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GENERALIZED STEVIĆ–SHARMA TYPE OPERATORS FROM H^∞ SPACE INTO BLOCH–TYPE SPACES

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Abstract. The boundedness and compactness of a Stević-Sharma type operator from H^∞ space into Bloch-type spaces are characterized.

1. Introduction

Denote by $\mathbb{N}$ the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$, $S(\mathbb{D})$ the family of all analytic self-maps of $\mathbb{D}$, and $H^\infty = H^\infty(\mathbb{D})$ the space of bounded analytic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

Suppose that μ is a radial weight, that is, a strictly positive continuous function on $\mathbb{D}$ which is radial (i.e., $\mu(z) = \mu(|z|)$ for each $z \in \mathbb{D}$). The Bloch-type space $\mathscr{B}_\mu$ consists of all $f \in H(\mathbb{D})$ such that $\sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty$. $\mathscr{B}_\mu$ becomes a Banach space under the norm

$$\|f\|_{\mathscr{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f'(z)|.$$

When $\mu(z) = 1-|z|^2$, the induced space $\mathscr{B}_\mu$ is the classical Bloch space. The little Bloch-type space $\mathscr{B}_{\mu,0}$ consists of those functions f in $\mathscr{B}_\mu$ satisfying the condition

$$\lim_{|z|\to 1} \mu(z)|f'(z)| = 0.$$

It is known that $\mathscr{B}_{\mu,0}$ is a closed subspace of $\mathscr{B}_\mu$. For some investigations on Bloch-type spaces and operators on them see for instance [3, 6, 8, 10, 11, 13, 14, 15, 16, 17, 19, 20, 21, 27, 28, 32, 34].

Let $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the multiplication and composition operators are defined respectively by $M_\psi f = \psi \cdot f$ and $C_\varphi f = f \circ \varphi$, where $f \in H(\mathbb{D})$. The product of them is known as the weighted composition operator $W_{\psi,\varphi} f = \psi \cdot f \circ \varphi$, which has

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been extensively studied. The differentiation operator D, which is defined by $Df(z) = f'(z)$ for $f \in H(\mathbb{D})$, plays an important role in operator theory and many other different areas of mathematics.

The first papers on product-type operators including the differentiation operator dealt with the operators DC_φ and $C_\varphi D$, (see, for example, [9, 12, 13, 17, 18]). In [25, 26], Stević et al. introduced the following so-called Stević-Sharma operator

$$T_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}),$$

where $u, v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. By taking some specific choices of the involving symbols, we can obtain the general product-type operators:

$$M_uC_\varphi = T_{u,0,\varphi}, \quad C_\varphi M_u = T_{u\circ\varphi,0,\varphi}, \quad M_uD = T_{0,u,id},$$

$$DM_u = T_{u',u,id}, \quad C_\varphi D = T_{0,1,\varphi}, \quad DC_\varphi = T_{0,\varphi',\varphi},$$

$$M_uC_\varphi D = T_{0,u,\varphi}, \quad M_uDC_\varphi = T_{0,u\varphi',\varphi}, \quad C_\varphi M_uD = T_{0,u\circ\varphi,\varphi},$$

$$DM_uC_\varphi = T_{u',u\varphi',\varphi}, \quad C_\varphi DM_u = T_{u'\circ\varphi,u\circ\varphi,\varphi}, \quad DC_\varphi M_u = T_{\varphi'(u'\circ\varphi),\varphi'(u\circ\varphi),\varphi}.$$

For this reason, Stević-Sharma operator is particularly important and has aroused great interest of experts (see, e.g., [4, 15, 28, 29, 34, 30] and the references therein).

In [27], Stević et al. introduced the following product-type operator

$$T^n_{u,v,\varphi}f(z) = u(z)f^{(n)}(\varphi(z)) + v(z)f^{(n+1)}(\varphi(z)), \quad n \in \mathbb{N}_0, \tag{1}$$

and investigated its boundedness and compactness from a general space to Bloch-type space. Subsequently, Abbasi in [1] studied the boundedness, compactness and essential norm of $T^n_{u,v,\varphi}$ from Hardy space to nth weighted-type space. Abbasi and Zhu et al. in [3, 33] characterized the boundedness, compactness and essential norm of $T^n_{u,v,\varphi}$ from or to Zygmund-type space. The second author et al. investigated the boundedness and compactness of $T^n_{u,v,\varphi}$ from Hardy space [6] and $Q_k(p,q)$ space [8] to Zygmund-type space or Bloch-type space. Since the publication of [27] have also appeared several extensions of operator (1) on the unit disc, as well as on the unit ball (see, for example, [2, 7, 22, 23, 24]).

