_{\omega}}:=\|\mathscr{M}f\|_{L^{p(\cdot)}_{\omega}}$.

DEFINITION 3. Let $p(\cdot):\mathbb{R}^n\to(0,\infty)$, $0<p_-\leqslant p_+<\infty$, $p_0>1$, and $\omega\in\mathscr{W}_{p(\cdot)}$. Fix an integer $d\geqslant 1$. A function $a(\cdot)$ on $\mathbb{R}^n$ is called a $\omega-(p(\cdot),p_0,d)$ atom if there exists a cube Q such that

$a_1)$ $\operatorname{supp}(a)\subset Q$,

$a_2)$ $\|a\|_{L^{p_0}}\leqslant\frac{|Q|^{\frac{1}{p_0}}}{\|\chi_Q\|_{L^{p(\cdot)}_{\omega}}}$,

$a_3)$ $\int x^\beta a(x)\,dx=0$ for all $|\beta|\leqslant d$.

Next, we introduce two indices, which are related to the intrinsic structure of the atomic decomposition of $H_{\omega}^{p(\cdot)}$. Given $\omega \in \mathscr{W}_{p(\cdot)}$, we write

$$s_{\omega,p(\cdot)} := \inf\left\{ s \geqslant 1 : M \text{ is bounded on } L_{\omega^{-1/s}}^{(sp(\cdot))'} \right\}$$

and

$$\mathbb{S}_{\omega,p(\cdot)} := \left\{ s \geqslant 1 : M \text{ is bounded on } L_{\omega^{-\kappa/s}}^{(sp(\cdot))'/\kappa} \text{ for some } \kappa > 1 \right\}.$$

Then, for every fixed $s \in \mathbb{S}_{\omega,p(\cdot)}$, we define

$$\kappa_{\omega,p(\cdot)}^{s} := \sup\left\{ \kappa > 1 : M \text{ is bounded on } L_{\omega^{-\kappa/s}}^{(sp(\cdot))'/\kappa} \right\}.$$

The index $\kappa_{\omega,p(\cdot)}^{s}$ is used to measure the left-openness of the boundedness of M on the family $\left\{ L_{\omega^{-\kappa/s}}^{(sp(\cdot))'/\kappa} \right\}_{\kappa>1}$. The index $s_{\omega,p(\cdot)}$ is related to the vanishing moment condition and the index $\kappa_{\omega,p(\cdot)}^{s}$ is related to the size condition of the atoms (see [16, Theorems 5.3 and 6.3]).

The following three results are crucial in obtaining the $H_{\omega}^{p(\cdot)} - L_{\omega}^{q(\cdot)}$ boundedness of the Riesz potential. The first is a supporting result [16, Lemma 5.4], the second is a combination of [16, Theorem 6.2] and [27, Theorem 3.1], the last one was proved in [17, Proposition 2.1].

PROPOSITION 2. ([16, Lemma 5.4]) *Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a measurable function with $0 < p_- \leqslant p_+ < \infty$ and $\omega \in \mathscr{W}_{p(\cdot)}$. Let $s \in \mathbb{S}_{\omega,p(\cdot)}$ and $\{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence of scalars. If $r > (\kappa_{\omega,p(\cdot)}^{s})'$, $\{b_j\}_{j\in\mathbb{N}}$ is a sequence in L^r such that, for every j, $\operatorname{supp}(b_j)$ is contained in a cube Q_j and*

$$\|b_j\|_{L^r} \leqslant A_j |Q_j|^{1/r},$$

where the A_j's are positive scalars for all $j \in \mathbb{N}$, then

$$\left\| \sum_{j\in\mathbb{N}} \lambda_j b_j \right\|_{L_{\omega^{1/s}}^{sp(\cdot)}} \leqslant C \left\| \sum_{j\in\mathbb{N}} A_j |\lambda_j| \chi_{Q_j} \right\|_{L_{\omega^{1/s}}^{sp(\cdot)}}$$

for some $C > 0$ independent of $\{\lambda_j\}_{j\in\mathbb{N}}$, $\{b_j\}_{j\in\mathbb{N}}$ and $\{A_j\}_{j\in\mathbb{N}}$.

THEOREM 1. *Let $1 < p_0 < \infty$, $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a measurable function with $0 < p_- \leqslant p_+ < \infty$ and $\omega \in \mathscr{W}_{p(\cdot)}$. Then, for every $s > 1$ fixed, and every $f \in H_{\omega}^{p(\cdot)}(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ there exist a sequence of scalars $\{\lambda_j\}$, a sequence of cubes $\{Q_j\}$ and $\omega - (p(\cdot), p_0, \lfloor ns_{\omega,p(\cdot)} - n \rfloor)$ atoms a_j supported on Q_j such that*

$$\left\| \sum_j \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L_{\omega}^{p(\cdot)}}} \right)^{\theta} \chi_{Q_j} \right\|_{L_{\omega^{\theta}}^{p(\cdot)/\theta}}^{1/\theta} \leqslant C \|f\|_{H_{\omega}^{p(\cdot)}}, \text{ for all } 0 < \theta < \infty, \tag{2}$$

and $f = \sum_j \lambda_j a_j$ converges in $L^s(\mathbb{R}^n)$.

Proof. The existence of a such atomic decomposition as well as the validity of inequality in (2) are guaranteed by [16, Theorem 6.2]. The construction of a such atomic decomposition is analogous to that given for classical Hardy spaces (see [31, Chapter III]). So, following the proof in [27, Theorem 3.1], we obtain the convergence of the atomic series to f in $L^s(\mathbb{R}^n)$. $\square$

We define the set $\mathscr{S}_0(\mathbb{R}^n)$ by

$$\mathscr{S}_0(\mathbb{R}^n) = \left\{\varphi \in \mathscr{S}(\mathbb{R}^n) : \int x^\beta \varphi(x)dx = 0, \text{ for all } \beta \in \mathbb{N}_0^n\right\}.$$

We say that an exponent function $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ such that $0 < p_- \leqslant p_+ < \infty$ belongs to $\mathscr{P}^{log}(\mathbb{R}^n)$, if there exist two positive constants C and C_∞ such that $p(\cdot)$ satisfies the local log-Hölder continuity condition, i.e.:

$$|p(x) - p(y)| \leqslant \frac{C}{-\log(|x-y|)}, \quad |x-y| \leqslant \frac{1}{2},$$

and is log-Hölder continuous at infinity, i.e.:

$$|p(x) - p_\infty| \leqslant \frac{C_\infty}{\log(e+|x|)}, \quad x \in \mathbb{R}^n,$$

for some $p_- \leqslant p_\infty \leqslant p_+$.