In [7] we considered the following Stević-Sharma type operator

$$T^{m,n}_{u,v,\varphi}f(z) = u(z)f^{(m)}(\varphi(z)) + v(z)f^{(n)}(\varphi(z)), \quad m \in \mathbb{N}_0, \quad n \in \mathbb{N}. \tag{2}$$

Without loss of generality, we may let $m < n$. Note that when $m = 0$ and $n = 1$, we get the classical Stević-Sharma operator. Here we investigate the boundedness and compactness of the operators $T^{m,n}_{u,v,\varphi}$ from H^∞ space into Bloch-type spaces $\mathscr{B}_\mu$.

Throughout this paper, for nonnegative quantities A and B, we use the abbreviation $A \lesssim B$ or $B \gtrsim A$ if there exists a positive constant C independent of A and B such that $A \leqslant CB$.

2. Preliminaries

In this section, we formulate some lemmas which will be used in the proofs of the main results. The first one is well-known (see [31]).

LEMMA 1. *Let* $f \in H^\infty$ *and* $k \in \mathbb{N}$. *Then*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^k|f^{(k)}(z)| \lesssim \|f\|_\infty.$$

For any $w \in \mathbb{D}$ and $j \in \mathbb{N}$, set

$$f_{j,w}(z) = \frac{(1-|w|^2)^j}{(1-\overline{w}z)^j}, \quad z \in \mathbb{D}. \tag{3}$$

It is evident that $f_{j,w} \in H^\infty$ and $\sup_{w\in\mathbb{D}} \|f_{j,w}\|_\infty \lesssim 1$ for every $j \in \mathbb{N}$. Moreover, we easily see that $f_{j,w}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$.

LEMMA 2. *Let* $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ *and* $m+1 < n$. *For any* $0 \neq w \in \mathbb{D}$ *and* $i,k \in \{m, m+1, n, n+1\}$, *there exists a function* $g_{i,w} \in H^\infty$ *such that*

$$g_{i,w}^{(k)}(w) = \frac{\overline{w}^k \delta_{ik}}{(1-|w|^2)^k},$$

where δ_{ik} *is the Kronecker delta symbol.*

Proof. For any $w \in \mathbb{D}\setminus\{0\}$ and constants c_1, c_2, c_3, c_4, let

$$g_w(z) = \sum_{j=1}^{4} c_j f_{j,w}(z),$$

where $f_{j,w}$ is defined in (3). For each $i \in \{m, m+1, n, n+1\}$, the system of linear equations

$$\begin{cases} g_w^{(m)}(w) = \left(m!c_1 + (m+1)!c_2 + \frac{(m+2)!}{2}c_3 + \frac{(m+3)!}{6}c_4\right)\frac{\overline{w}^m}{(1-|w|^2)^m} = \frac{\overline{w}^m\delta_{im}}{(1-|w|^2)^m} \\ g_w^{(m+1)}(w) = \left((m+1)!c_1 + (m+2)!c_2 + \frac{(m+3)!}{2}c_3 + \frac{(m+4)!}{6}c_4\right)\frac{\overline{w}^{m+1}}{(1-|w|^2)^{m+1}} = \frac{\overline{w}^{m+1}\delta_{i(m+1)}}{(1-|w|^2)^{m+1}} \\ g_w^{(n)}(w) = \left(n!c_1 + (n+1)!c_2 + \frac{(n+2)!}{2}c_3 + \frac{(n+3)!}{6}c_4\right)\frac{\overline{w}^n}{(1-|w|^2)^n} = \frac{\overline{w}^n\delta_{in}}{(1-|w|^2)^n} \\ g_w^{(n+1)}(w) = \left((n+1)!c_1 + (n+2)!c_2 + \frac{(n+3)!}{2}c_3 + \frac{(n+4)!}{6}c_4\right)\frac{\overline{w}^{n+1}}{(1-|w|^2)^{n+1}} = \frac{\overline{w}^{n+1}\delta_{i(n+1)}}{(1-|w|^2)^{n+1}} \end{cases}$$

has a unique solution $c_{i,j}, j \in \{1,2,3,4\}$ that is independent from w, since the determinant of coefficient matrix

$$\begin{vmatrix} m! & (m+1)! & \frac{(m+2)!}{2} & \frac{(m+3)!}{6} \\ (m+1)! & (m+2)! & \frac{(m+3)!}{2} & \frac{(m+4)!}{6} \\ n! & (n+1)! & \frac{(n+2)!}{2} & \frac{(n+3)!}{6} \\ (n+1)! & (n+2)! & \frac{(n+3)!}{2} & \frac{(n+4)!}{6} \end{vmatrix} = \frac{1}{12}m!(m+1)!n!(n+1)!(n-m)^2[(n-m)^2-1] \neq 0.$$

For such chosen numbers $c_{i,j}$, $j \in \{1,2,3,4\}$ the function

$$g_{i,w}(z) := \sum_{j=1}^{4} c_{i,j} f_{j,w}(z)$$

satisfies the desired conditions. $\square$

The following lemma can be proved similar to [5, Proposition 3.11], so we omit the details.

LEMMA 3. *Let* $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ *and* μ *be a radial weight such that the operator* $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ *is bounded. Then* $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ *is compact if and only if* $\|T^{m,n}_{u,v,\varphi} f_k\|_{\mathscr{B}_\mu} \to 0$ *as* $k \to \infty$ *for each bounded sequence* $\{f_k\}_{k\in\mathbb{N}}$ *in* H^∞ *which converges to zero uniformly on compact subsets of* $\mathbb{D}$ *as* $k \to \infty$.

3. The operator $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$

In this section, we first give some necessary and sufficient conditions for the generalized Stević-Sharma type operators $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ to be bounded in different cases involving m and n. For simplicity of the expressions, we write

$$\begin{aligned} E_m(z) &= u'(z), \\ E_{m+1}(z) &= u(z)\varphi'(z), \\ E_n(z) &= v'(z), \\ E_{n+1}(z) &= v(z)\varphi'(z). \end{aligned}$$

THEOREM 1. *Let* $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $m+1 < n$, I *be the set* $\{m,m+1,n,n+1\}$ *and* μ *be a radial weight. Then the following statements are equivalent.*

(i) *The operator* $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ *is bounded.*

(ii)

$$\sum_{j=1}^{4} \sup_{w\in\mathbb{D}} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathscr{B}_\mu} < \infty \quad \textit{and} \quad \sum_{i\in I} \sup_{z\in\mathbb{D}} \mu(z)|E_i(z)| < \infty,$$

where $f_{j,w}$ *are defined in* (3).

(iii)

$$\sum_{i\in I} \sup_{z\in\mathbb{D}} \frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i} < \infty.$$

Proof. (i)$\Rightarrow$(ii). Assume that $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ is bounded. Since for each $w \in \mathbb{D}$ and $j \in \{1,2,3,4\}$, $\|f_{j,w}\|_\infty \lesssim 1$, we have $\sup_{w\in\mathbb{D}} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathscr{B}_\mu} < \infty$, and consequently

$$\sum_{j=1}^{4} \sup_{w\in\mathbb{D}} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathscr{B}_\mu} < \infty.$$

Taking $f_m(z) = z^m \in H^\infty$, by the boundedness of $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ we get

$$\infty > \|T^{m,n}_{u,v,\varphi} f_m\|_{\mathscr{B}_\mu} \geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T^{m,n}_{u,v,\varphi} f_m)'(z)| = \sup_{z\in\mathbb{D}} \mu(z)|E_m(z)|m!,$$

which yields

$$\sup_{z\in\mathbb{D}} \mu(z)|E_m(z)| < \infty. \tag{4}$$

Applying the operator $T^{m,n}_{u,v,\varphi}$ to $f_{m+1}(z) = z^{m+1} \in H^\infty$ we have