THEOREM 2. ([17, Proposition 2.1]) *Let* $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *with* $0 < p_- \leqslant p_+ < \infty$. *If* $\omega \in \mathscr{W}_{p(\cdot)}$, *then* $\mathscr{S}_0(\mathbb{R}^n) \subset H_\omega^{p(\cdot)}(\mathbb{R}^n)$ *densely.*

3. Auxiliary results

Next, we will prove two basics properties of the set $\mathscr{W}_{p(\cdot)}$, which will allow us to get the main result of Section 5.

PROPOSITION 3. *Let* $0 < \alpha < n$ *and let* $q(\cdot) : \mathbb{R}^n \to (0,\infty)$ *be a measurable function such that* $0 < q_- \leqslant q_+ < \infty$. *If* $\omega \in \mathscr{W}_{q(\cdot)}$ *and* $\frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n}$, *then* $\omega \in \mathscr{W}_{p(\cdot)}$. *Moreover,* $s_{\omega,p(\cdot)} \leqslant s_{\omega,q(\cdot)} + \frac{\alpha}{n}$.

Proof. By Definition 1, $\omega \in \mathscr{W}_{q(\cdot)}$ if and only if

(i) there exists $0 < q_* < \min\{1, q_-\}$ such that $\|\chi_Q\|_{L^{q(\cdot)/q_*}_{\omega^{q_*}}} < \infty$, and

$$\|\chi_Q\|_{L^{(q(\cdot)/q_*)'}_{\omega^{-q_*}}} < \infty, \text{ for all cube } Q;$$

(ii) there exist $\kappa > 1$ and $s > 1$ such that Hardy-Littlewood maximal operator M is bounded on $L^{(sq(\cdot))'/\kappa}_{\omega^{-\kappa/s}}$.

We define $\frac{1}{p_*} := \frac{1}{q_*} + \frac{\alpha}{n}$. Since $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{p_*} - \frac{1}{q_*}$, it follows that $0 < p_* < \min\{1, p_-\}$ and $\frac{p_*}{q_*}\left(\frac{p(\cdot)}{p_*}\right)' = \left(\frac{q(\cdot)}{q_*}\right)'$. So, from Lemma 1 and (i) above, we have

$$\|\chi_Q\|_{L^{\left(\frac{p(\cdot)}{p_*}\right)'}_{\omega^{-p_*}}} = \|\chi_Q\|^{\frac{p_*}{q_*}}_{L^{\frac{p_*}{q_*}\left(\frac{p(\cdot)}{p_*}\right)'}_{\omega^{-q_*}}} = \|\chi_Q\|^{\frac{p_*}{q_*}}_{L^{\left(\frac{q(\cdot)}{q_*}\right)'}_{\omega^{-q_*}}} < \infty.$$

Being $p(\cdot) \leqslant q(\cdot)$, by [6, Corollary 2.48], Lemma 1 and (i), we have

$$\begin{aligned}\|\chi_Q\|^{q_*/p_*}_{L^{p(\cdot)/p_*}_{\omega^{p_*}}} &= \|\chi_Q\, \omega^{p_*}\|^{q_*/p_*}_{L^{p(\cdot)/p_*}} \leqslant (1+|Q|)^{q_*/p_*}\|\chi_Q\, \omega^{p_*}\|^{q_*/p_*}_{L^{q(\cdot)/p_*}} \\ &= (1+|Q|)^{q_*/p_*}\|\chi_Q\|_{L^{q(\cdot)/q_*}_{\omega^{q_*}}} < \infty.\end{aligned}$$

From (ii) follows that the maximal operator M is bounded on $L^{(sq(\cdot))'}_{\omega^{-1/s}}$, with $s > 1$. We fix $r > s + \frac{\alpha}{n}$, and define $q_0 := \frac{r}{s}$ and $\frac{1}{p_0} := \frac{1}{q_0} + \frac{\alpha}{nr}$. It is clear that $\frac{r}{p_0} > 1$, $\frac{q_0}{p_0} > 1$ and $\frac{p_0}{q_0}\left(\frac{r}{p_0}p(\cdot)\right)' = \left(\frac{r}{q_0}q(\cdot)\right)' = (sq(\cdot))'$. Thus, for $\tilde{s} := \frac{r}{p_0}$ and $\tilde{\kappa} := \frac{q_0}{p_0}$, we obtain

$$\|Mf\|_{L^{(\tilde{s}p(\cdot))'/\tilde{\kappa}}_{\omega^{-\tilde{\kappa}/\tilde{s}}}} = \|Mf\|_{L^{(sq(\cdot))'}_{\omega^{-1/s}}} \leqslant C\|f\|_{L^{(sq(\cdot))'}_{\omega^{-1/s}}} = C\|f\|_{L^{(\tilde{s}p(\cdot))'/\tilde{\kappa}}_{\omega^{-\tilde{\kappa}/\tilde{s}}}}.$$

So, $\omega \in \mathscr{W}_{p(\cdot)}$ and $s_{\omega,p(\cdot)} \leqslant s_{\omega,q(\cdot)} + \frac{\alpha}{n}$. $\square$

The following necessary condition is due to Cruz-Uribe, Fiorenza and Neugebauer (see [5, Theorem 1.5]). It should be compared to the Muckenhoupt $\mathscr{A}_p$ condition from the study of weighted norm inequalities (see [13, Chapter 7]).

LEMMA 3. *Given a weight ω and an exponent function $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ such that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}_w(\mathbb{R}^n)$, then, there exists a positive constant C such that for every cube $Q \subset \mathbb{R}^n$,*

$$\|\chi_Q\|_{L^{p(\cdot)}_\omega}\|\chi_Q\|_{L^{p'(\cdot)}_{\omega^{-1}}} \leqslant C|Q|. \tag{3}$$

DEFINITION 4. Given an exponent function $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ and a weight ω, we write $\omega \in \mathscr{A}_{p(\cdot)}$ if ω satisfies (3).