$$\begin{aligned}\infty > \|T^{m,n}_{u,v,\varphi} f_{m+1}\|_{\mathscr{B}_\mu} &\geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T^{m,n}_{u,v,\varphi} f_{m+1})'(z)| \\ &= \sup_{z\in\mathbb{D}} \mu(z)|E_m(z)\varphi(z)(m+1)! + E_{m+1}(z)(m+1)!| \\ &\geqslant \sup_{z\in\mathbb{D}} \mu(z)|E_{m+1}(z)|(m+1)! - \sup_{z\in\mathbb{D}} \mu(z)|E_m(z)\varphi(z)|(m+1)!,\end{aligned}$$

from which along with (4) and the fact that $|\varphi(z)| < 1$ it follows that

$$\sup_{z\in\mathbb{D}} \mu(z)|E_{m+1}(z)| < \infty. \tag{5}$$

By using the function $f_n(z) = z^n \in H^\infty$, we obtain

$$\begin{aligned}\infty > &\|T^{m,n}_{u,v,\varphi} f_n\|_{\mathscr{B}_\mu} \geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T^{m,n}_{u,v,\varphi} f_n)'(z)| \\ = &\sup_{z\in\mathbb{D}} \mu(z)\left|E_m(z)\varphi(z)^{n-m}\frac{n!}{(n-m)!} + E_{m+1}(z)\varphi(z)^{n-m-1}\frac{n!}{(n-m-1)!} + E_n(z)n!\right|,\end{aligned}$$

from which along with (4), (5), the triangle inequality and the fact that $|\varphi(z)| < 1$ gives

$$\sup_{z\in\mathbb{D}} \mu(z)|E_n(z)| < \infty. \tag{6}$$

Taking $f_{n+1}(z) = z^{n+1} \in H^\infty$, similarly we have

$$\begin{aligned}\infty > &\|T^{m,n}_{u,v,\varphi} f_{n+1}\|_{\mathscr{B}_\mu} \geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T^{m,n}_{u,v,\varphi} f_{n+1})'(z)| \\ = &\sup_{z\in\mathbb{D}} \mu(z)\Big|E_m(z)\varphi(z)^{n-m+1}\frac{(n+1)!}{(n-m+1)!} + E_{m+1}(z)\varphi(z)^{n-m}\frac{(n+1)!}{(n-m)!} \\ &+ E_n(z)\varphi(z)(n+1)! + E_{n+1}(z)(n+1)!\Big|,\end{aligned}$$

from which along with (4), (5), (6), the triangle inequality and the fact that $|\varphi(z)| < 1$ it follows that

$$\sup_{z\in\mathbb{D}} \mu(z)|E_{n+1}(z)| < \infty. \tag{7}$$

Combining (4)–(7) we can see that

$$\sum_{i\in I}\sup_{z\in\mathbb{D}}\mu(z)|E_i(z)|<\infty.$$

(ii)$\Rightarrow$(iii). Suppose that (ii) holds. For each $i\in I$ and $\varphi(w)\neq 0$, Lemma 2 says that there exist constants $c_{i,j}, j\in\{1,2,3,4\}$ such that

$$g_{i,\varphi(w)}(z)=\sum_{j=1}^{4}c_{i,j}f_{j,\varphi(w)}(z)\in H^\infty, \tag{8}$$

and

$$g^{(k)}_{i,\varphi(w)}(z)=\frac{\overline{\varphi(w)}^k\delta_{ik}}{(1-|\varphi(w)|^2)^k},$$

where $f_{j,\varphi(w)}$ are defined in (3) and $k\in I$. Then we have

$$\begin{aligned}\infty>\sum_{j=1}^{4}\sup_{w\in\mathbb{D}}\|T^{m,n}_{u,v,\varphi}f_{j,\varphi(w)}\|_{\mathscr{B}_\mu}&\gtrsim\sup_{w\in\mathbb{D}}\|T^{m,n}_{u,v,\varphi}g_{i,\varphi(w)}\|_{\mathscr{B}_\mu}\\ &\geqslant\mu(w)\big|(T^{m,n}_{u,v,\varphi}g_{i,\varphi(w)})'(w)\big|=\frac{\mu(w)|E_i(w)||\varphi(w)|^i}{(1-|\varphi(w)|^2)^i}.\end{aligned}\tag{9}$$