LEMMA 4. *Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a measurable function with $0 < p_- \leqslant p_+ < \infty$. If $\omega \in \mathscr{W}_{p(\cdot)}$, then, for every cube $Q \subset \mathbb{R}^n$,*

$$\|\chi_{2Q}\|_{L^{p(\cdot)}_\omega} \approx \|\chi_Q\|_{L^{p(\cdot)}_\omega}.$$

Proof. By the order preserving property of the norm $\|\cdot\|_{L^{p(\cdot)}_\omega}$, we have that

$$\|\chi_Q\|_{L^{p(\cdot)}_\omega} \leqslant \|\chi_{2Q}\|_{L^{p(\cdot)}_\omega}. \tag{4}$$

On the other hand, since $\omega \in \mathscr{W}_{p(\cdot)}$, the maximal operator is bounded on $L^{(sp(\cdot))'}_{\omega^{-1/s}}$. Then, by Lemma 3 (exchanging the roles of $p(\cdot)$ and $p'(\cdot)$), (4) above, and Hölder's inequality applied to $|Q| = \int \chi_Q(x)dx$ (see Lemma 2), we have that

$$\|\chi_{2Q}\|^{1/s}_{L^{p(\cdot)}_{\omega}} = \|\chi_{2Q}\|_{L^{sp(\cdot)}_{\omega^{1/s}}} \leqslant C|Q|\|\chi_{2Q}\|^{-1}_{L^{(sp(\cdot))'}_{\omega^{-1/s}}} \leqslant C|Q|\|\chi_Q\|^{-1}_{L^{(sp(\cdot))'}_{\omega^{-1/s}}}$$
$$\leqslant C\|\chi_Q\|_{L^{sp(\cdot)}_{\omega^{1/s}}} = C\|\chi_Q\|^{1/s}_{L^{p(\cdot)}_{\omega}}.$$

This completes the proof. □

4. Off-diagonal Fefferman-Stein inequality

We apply extrapolation techniques to obtain an "off-diagonal" version of the Fefferman-Stein vector-valued maximal inequality on $L^{p(\cdot)}_{\omega}$. The following result generalizes Theorem 3.1 obtained in [16].

THEOREM 3. *Let $0 \leqslant \alpha < n$, $1 < u < \infty$ and let $q(\cdot): \mathbb{R}^n \to (0,\infty)$ be a measurable function with $0 < q_- \leqslant q_+ < \infty$. If $\omega \in \mathscr{W}_{q(\cdot)}$, then for $\frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n}$ and any $r > s_{\omega,q(\cdot)} + \frac{\alpha}{n}$,*

$$\left\| \left(\sum_{j\in\mathbb{N}} (M_{\frac{\alpha}{r}} f_j)^u \right)^{1/u} \right\|_{L^{rq(\cdot)}_{\omega^{1/r}}} \lesssim \left\| \left(\sum_{j\in\mathbb{N}} |f_j|^u \right)^{1/u} \right\|_{L^{rp(\cdot)}_{\omega^{1/r}}}, \tag{5}$$

holds for all sequences of bounded measurable functions with compact support $\{f_j\}_{j=1}^{\infty}$.

Proof. Given $r > s_{\omega,q(\cdot)} + \frac{\alpha}{n}$, from the definition of $s_{\omega,q(\cdot)}$, we have $s > s_{\omega,q(\cdot)}$ such that $s + \frac{\alpha}{n} < r$ and M is bounded on $L^{(sq(\cdot))'}_{w^{-1/s}}(\mathbb{R}^n)$. Define

$$\mathscr{F} = \left\{ \left(\left(\sum_{j=1}^{K} (M_{\frac{\alpha}{r}} f_j)^u \right)^{1/u}, \left(\sum_{j=1}^{K} |f_j|^u \right)^{1/u} \right) : K \in \mathbb{N}, \{f_j\}_{j=1}^{K} \subset L^{\infty}_{comp} \right\},$$

where L^{∞}_{comp} denotes the set of bounded functions with compact support.

Let $q_0 = \frac{r}{s}$ and let p_0 be defined by $\frac{1}{p_0} := \frac{1}{q_0} + \frac{\alpha}{nr}$. Since $r > s + \frac{\alpha}{n}$ we have that $1 < p_0 < \frac{nr}{\alpha}$. From [24, Theorem 3] follows that there exists an universal constant $C > 0$ such that for any $(F,G) \in \mathscr{F}$ and any $v \in \mathscr{A}_1$ (for the definition of the $\mathscr{A}_1$ class, the reader may refer to [13, Chapter 7])

$$\int [F(x)]^{q_0} v(x)\, dx \leqslant C \left(\int [G(x)]^{p_0} [v(x)]^{p_0/q_0}\, dx \right)^{q_0/p_0}, \tag{6}$$

since $v \in \mathscr{A}_1$ implies that $v^{1/q_0} \in \mathscr{A}_{p_0,q_0}$ (for the definition of the $\mathscr{A}_{p,q}$ class, see [24, inequality (1.1)]). On the other hand, by Proposition 1, we have

$$\|F\|^{q_0}_{L^{rq(\cdot)}_{\omega^{1/r}}} = \|F^{q_0}\|_{L^{sq(\cdot)}_{\omega^{1/s}}} \leqslant C \sup \left\{ \int_{\mathbb{R}^n} |[F(x)]^{q_0} g(x)|\, dx : \|g\|_{L^{(sq(\cdot))'}_{\omega^{-1/s}}} \leqslant 1 \right\} \tag{7}$$

for some constant $C > 0$.

Let $\mathscr{R}$ be the operator defined on $L^{(sq(\cdot))'}_{\omega^{-1/s}}$ by

$$\mathscr{R}g(x) = \sum_{k=0}^{\infty} \frac{M^k g(x)}{2^k \|M\|^k_{L^{(sq(\cdot))'}_{\omega^{-1/s}}}},$$

where, for $k \geqslant 1$, M^k denotes k iterations of the Hardy-Littlewood maximal operator M, $M^0 = M$, and $\|M\|_{L^{(sq(\cdot))'}_{\omega^{-1/s}}}$ is the operator norm of the maximal operator M on $L^{(sq(\cdot))'}_{\omega^{-1/s}}$. It follows immediately from this definition that:

(i) if g is non-negative, $g(x) \leqslant \mathscr{R}g(x)$ a.e. $x \in \mathbb{R}^n$;

(ii) $\|\mathscr{R}g\|_{L^{(sq(\cdot))'}_{\omega^{-1/s}}} \leqslant 2\|g\|_{L^{(sq(\cdot))'}_{\omega^{-1/s}}}$;

(iii) $\mathscr{R}g \in \mathscr{A}_1$ with $[\mathscr{R}g]_{\mathscr{A}_1} \leqslant 2\|M\|_{L^{(sq(\cdot))'}_{\omega^{-1/s}}}$.

Since F is non-negative, we can take the supremum in (7) over those non-negative g only. For any fixed non-negative $g \in L^{(sq(\cdot))'}_{\omega^{-1/s}}$, by (i) above we have that

$$\int [F(x)]^{q_0} g(x) dx \leqslant \int [F(x)]^{q_0} (\mathscr{R}g)(x) dx. \tag{8}$$

Then (iii) and (6), and Hölder's inequality yield

$$\begin{aligned} \int [F(x)]^{q_0} (\mathscr{R}g)(x) dx &\leqslant C \left(\int [G(x)]^{p_0} [(\mathscr{R}g)(x)]^{p_0/q_0} dx \right)^{q_0/p_0} \\ &\leqslant C \|G^{p_0}\|^{q_0/p_0}_{L^{rp(\cdot)/p_0}_{\omega^{p_0/r}}} \|(\mathscr{R}g)^{p_0/q_0}\|^{q_0/p_0}_{L^{(rp(\cdot)/p_0)'}_{\omega^{-p_0/r}}} \\ &= C \|G\|^{q_0}_{L^{rp(\cdot)}_{\omega^{1/r}}} \|\mathscr{R}g\|_{L^{\frac{p_0}{q_0}\left(\frac{rp(\cdot)}{p_0}\right)'}_{\omega^{-q_0/r}}} \end{aligned} \tag{9}$$

being $\frac{1}{rp(\cdot)} - \frac{1}{rq(\cdot)} = \frac{1}{p_0} - \frac{1}{q_0}$ and $q_0 = \frac{r}{s}$, we have $\frac{p_0}{q_0}\left(\frac{r}{p_0}p(\cdot)\right)' = \left(\frac{r}{q_0}q(\cdot)\right)' = (sq(\cdot))'$, so

$$= C \|G\|^{q_0}_{L^{rp(\cdot)}_{\omega^{1/r}}} \|\mathscr{R}g\|_{L^{(sq(\cdot))'}_{\omega^{-1/s}}}$$

now, (ii) gives

$$\leqslant C \|G\|^{q_0}_{L^{rp(\cdot)}_{\omega^{1/r}}} \|g\|_{L^{(sq(\cdot))'}_{\omega^{-1/s}}}.$$

Thus, (8) and (9) lead to

$$\int [F(x)]^{q_0} g(x)dx \leqslant C\|G\|^{q_0}_{L^{rp(\cdot)}_{\omega^{1/r}}}, \tag{10}$$

for all non-negative g with $\|g\|_{L^{(sq(\cdot))'}_{\omega^{-1/s}}} \leqslant 1$. Then, (7) and (10) give (5) for all finite sequences $\{f_j\}_{j=1}^K \subset L^\infty_{comp}$. Finally, by passing to the limit, we obtain (5) for all infinite sequences $\{f_j\}_{j=1}^\infty \subset L^\infty_{comp}$. □

COROLLARY 1. *Let $0<\alpha<n$, $q(\cdot):\mathbb{R}^n \to (0,\infty)$ be a measurable function with $0<q_- \leqslant q_+ < \infty$ and $\omega \in \mathscr{W}_{q(\cdot)}$. If $\frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n}$ and $\|\chi_Q\|_{L^{q(\cdot)}_\omega} \approx |Q|^{-\alpha/n}\|\chi_Q\|_{L^{p(\cdot)}_\omega}$ for every cube Q, then for any sequence of scalars $\{\lambda_j\}_{j\in\mathbb{N}}$, any family of cubes $\{Q_j\}_{j\in\mathbb{N}}$, and any $\theta\in(0,\infty)$ fixed we have*

$$\left\|\sum_{j\in\mathbb{N}}\left(\frac{|\lambda_j|\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{q(\cdot)}_\omega}}\right)^\theta\right\|^{1/\theta}_{L^{q(\cdot)/\theta}_{\omega^\theta}} \lesssim \left\|\sum_{j\in\mathbb{N}}\left(\frac{|\lambda_j|\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\right)^\theta\right\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}}.$$

Proof. Since $\|\chi_Q\|_{L^{q(\cdot)}_\omega} \approx |Q|^{-\alpha/n}\|\chi_Q\|_{L^{p(\cdot)}_\omega}$ for every cube Q we obtain

$$\begin{aligned}\left\|\sum_j\left(\frac{|\lambda_j|\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{q(\cdot)}_\omega}}\right)^\theta\right\|^{1/\theta}_{L^{q(\cdot)/\theta}_{\omega^\theta}} &= \left\|\left\{\sum_j\left(\frac{|\lambda_j|\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{q(\cdot)}_\omega}}\right)^\theta\right\}^{1/\theta}\right\|_{L^{q(\cdot)}_\omega} \\ &\lesssim \left\|\left\{\sum_j\left(\frac{|\lambda_j||Q_j|^{\alpha/n}\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\right)^\theta\right\}^{1/\theta}\right\|_{L^{q(\cdot)}_\omega},\end{aligned}$$

it is easy to check that $|Q_j|^{\alpha/n}\chi_{Q_j}(x) \leqslant M_{\frac{\alpha\theta}{N}}(\chi_{Q_j})^{\frac{N}{\theta}}(x)$ for all j and all $N\in\mathbb{N}$, so

$$\begin{aligned}\lesssim \left\|\left\{\sum_j\left(\frac{|\lambda_j|M_{\frac{\alpha\theta}{N}}(\chi_{Q_j})^{\frac{N}{\theta}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\right)^\theta\right\}^{1/\theta}\right\|_{L^{q(\cdot)}_\omega} &= \left\|\left\{\sum_j\left(\frac{|\lambda_j|^\theta M_{\frac{\alpha\theta}{N}}(\chi_{Q_j})^N}{\|\chi_{Q_j}\|^\theta_{L^{p(\cdot)}_\omega}}\right)\right\}^{1/\theta}\right\|_{L^{q(\cdot)}_\omega} \\ &= \left\|\left\{\sum_j\left(\frac{|\lambda_j|^\theta M_{\frac{\alpha\theta}{N}}(\chi_{Q_j})^N}{\|\chi_{Q_j}\|^\theta_{L^{p(\cdot)}_\omega}}\right)\right\}^{1/N}\right\|^{N/\theta}_{L^{Nq(\cdot)/\theta}_{\omega^{\theta/N}}},\end{aligned}$$

taking N such that $N/\theta > s_{\omega,q(\cdot)} + \frac{\alpha}{n}$, by Theorem 3, we get

$$\lesssim \left\|\left\{\sum_j\left(\frac{|\lambda_j|^\theta\chi_{Q_j}}{\|\chi_{Q_j}\|^\theta_{L^{p(\cdot)}_\omega}}\right)\right\}^{1/N}\right\|^{N/\theta}_{L^{Np(\cdot)/\theta}_{\omega^{\theta/N}}} = \left\|\sum_j\left(\frac{|\lambda_j|\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\right)^\theta\right\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}}.$$

This completes the proof. □

5. Weighted variable estimates for Riesz potential

Let $0 < \alpha < n$. The *Riesz potential* of order α is the fractional operator I_α defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n} dy, \; x \in \mathbb{R}^n, \tag{11}$$

$f \in \mathscr{S}(\mathbb{R}^n)$. A well known result of Sobolev gives the boundedness of I_α from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. In [2] Capone, Cruz Uribe and Fiorenza extend this result to the case of Lebesgue spaces with variable exponents $L^{p(\cdot)}(\mathbb{R}^n)$. The behavior of Riesz potential on variable Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$ was studied by Urciuolo and the author [30, 28].