From (9) it follows that for each $i\in I$,

$$\begin{aligned}\sup_{w\in\mathbb{D}}\frac{\mu(w)|E_i(w)|}{(1-|\varphi(w)|^2)^i}&\leqslant\sup_{|\varphi(w)|>\frac12}\frac{\mu(w)|E_i(w)|}{(1-|\varphi(w)|^2)^i}+\sup_{|\varphi(w)|\leqslant\frac12}\frac{\mu(w)|E_i(w)|}{(1-|\varphi(w)|^2)^i}\\ &\leqslant 2^i\sup_{|\varphi(w)|>\frac12}\frac{\mu(w)|E_i(w)||\varphi(w)|^i}{(1-|\varphi(w)|^2)^i}+\left(\frac43\right)^i\sup_{|\varphi(w)|\leqslant\frac12}\mu(w)|E_i(w)|\\ &<\infty.\end{aligned}$$

Consequently,

$$\sum_{i\in I}\sup_{z\in\mathbb{D}}\frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i}<\infty.$$

(iii)$\Rightarrow$(i). Assume that (iii) holds. For any $f\in H^\infty$, using Lemma 1 we have

$$\mu(z)|(T^{m,n}_{u,v,\varphi}f)'(z)|\leqslant\sum_{i\in I}\mu(z)|E_i(z)||f^{(i)}(\varphi(z))|\lesssim\|f\|_\infty\sum_{i\in I}\frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i}. \tag{10}$$

Besides, we have

$$\begin{aligned}|(T^{m,n}_{u,v,\varphi}f)(0)|\leqslant&|u(0)f^{(m)}(\varphi(0))|+|v(0)f^{(n)}(\varphi(0))|\\ &\lesssim\left(\frac{|u(0)|}{(1-|\varphi(0)|^2)^m}+\frac{|v(0)|}{(1-|\varphi(0)|^2)^n}\right)\|f\|_\infty.\end{aligned}$$

Hence $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ is bounded. The proof is completed. $\square$

When $m+1=n$, as in the proof of Lemma 2, we conclude similarly that for any $0 \neq w \in \mathbb{D}$ and $i,k \in \{m,n,n+1\}$, there exist constants $d_{i,j}$, $j \in \{1,2,3\}$ such that the function $h_{i,w} = \sum_{j=1}^{3} d_{i,j} f_{j,w}(z) \in H^\infty$ satisfies

$$h^{(k)}_{i,w}(w) = \frac{\overline{w}^k \delta_{ik}}{(1-|w|^2)^k}.$$

By this and in the same way as Theorem 1, we can get the following conclusion.

THEOREM 2. *Let $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $m+1=n$, J be the set $\{m,n+1\}$ and μ be a radial weight. Then the following statements are equivalent.*

(i) *The operator $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ is bounded.*

(ii)

$$\sum_{j=1}^{3} \sup_{w\in\mathbb{D}} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathscr{B}_\mu} < \infty,$$

and

$$\sum_{i\in J} \sup_{z\in\mathbb{D}} \mu(z)|E_i(z)| + \sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)+v'(z)| < \infty,$$

where $f_{j,w}$ are defined in (3).

(iii)

$$\sum_{i\in J} \sup_{z\in\mathbb{D}} \frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i} + \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^n} < \infty.$$

We shall now describe the compactness of $T^{m,n}_{u,v,\varphi}$ acting from H^∞ to $\mathscr{B}_\mu$.

THEOREM 3. *Let $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $m+1<n$, I be the set $\{m,m+1,n,n+1\}$ and μ be a radial weight. Suppose that $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ is bounded, then the following statements are equivalent.*

(i) *The operator $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ is compact.*

(ii)

$$\sum_{j=1}^{4} \lim_{|\varphi(w)|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,\varphi(w)}\|_{\mathscr{B}_\mu} = 0,$$

where $f_{j,\varphi(w)}$ are defined in (3).