In this section we will prove that the Riesz potential I_α is bounded from weighted variable Hardy spaces into weighted variable Lebesgue spaces. The main tools that we will use are Theorem 3, Corollary 1 and Theorem 1.

THEOREM 4. *Let $0 < \alpha < n$, $q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $0 < q_- \leqslant q_+ < \infty$, and $\omega \in \mathscr{W}_{q(\cdot)}$. If $\frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n}$ and $\|\chi_Q\|_{L^{q(\cdot)}_\omega} \approx |Q|^{-\alpha/n} \|\chi_Q\|_{L^{p(\cdot)}_\omega}$ for every cube Q, then the Riesz potential I_α given by* (11) *can be extended to a bounded operator $H^{p(\cdot)}_\omega(\mathbb{R}^n) \to L^{q(\cdot)}_\omega(\mathbb{R}^n)$.*

Proof. Let $\omega \in \mathscr{W}_{q(\cdot)}$, by Definition 1, there exists $0 < \theta < 1$ such that $\frac{1}{\theta} \in \mathbb{S}_{\omega,q(\cdot)}$. Now, we take $q_0 > \frac{n}{n-\alpha}$ such that $q_0 > \theta \left(\kappa^{1/\theta}_{\omega,q(\cdot)}\right)'$, and define $\frac{1}{p_0} := \frac{1}{q_0} + \frac{\alpha}{n}$. By Proposition 3, for $\frac{1}{p(\cdot)} = \frac{1}{q(\cdot)} + \frac{\alpha}{n}$, we have that $\mathscr{W}_{q(\cdot)} \subset \mathscr{W}_{p(\cdot)}$ and $s_{\omega,p(\cdot)} \leqslant s_{\omega,q(\cdot)} + \frac{\alpha}{n}$. So, given $f \in S_0(\mathbb{R}^n)$, from Theorem 1 (taking $s = p_0$) and since one can always choose atoms with additional vanishing moment, we have that there exist a sequence of real numbers $\{\lambda_j\}_{j=1}^\infty$, a sequence of cubes $Q_j = Q(z_j, r_j)$ centered at z_j with side length r_j and $\omega - (p(\cdot), p_0, \lfloor n s_{\omega,q(\cdot)} + \alpha - n \rfloor)$ atoms a_j supported on Q_j, satisfying

$$\left\| \sum_j \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}} \right)^\theta \chi_{Q_j} \right\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}} \lesssim \|f\|_{H^{p(\cdot)}_\omega}, \tag{12}$$

and $f = \sum_j \lambda_j a_j$ converges in $L^{p_0}(\mathbb{R}^n)$. By Sobolev's Theorem we have that I_α is bounded from $L^{p_0}(\mathbb{R}^n)$ into $L^{q_0}(\mathbb{R}^n)$, so

$$|I_\alpha f(x)| \leqslant \sum_j |\lambda_j| |I_\alpha a_j(x)|, \;\; a.e. \; x \in \mathbb{R}^n.$$

Then

$$\|I_\alpha f\|_{L^{q(\cdot)}_\omega} \lesssim \left\| \sum_j |\lambda_j| \chi_{2Q_j} \cdot I_\alpha a_j \right\|_{L^{q(\cdot)}_\omega} + \left\| \sum_j |\lambda_j| \chi_{\mathbb{R}^n \setminus 2Q_j} \cdot I_\alpha a_j \right\|_{L^{q(\cdot)}_\omega} = J_1 + J_2, \tag{13}$$

where $2Q_j = Q(z_j, 2r_j)$. To estimate J_1, we again apply Sobolev's Theorem and obtain

$$\left\|(I_\alpha a_j)^\theta\right\|_{L^{q_0/\theta}(2Q_j)} = \|I_\alpha a_j\|^\theta_{L^{q_0}(2Q_j)} \lesssim \|a_j\|^\theta_{L^{p_0}}$$
$$\lesssim \frac{|Q_j|^{\frac{\theta}{p_0}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)/\theta}_{\omega^\theta}}} \lesssim \frac{|2Q_j|^{\frac{\theta}{q_0}}}{\|\chi_{2Q_j}\|_{L^{q(\cdot)/\theta}_{\omega^\theta}}},$$

where θ, q_0 and p_0 are given above. The last inequality follows from the condition $\|\chi_Q\|_{L^{q(\cdot)}_\omega} \approx |Q|^{-\alpha/n}\|\chi_Q\|_{L^{p(\cdot)}_\omega}$ assumed for every cube Q and Lemma 4 applied to the exponent $q(\cdot)$. Since $\frac{1}{\theta} \in \mathbb{S}_{\omega,q(\cdot)}$ and $q_0 > \theta\left(\kappa^{1/\theta}_{\omega,q(\cdot)}\right)'$, we apply the θ-inequality and Proposition 2 with $b_j = \left(\chi_{2Q_j}\cdot I_\alpha(a_j)\right)^\theta$ and $A_j = \|\chi_{2Q_j}\|^{-1}_{L^{q(\cdot)/\theta}_{\omega^\theta}}$ to obtain

$$J_1 \lesssim \left\|\sum_j \left(|\lambda_j|\chi_{2Q_j}\cdot I_\alpha a_j\right)^\theta\right\|^{1/\theta}_{L^{q(\cdot)/\theta}_{\omega^\theta}} \lesssim \left\|\sum_j\left(\frac{|\lambda_j|}{\|\chi_{2Q_j}\|_{L^{q(\cdot)}_\omega}}\right)^\theta \chi_{2Q_j}\right\|^{1/\theta}_{L^{q(\cdot)/\theta}_{\omega^\theta}}.$$

Being $\chi_{2Q_j} \leqslant M(\chi_{Q_j})^2$, by Lemma 4 and Theorem 3, we have

$$J_1 \lesssim \left\|\left\{\sum_j\left(\frac{|\lambda_j|^{\theta/2}}{\|\chi_{Q_j}\|^{\theta/2}_{L^{q(\cdot)}_\omega}}M(\chi_{Q_j})\right)^2\right\}^{1/2}\right\|^{2/\theta}_{L^{2q(\cdot)/\theta}_{\omega^{\theta/2}}} \lesssim \left\|\sum_j\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{q(\cdot)}_\omega}}\right)^\theta \chi_{Q_j}\right\|^{1/\theta}_{L^{q(\cdot)/\theta}_{\omega^\theta}}.$$

Corollary 1 and (12) give

$$J_1 \lesssim \left\|\sum_j\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\right)^\theta \chi_{Q_j}\right\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}} \lesssim \|f\|_{H^{p(\cdot)}_\omega}. \tag{14}$$

Now, we estimate J_2. Let $d = \lfloor ns_{\omega,q(\cdot)} + \alpha - n\rfloor$, and let $a_j(\cdot)$ be a $\omega-(p(\cdot),p_0,d)$ atom supported on the cube $Q_j = Q(z_j, r_j)$. In view of the moment condition of $a_j(\cdot)$ we obtain

$$I_\alpha a_j(x) = \int_{Q_j}\left(|x-y|^{\alpha-n} - q_d(x,y)\right)a(y)dy, \quad \textit{for all } x \notin 2Q_j,$$

where q_d is the degree d Taylor polynomial of the function $y \to |x-y|^{\alpha-n}$ expanded around z_j. By the standard estimate of the remainder term of the Taylor expansion, there exists ξ between y and z_j such that

$$\left||x-y|^{\alpha-n} - q_d(x,y)\right| \leqslant C|y-z_j|^{d+1}|x-\xi|^{-n+\alpha-d-1},$$

for any $y \in Q_j$ and any $x \notin 2Q_j$, since $|x-\xi| \geqslant \dfrac{|x-z_j|}{1+\sqrt{n}}$, we get

$$\left||x-y|^{\alpha-n} - q_d(x,y)\right| \leqslant Cr^{d+1}|x-z_j|^{-n+\alpha-d-1},$$

this inequality and the condition $a_2)$ of the atom allow us to conclude that

$$\begin{aligned} |I_\alpha a_j(x)| &\lesssim \|a_j\|_1\, r^{d+1}|x-z_j|^{-n+\alpha-d-1} \\ &\lesssim |Q_j|^{1-\frac{1}{p_0}}\|a_j\|_{p_0} r^{d+1}|x-z_j|^{-n+\alpha-d-1} \\ &\lesssim \frac{r^{n+d+1}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}|x-z_j|^{-n+\alpha-d-1} \\ &\lesssim \frac{\left[M_{\frac{\alpha n}{n+d+1}}(\chi_{Q_j})(x)\right]^{\frac{n+d+1}{n}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}, \quad \text{for all } x \notin 2Q_j. \end{aligned} \tag{15}$$

We put $r = \frac{n+d+1}{n}$, thus (15) leads to

$$J_2 \lesssim \left\|\left\{\sum_j \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\left[M_{\frac{\alpha}{r}}(\chi_{Q_j})\right]^r\right\}^{1/r}\right\|^r_{L^{rq(\cdot)}_{\omega^{1/r}}}.$$

Since

$$r = \frac{n+\lfloor ns_{\omega,q(\cdot)}+\alpha-n\rfloor+1}{n} > s_{\omega,q(\cdot)} + \frac{\alpha}{n},$$

to apply Theorem 3, with $u=r$, we obtain

$$J_2 \lesssim \left\|\left\{\sum_j \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\chi_{Q_j}\right\}^{1/r}\right\|^r_{L^{rp(\cdot)}_{\omega^{1/r}}} = \left\|\sum_j \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\chi_{Q_j}\right\|_{L^{p(\cdot)}_\omega}.$$

Being $0<\theta<1$, the θ-inequality and (12) give

$$J_2 \lesssim \left\|\sum_j\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}_\omega}}\right)^\theta \chi_{Q_j}\right\|^{1/\theta}_{L^{p(\cdot)/\theta}_{\omega^\theta}} \lesssim \|f\|_{H^{p(\cdot)}_\omega}. \tag{16}$$

Hence, (13), (14) and (16) yield

$$\|I_\alpha f\|_{L^{q(\cdot)}_\omega} \lesssim \|f\|_{H^{p(\cdot)}_\omega}, \text{ for all } f \in \mathscr{S}_0(\mathbb{R}^n).$$

Finally, by taking into account that $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, $0 < p_- \leqslant p_+ < \infty$ and $\omega \in \mathscr{W}_{p(\cdot)}$, the theorem follows from Theorem 2. $\square$

6. Example: power weights

In this section we give examples of power weights w satisfying, for certain exponents $q(\cdot)$, that: for every cube $Q \subset \mathbb{R}^n$

$$\|\chi_Q\|_{L^{q(\cdot)}_\omega} \approx |Q|^{-\alpha/n} \|\chi_Q\|_{L^{p(\cdot)}_\omega},$$

where $0 < \alpha < n$ and $\frac{1}{p(\cdot)} = \frac{1}{q(\cdot)} + \frac{\alpha}{n}$. For them, we first introduce the reverse Hölder condition for a weight w.

A weight ω satisfies the *reverse Hölder inequality* with exponent $s > 1$, denoted by $\omega \in RH_s$, if there exists a constant $C > 0$ such that for every cube Q,

$$\left(\frac{1}{|Q|} \int_Q [\omega(x)]^s dx \right)^{\frac{1}{s}} \leqslant C \frac{1}{|Q|} \int_Q \omega(x) dx;$$

the best possible constant is denoted by $[\omega]_{RH_s}$. We observe that if $\omega \in RH_s$, then by Hölder's inequality, $\omega \in RH_t$ for all $1 < t < s$, and $[\omega]_{RH_t} \leqslant [\omega]_{RH_s}$.

LEMMA 5. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $0 < p_- \leqslant p_+ < \infty$, $\gamma \in \mathbb{R}$, and let $f \in L^1_{loc}(\mathbb{R}^n)$ be a function such that $|f(x)| \lesssim (1+|x|)^\gamma$. Then $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \|f\|_{L^{p_\infty}(\mathbb{R}^n)}$.*

Proof. We take $s > 1/p_-$. Then, by applying [9, Lemma 2.7] with $\omega \equiv 1$, we get $\|f\|^{1/s}_{L^{p(\cdot)}} = \||f|^{1/s}\|_{L^{sp(\cdot)}} \approx \||f|^{1/s}\|_{L^{sp_\infty}} = \|f\|^{1/s}_{L^{p_\infty}}$. □

The following results talk about the size of the cubes in the $L^{p(\cdot)}_\omega$-norm. We recall that $\ell(Q)$ denotes the side length of the cube Q.