(iii)

$$\sum_{i\in I} \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i} = 0.$$

Proof. (i)$\Rightarrow$(ii). Suppose that $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ is compact. Let $\{w_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(w_k)| \to 1$ as $k\to\infty$. Set $f_{j,k} = f_{j,\varphi(w_k)}$, where $j \in \{1,2,3,4\}$, then $f_{j,k}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. By using Lemma 3, we have

$$\lim_{k\to\infty}\|T^{m,n}_{u,v,\varphi} f_{j,k}\|_{\mathscr{B}_\mu} = 0,$$

from which along with the fact that $|\varphi(w_k)| \to 1$ as $k\to\infty$ it follows that

$$\lim_{|\varphi(w)|\to 1}\|T^{m,n}_{u,v,\varphi} f_{j,\varphi(w)}\|_{\mathscr{B}_\mu} = 0,$$

which yields that (ii) holds.

(ii)$\Rightarrow$(iii). Assume that (ii) holds. If $\|\varphi\|_\infty < 1$, then (iii) automatically holds. Now we consider the case $\|\varphi\|_\infty = 1$. Since $T^{m,n}_{u,v,\varphi} : H^\infty \to \mathscr{B}_\mu$ is bounded, letting $|\varphi(w)| \to 1$ in (9) gives

$$\lim_{|\varphi(w)|\to 1}\frac{\mu(w)|E_i(w)|}{(1-|\varphi(w)|^2)^i} = 0,$$

where $i\in I$. Therefore,

$$\sum_{i\in I}\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i} = 0.$$

(iii)$\Rightarrow$(i). Suppose that (iii) holds. Then for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i} < \varepsilon, \quad i\in I,$$

whenever $\delta < |\varphi(z)| < 1$. Moreover, we have $L_i := \sup_{z\in\mathbb{D}}\mu(z)|E_i(z)| < \infty$ for each $i\in I$ by Theorem 1.

Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in H^∞ such that $\sup_{k\in\mathbb{N}}\|f_k\|_\infty \lesssim 1$ and $f_k \to 0$ uniformly on compact subset of $\mathbb{D}$ as $k\to\infty$. Applying Lemma 1 we have

$$\begin{aligned}
&\|T^{m,n}_{u,v,\varphi} f_k\|_{\mathscr{B}_\mu}\\
=&|(T^{m,n}_{u,v,\varphi} f_k)(0)| + \sup_{z\in\mathbb{D}}\mu(z)|(T^{m,n}_{u,v,\varphi} f_k)'(z)|\\
\leqslant&|u(0)f_k^{(m)}(\varphi(0))| + |v(0)f_k^{(n)}(\varphi(0))| + \sum_{i\in I}\sup_{z\in\mathbb{D}}\mu(z)|E_i(z)f_k^{(i)}(\varphi(z))|\\
\leqslant&|u(0)||f_k^{(m)}(\varphi(0))| + |v(0)||f_k^{(n)}(\varphi(0))|\\
&+\sum_{i\in I}\sup_{|\varphi(z)|\leqslant\delta}\mu(z)|E_i(z)||f_k^{(i)}(\varphi(z))| + \sum_{i\in I}\sup_{\delta<|\varphi(z)|<1}\mu(z)|E_i(z)||f_k^{(i)}(\varphi(z))|\\
\lesssim&|u(0)||f_k^{(m)}(\varphi(0))| + |v(0)||f_k^{(n)}(\varphi(0))|\\
&+\sum_{i\in I}L_i\sup_{|\varphi(z)|\leqslant\delta}|f_k^{(i)}(\varphi(z))| + \sum_{i\in I}\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i}\\
\leqslant&|u(0)||f_k^{(m)}(\varphi(0))| + |v(0)||f_k^{(n)}(\varphi(0))| + \sum_{i\in I}L_i\sup_{|w|\leqslant\delta}|f_k^{(i)}(w)| + 4\varepsilon.
\end{aligned} \tag{11}$$

Since $f_k \to 0$ uniformly on compact subset of $\mathbb{D}$ as $k \to \infty$, we conclude that for $i \in I$, $f_k^{(i)}$ also do by Cauchy's estimate. In particular, $\{\varphi(0)\}$ and $\{w : |w| \leqslant \delta\}$ are compact subsets of $\mathbb{D}$, hence letting $k \to \infty$ in (11) yields

$$\lim_{k\to\infty} \|T_{u,v,\varphi}^{m,n} f_k\|_{\mathscr{B}_\mu} \leqslant 4\varepsilon.$$

By the arbitrariness of ε we can see that $\lim_{k\to\infty} \|T_{u,v,\varphi}^{m,n} f_k\|_{\mathscr{B}_\mu} = 0$, from which by Lemma 3 we deduce that $T_{u,v,\varphi}^{m,n} : H^\infty \to \mathscr{B}_\mu$ is compact. $\square$

By using similar arguments it is proved the following theorem. We omit the details.