LEMMA 6. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < \infty$ and $p_\infty = p_-$, and let ω be a weight such that $\omega(x) \lesssim (1+|x|)^\gamma$ for some $\gamma \in \mathbb{R}$. If $\omega \in RH_{p_+}$, then for every cube Q we have*

$$\|\chi_Q\|_{L^{p(\cdot)}_\omega} \approx \begin{cases} [\omega^{p_-}(Q)]^{1/p_-}, & \text{if } \ell(Q) > 1 \\ \left[\omega^{p_-(Q)}(Q)\right]^{1/p_-(Q)}, & \text{if } \ell(Q) \leqslant 1 \end{cases}. \tag{17}$$

Proof. Applying Lemma 5, with $f = \chi_Q \omega$, we obtain

$$\|\chi_Q\|_{L^{p(\cdot)}_\omega} = \|\chi_Q \omega\|_{L^{p(\cdot)}} \approx [\omega^{p_-}(Q)]^{1/p_-}, \tag{18}$$

for all cube Q.

From (18) and since $1 < p_- \leqslant p_-(Q)$, by Hölder's inequality, we have

$$\|\chi_Q\|^{p_-}_{L^{p(\cdot)}_\omega} \approx \int_Q [\omega(x)]^{p_-} dx \leqslant \left[\omega^{p_-(Q)}(Q)\right]^{\frac{p_-}{p_-(Q)}} \|\chi_Q\|_{\left(\frac{p_-(Q)}{p_-}\right)'} \leqslant \left[\omega^{p_-(Q)}(Q)\right]^{\frac{p_-}{p_-(Q)}},$$

if $\ell(Q) \leqslant 1$. So,

$$\|\chi_Q\|_{L^{p(\cdot)}_\omega} \lesssim \left[\omega^{p_-(Q)}(Q)\right]^{\frac{1}{p_-(Q)}}, \tag{19}$$

holds for every cube Q with $\ell(Q) \leqslant 1$.

On the other hand, $\omega \in RH_{p_+}$ and $p_-(Q) \leqslant p_+$, then $\omega \in RH_{p_-(Q)}$ and

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q [\omega(x)]^{p_-(Q)}dx\right)^{1/p_-(Q)} &\leqslant C|Q|^{-1}\int_Q \omega(x)dx \\ &\leqslant C|Q|^{-1}\|\chi_Q\,\omega\|_{L^{p(\cdot)}}\|\chi_Q\|_{L^{p'(\cdot)}},\end{aligned}$$

[25, Lemma 2.2] gives $\|\chi_Q\|_{L^{p'(\cdot)}} \approx |Q|^{1/p'_+(Q)}$ if $\ell(Q) \leqslant 1$, being $p'_+(Q) = (p_-(Q))'$, we obtain

$$\left[\omega^{p_-(Q)}(Q)\right]^{1/p_-(Q)} \lesssim \|\chi_Q\|_{L^{p(\cdot)}_\omega}, \quad \text{if } \ \ell(Q) \leqslant 1. \tag{20}$$

Finally, (18), (19) and (20) give (17). □

COROLLARY 2. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $0 < p_- \leqslant p_+ < \infty$ and $p_\infty = p_-$, and let $s > 1/p_-$. If $\omega(x) \lesssim (1+|x|)^\gamma$ for some $\gamma \in \mathbb{R}$ and $\omega^{1/s} \in RH_{sp_+}$, then (17) holds.*

Proof. Since $(sp(\cdot))_+ = sp_+$, $(sp(\cdot))_- = sp_-$ and $(sp(\cdot))_-(Q) = sp_-(Q)$, Lemma 6, applied to $sp(\cdot)$ and $\omega^{1/s}$, gives

$$\|\chi_Q\|^{1/s}_{L^{p(\cdot)}_\omega} = \|\chi_Q\|_{L^{sp(\cdot)}_{\omega^{1/s}}} \approx \begin{cases} [\omega^{p_-}(Q)]^{1/sp_-}, & \text{if } \ \ell(Q) > 1 \\ \left[\omega^{p_-(Q)}(Q)\right]^{1/sp_-(Q)}, & \text{if } \ \ell(Q) \leqslant 1 \end{cases}.$$

So, the lemma follows. □

LEMMA 7. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < \infty$ and $p_\infty = p_+$. If $\omega \in RH_{p_+}$, then for every cube Q we have*

$$\|\chi_Q\|_{L^{p(\cdot)}_\omega} \approx \begin{cases} [\omega^{p_+}(Q)]^{1/p_+}, & \text{if } \ \ell(Q) > 1 \\ \left[\omega^{p_+(Q)}(Q)\right]^{1/p_+(Q)}, & \text{if } \ \ell(Q) \leqslant 1 \end{cases}. \tag{21}$$

Proof. By hypothesis $p_\infty = p_+$, thus there exists $C_\infty > 0$ such that

$$0 \leqslant p_+ - p(x) \leqslant \frac{C_\infty}{\log(e+|x|)}, \quad \text{for all } x \in \mathbb{R}^n.$$

Let $\lambda = [\omega^{p_+}(Q)]^{1/p_+}$, by applying [5, Lemma 2.8] with $t = 2/p_-$, we have

$$\begin{aligned}\int_Q \left(\frac{\omega(x)}{\lambda}\right)^{p(x)} dx &\leqslant C_{2/p_-}\int_Q \left(\frac{\omega(x)}{\lambda}\right)^{p_+} dx + \int_Q \frac{dx}{(e+|x|)^{2n}} \\ &\leqslant C_{2/p_-} + \int_{\mathbb{R}^n} \frac{dx}{(e+|x|)^{2n}} =: M < \infty.\end{aligned}$$

So, from the definition of the "norm" $\|\cdot\|_{L^{p(\cdot)}_\omega}$ we have

$$\|\chi_Q\|_{L^{p(\cdot)}_\omega} \leqslant \max\{M^{1/p_-}, M^{1/p_+}\}[\omega^{p_+}(Q)]^{1/p_+}, \text{ for all cube } Q. \tag{22}$$

Now, if $\ell(Q) > 1$, the condition $\omega \in RH_{p_+}$, Hölder's inequality and [25, Lemma 2.2] give