THEOREM 4. *Let $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $m+1 = n$, J be the set $\{m, n+1\}$ and μ be a radial weight. Suppose that $T_{u,v,\varphi}^{m,n} : H^\infty \to \mathscr{B}_\mu$ is bounded, then the following statements are equivalent.*

(i) *The operator $T_{u,v,\varphi}^{m,n} : H^\infty \to \mathscr{B}_\mu$ is compact.*

(ii)

$$\sum_{j=1}^{3} \lim_{|\varphi(w)|\to 1} \|T_{u,v,\varphi}^{m,n} f_{j,\varphi(w)}\|_{\mathscr{B}_\mu} = 0,$$

where $f_{j,\varphi(w)}$ are defined in (3).

(iii)

$$\sum_{i\in J} \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i} + \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^n} = 0.$$

4. The operator $T_{u,v,\varphi}^{m,n} : H^\infty \to \mathscr{B}_{\mu,0}$

In this section, we study the boundedness and compactness of $T_{u,v,\varphi}^{m,n}$ from H^∞ to the little Bloch-type spaces $\mathscr{B}_{\mu,0}$.

THEOREM 5. *Let $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $m+1 < n$, I be the set $\{m, m+1, n, n+1\}$ and μ be a radial weight. If the operator $T_{u,v,\varphi}^{m,n} : H^\infty \to \mathscr{B}_{\mu,0}$ is bounded, then*

$$\sum_{i\in I} \lim_{|z|\to 1} \mu(z)|E_i(z)| = 0. \tag{12}$$

Proof. Suppose that $T_{u,v,\varphi}^{m,n} : H^\infty \to \mathscr{B}_{\mu,0}$ is bounded. It is evident that $T_{u,v,\varphi}^{m,n} : H^\infty \to \mathscr{B}_\mu$ is bounded, and for each $f \in H^\infty$, we have $T_{u,v,\varphi}^{m,n} f \in \mathscr{B}_{\mu,0}$. Taking $f_m(z) = z^m \in H^\infty$, we obtain

$$\lim_{|z|\to 1} \mu(z)|E_m(z)| = \frac{1}{m!} \lim_{|z|\to 1} \mu(z)|(T_{u,v,\varphi}^{m,n} f_m)'(z)| = 0. \tag{13}$$

By using the function $f_{m+1}(z) = z^{m+1} \in H^\infty$, we obtain

$$\begin{aligned} 0 &= \frac{1}{(m+1)!} \lim_{|z|\to 1} \mu(z)|(T^{m,n}_{u,v,\varphi} f_{m+1})'(z)| \\ &= \lim_{|z|\to 1} \mu(z)|E_m(z)\varphi(z) + E_{m+1}(z)| \\ &\geqslant \lim_{|z|\to 1} \mu(z)|E_{m+1}(z)| - \lim_{|z|\to 1} \mu(z)|E_m(z)\varphi(z)|, \end{aligned}$$

from which along with (13) and the fact that $|\varphi(z)| < 1$ it follows that

$$\lim_{|z|\to 1} \mu(z)|E_{m+1}(z)| = 0. \tag{14}$$

Taking $f_n(z) = z^n$, we have

$$\begin{aligned} \lim_{|z|\to 1} \mu(z) \Big| & E_m(z)\varphi(z)^{n-m} \frac{n!}{(n-m)!} \\ & + E_{m+1}(z)\varphi(z)^{n-m-1} \frac{n!}{(n-m-1)!} + E_n(z)n! \Big| \\ = & \lim_{|z|\to 1} \mu(z)|(T^{m,n}_{u,v,\varphi} f_n)'(z)| = 0, \end{aligned}$$

from which along with (13), (14), the triangle inequality, and the fact that $|\varphi(z)| < 1$ yields that