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q [\omega(x)]^{p_+}dx\right)^{1/p_+} &\leqslant \frac{C}{|Q|}\int_Q \omega(x)dx \\ &\leqslant \frac{C}{|Q|}\|\chi_Q\,\omega\|_{L^{p(\cdot)}}\|\chi_Q\|_{L^{p'(\cdot)}} \\ &\leqslant \frac{C}{|Q|}\|\chi_Q\,\omega\|_{L^{p(\cdot)}}|Q|^{1-\frac{1}{p_+}}.\end{aligned}$$

Consequently,

$$[\omega^{p_+}(Q)]^{1/p_+} \leqslant C\|\chi_Q\|_{L^{p(\cdot)}_\omega}, \text{ if } \ell(Q) > 1. \tag{23}$$

Then, (22) and (23) give

$$\|\chi_Q\|_{L^{p(\cdot)}_\omega} \approx [\omega^{p_+}(Q)]^{1/p_+}, \text{ if } \ell(Q) > 1. \tag{24}$$

Now, we study the case $\ell(Q) \leqslant 1$. It is easy to check that

$$0 \leqslant p_+(Q) - p(x) \leqslant \frac{C}{\log(e+|x|)}, \text{ for all } x \in Q.$$

Since $1 < p_+(Q) \leqslant p_+$, we have that $\omega \in RH_{p_+(Q)}$. Then, by reasoning as above, we get

$$\|\chi_Q\|_{L^{p(\cdot)}_\omega} \approx \left[\omega^{p_+(Q)}(Q)\right]^{1/p_+(Q)}, \text{ if } \ell(Q) \leqslant 1. \tag{25}$$

Finally, (24) and (25) give (21). □

COROLLARY 3. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $0 < p_- \leqslant p_+ < \infty$ and $p_\infty = p_+$, and let $s > 1/p_-$. If $\omega^{1/s} \in RH_{sp_+}$, then (21) holds.*

Proof. The proof follows from Lemmas 1 and 7. □

Now, we are in a position to present our example.

EXAMPLE 1. For $\gamma > -n$, let $w_\gamma(x) = |x|^\gamma$ with $x \in \mathbb{R}^n \setminus \{0\}$. From the estimates obtained in [13, Example 7.1.6, pp. 505], we observe that

$$\omega_\gamma \in \bigcap_{t>1} RH_t, \text{ if } \gamma \geqslant 0 \tag{26}$$

and

$$\omega_\gamma \in \bigcap_{1<t<n/|\gamma|} RH_t, \text{ if } -n < \gamma < 0. \tag{27}$$

Moreover, for $0 \leqslant \alpha < n$, $0 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and each cube Q we have that

$$[\omega_\gamma^p(Q)]^{-1/p}[\omega_\gamma^q(Q)]^{1/q} = [\omega_{\gamma p}(Q)]^{-1/p}[\omega_{\gamma q}(Q)]^{1/q} \approx |Q|^{-\alpha/n}, \quad \text{for every } \gamma \geqslant 0 \tag{28}$$

and

$$[\omega_\gamma^p(Q)]^{-1/p}[\omega_\gamma^q(Q)]^{1/q} \approx |Q|^{-\alpha/n}, \text{ for every } -n < \gamma < 0 \text{ with } q < \frac{n}{|\gamma|}.$$

Let $q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $0 < q_\infty = q_- \leqslant q_+ < \infty$, and let $p(\cdot)$ be defined by $\frac{1}{p(\cdot)} := \frac{1}{q(\cdot)} + \frac{\alpha}{n}$. If $\gamma \geqslant 0$, then, by (28) and Corollary 2 (first applied to $q(\cdot)$ and then to $p(\cdot)$), we get

$$\|\chi_Q\|_{L^{q(\cdot)}_{\omega_\gamma}} \approx |Q|^{-\alpha/n}\|\chi_Q\|_{L^{p(\cdot)}_{\omega_\gamma}}.$$

It is clear that $q'(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, $0 < q'_- \leqslant q'_+ = q'_\infty < \infty$, $q'_+ = (q_-)'$ and $q'_+(Q) = (q_-(Q))'$. Then for every $0 \leqslant \gamma < \frac{n}{q'_+}$, by (26) and Corollary 2; and (27) and Corollary 3 (applied to ω_γ and $q(\cdot)$; and $\omega_{-\gamma}$ and $q'(\cdot)$ respectively), we obtain that

$$\|\chi_Q\|_{L^{q(\cdot)}_{\omega_\gamma}}\|\chi_Q\|_{L^{(q(\cdot))'}_{\omega_{-\gamma}}} \approx \left\{ \begin{array}{ll} \left[\omega_\gamma^{q_-}(Q)\right]^{1/q_-}[\omega_{-\gamma}^{q'_+}(Q)]^{1/q'_+}, & \text{if } \ell(Q) > 1 \\ \left[\omega_\gamma^{q_-(Q)}(Q)\right]^{1/q_-(Q)}\left[\omega_{-\gamma}^{q'_+(Q)}(Q)\right]^{1/q'_+(Q)}, & \text{if } \ell(Q) \leqslant 1 \end{array} \right\}$$
$$\approx |Q|,$$

for all cube $Q \subset \mathbb{R}^n$, where the last estimate follows from [13, Example 7.1.6, pp. 505]. In particular, for $s > 1/q_-$ and $0 \leqslant \gamma < \frac{sn}{(sq(\cdot))'_+}$, we have

$$\|\chi_Q\|_{L^{sq(\cdot)}_{\omega_{\gamma/s}}}\|\chi_Q\|_{L^{(sq(\cdot))'}_{\omega_{-\gamma/s}}} \approx |Q|.$$

Since $\omega_{\pm\gamma}^{1/s} = \omega_{\pm\gamma/s}$, by [5, Theorem 1.5] we have that the maximal operator M is bounded on $L^{(sq(\cdot))'}_{\omega_\gamma^{-1/s}}$. By the left-openness property of the set $\mathscr{A}_{(sq(\cdot))'}$ (see definition 4 and [15, Theorem 3.3]), it follows that there exists a constant $\kappa > 1$ such that the maximal operator M is bounded on $L^{(sq(\cdot))'/\kappa}_{\omega_\gamma^{-\kappa/s}}$. So, $\omega_\gamma \in \mathscr{W}_{q(\cdot)}$ and $\|\chi_Q\|_{L^{q(\cdot)}_{\omega_\gamma}} \approx |Q|^{-\alpha/n}\|\chi_Q\|_{L^{p(\cdot)}_{\omega_\gamma}}$ for every cube $Q \subset \mathbb{R}^n$.

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