$$\lim_{|z|\to 1} \mu(z)|E_n(z)| = 0. \tag{15}$$

Applying the operator $T^{m,n}_{u,v,\varphi}$ to $f_{n+1}(z) = z^{n+1} \in H^\infty$ we get

$$\begin{aligned} \lim_{|z|\to 1} \Big| & E_m(z)\varphi(z)^{n-m+1} \frac{(n+1)!}{(n-m+1)!} + E_{m+1}(z)\varphi(z)^{n-m} \frac{(n+1)!}{(n-m)!} \\ & + E_n(z)\varphi(z)(n+1)! + E_{n+1}(z)(n+1)! \Big| \\ = & \lim_{|z|\to 1} \mu(z)|(T^{m,n}_{u,v,\varphi} f_{n+1})'(z)| = 0, \end{aligned}$$

from which along with (13), (14), (15), the triangle inequality, and the fact that $|\varphi(z)| < 1$ it follows that

$$\lim_{|z|\to 1} \mu(z)|E_{n+1}(z)| = 0. \tag{16}$$

Combining (13)–(16) we can see that (12) holds. $\square$

THEOREM 6. *Let $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $m\in\mathbb{N}_0$, $n\in\mathbb{N}$, $m+1=n$, J be the set $\{m,n+1\}$ and μ be a radial weight. If the operator $T^{m,n}_{u,v,\varphi}:H^\infty\to\mathscr{B}_{\mu,0}$ is bounded, then*

$$\sum_{i\in J}\lim_{|z|\to1}\mu(z)|E_i(z)|+\lim_{|z|\to1}\mu(z)|u(z)\varphi'(z)+v'(z)|=0.$$

For the compactness of $T^{m,n}_{u,v,\varphi}:H^\infty\to\mathscr{B}_{\mu,0}$, we have the following results.

THEOREM 7. *Let $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $m\in\mathbb{N}_0$, $n\in\mathbb{N}$, $m+1<n$, I be the set $\{m,m+1,n,n+1\}$ and μ be a radial weight. If the operator $T^{m,n}_{u,v,\varphi}:H^\infty\to\mathscr{B}_{\mu,0}$ is compact, then*

$$\sum_{i\in I}\lim_{|z|\to1}\frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i}=0. \tag{17}$$

Proof. Suppose that $T^{m,n}_{u,v,\varphi}:H^\infty\to\mathscr{B}_{\mu,0}$ is compact, then $T^{m,n}_{u,v,\varphi}:H^\infty\to\mathscr{B}_{\mu}$ is compact and $T^{m,n}_{u,v,\varphi}:H^\infty\to\mathscr{B}_{\mu,0}$ is bounded. By Theorem 5, for any $\varepsilon>0$, there exists $\eta\in(0,1)$ such that

$$\mu(z)|E_i(z)|<\varepsilon,\quad i\in I, \tag{18}$$

whenever $\eta<|z|<1$. From Theorem 3, for any $\varepsilon>0$, there exists $\delta\in(0,1)$ such that

$$\frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i}<\varepsilon,\quad i\in I, \tag{19}$$

for $\delta<|\varphi(z)|<1$. Therefore, when $\delta<|\varphi(z)|<1$ and $\eta<|z|<1$, we have (19) holds. If $|\varphi(z)|\leqslant\delta$ and $\eta<|z|<1$, by using (18) we get

$$\frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i}\leqslant\frac{\varepsilon}{(1-\delta^2)^i}\lesssim\varepsilon,\quad i\in I. \tag{20}$$

Combining (19) with (20), we can see that (17) holds by the arbitrariness of ε. □

THEOREM 8. *Let $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $m\in\mathbb{N}_0$, $n\in\mathbb{N}$, $m+1=n$, J be the set $\{m,n+1\}$ and μ be a radial weight. If the operator $T^{m,n}_{u,v,\varphi}:H^\infty\to\mathscr{B}_{\mu,0}$ is compact, then*

$$\sum_{i\in J}\lim_{|z|\to1}\frac{\mu(z)|E_i(z)|}{(1-|\varphi(z)|^2)^i}+\lim_{|z|\to1}\frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^n}=0.$$

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