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REGULARITY IN ORLICZ SPACES FOR QUASI–LINEAR ELLIPTIC EQUATIONS OF SCHRÖDINGER TYPE

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Abstract. In this paper, we generalize gradient estimates in Lebesgue spaces to Orlicz spaces for weak solutions of quasi-linear elliptic equations of Schrödinger type on a Reifenberg flat domain, under the condition that the coefficients are in John-Nirenberg space with small *BMO* semi-norms. We assume that the potential belongs to some certain reverse Hölder class. Our results improve the known results for such equations using a harmonic analysis-free technique.

1. Introduction and main results

In this paper, we consider the following quasi-linear elliptic equation of Schrödinger type

$$\begin{cases} \operatorname{div}((\mathbf{A}Du.Du)^{\frac{p-2}{2}}\mathbf{A}Du)+\mathbf{V}|u|^{p-2}u=\operatorname{div}(|f|^{p-2}f) & \text{in} \quad \Omega, \\ u=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a domain of $\mathbb{R}^n$. Throughout the paper, we always assume that the following assumptions hold true (for the convenience of presentation we shall refer to these assumptions as **Assumption (H)**):

- ◇ $n \geqslant 2$ and $p \in (1,\infty)$.
- ◇ The data f is a vector field which belongs to at least the space $L^p(\Omega;\mathbb{R}^n)$.
- ◇ The coefficient matrix $\mathbf{A}:\Omega \to \mathbb{R}^{n\times n}$ is symmetric and satisfies the following uniform ellipticity condition: there exists a positive constant Λ such that for all $\xi \in \mathbb{R}^n$, we have
$$\Lambda^{-1}|\xi|^2 \leqslant \mathbf{A}(x)\xi.\xi \leqslant \Lambda|\xi|^2.$$

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- ⋄ The potential $\mathbf{V}:\mathbb{R}^n \to \mathbb{R}$ is nonnegative and belongs to the reverse Hölder class RH_γ with

$$\begin{cases} \frac{n}{p} \leqslant \gamma < n & \text{if } p < n, \\ 1 < \gamma < n & \text{if } p \geqslant n. \end{cases} \tag{1.2}$$

It means that there exists a constant $C > 0$ such that the following inequality

$$\left(\frac{1}{|B|}\int_B \mathbf{V}(y)^\gamma dy\right)^{1/\gamma} \leqslant \frac{C}{|B|}\int_B \mathbf{V}(y)dy$$

holds for all balls B in $\mathbb{R}^n$.

We have some important remarks related to the reverse Hölder class as follows.

REMARK 1.1.

(i) The RH_γ class which is a wide class of functions including all nonnegative polynomials was introduced independently by Muckenhoupt [18] and Gehring [9] in the study of weighted norm inequalities and quasi-conformal mapping, respectively.

(ii) It is well-known that $V \in RH_\gamma$ if and only if $\mathbf{V} \in A_\infty$, where A_∞ is the class of Muckenhoupt weights on $\mathbb{R}^n$. In addition, if $\mathbf{V} \in A_\infty$ then we have

$$[\mathbf{V}]_{A_\infty} := \sup_B \left(\fint_B \mathbf{V}dx\right)\exp\left(\fint_B \log \mathbf{V}^{-1}dx\right) < \infty, \tag{1.3}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. We call $[\mathbf{V}]_{A_\infty}$ the A_∞ constant of $\mathbf{V}$. (See pages 7, 8 in [23] and pages 8, 9 in [17])

As usual, the solutions of problem (1.1) are taken in a weak sense. More precisely, we have the following definition.

DEFINITION 1.2. Let Assumption (H) hold true. A function $u \in W_0^{1,p}(\Omega)$ is called a weak solution of the problem (1.1) if for any $\varphi \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega (\mathbf{A}Du.Du)^{\frac{p-2}{2}}\mathbf{A}Du.D\varphi dx + \int_\Omega \mathbf{V}|u|^{p-2}u\varphi dx = \int_\Omega |f|^{p-2}f.D\varphi dx.$$

REMARK 1.3. Let u be a weak solution of the problem (1.1). Since the parameter γ satifies (1.2), it follows from the Sobolev's embedding inequality that

$$\left|\int_\Omega \mathbf{V}|u|^{p-2}u\varphi dx\right| \leqslant \left(\int_\Omega \mathbf{V}^{\frac{n}{p}}dx\right)^{\frac{p}{n}}\left(\int_\Omega |u|^{\frac{np}{n-p}}dx\right)^{\frac{(n-p)(p-1)}{np}}\left(\int_\Omega |\varphi|^{\frac{np}{n-p}}dx\right)^{\frac{n-p}{np}} < \infty.$$

Moreover, $\|Du\|_{L^p(\Omega)}$ and $\|Du\|_{L^p(\Omega)} + \|\mathbf{V}^{\frac{1}{p}}u\|_{L^p(\Omega)}$ are two equivalent norms, and the problem (1.1) has a unique weak solution $u \in W_0^{1,p}(\Omega)$ (see [17]).

Our main goal is to study how the regularity of f is reflected to the solutions of (1.1) in the setting of Orlicz spaces. For this purpose we need to add further assumptions to $\mathbf{A}$ and the domain Ω. These assumptions are related to following definitions.

DEFINITION 1.4. Let δ and R be two positive constants. A domain $\Omega \subset \mathbb{R}^n$ is said to be (δ,R)-Reifenberg flat if for every $x \in \partial\Omega$ and every $r \in (0,R]$, there exists a coordinate system $\{y_1,\cdots,y_n\}$ depending on r and x so that $x=0$ in this coordinate system and

$$B_r(x)\cap\{y_n>\delta r\}\subset B_r(0)\cap\Omega\subset B_r(x)\cap\{y_n>-\delta r\}.$$

DEFINITION 1.5. Let δ and R be two positive constants. We say that $\mathbf{A}:\Omega\to\mathbb{R}^{n\times n}$ is (δ,R)-vanishing if

$$\sup_{0<r\leqslant R}\sup_{x\in\mathbb{R}^n}\fint_{B_r(x)}|\mathbf{A}(y)-\mathbf{A}_{B_r(x)}|dy\leqslant\delta, \tag{1.4}$$

where $B_r(x)$ is the ball of radius r and center x, and $\mathbf{A}_{B_r(x)}=\fint_{B_r(x)}\mathbf{A}(y)dy$.

Now we are in position to state main results of this work.

THEOREM 1.6. [A priori estimate] *Let Assumption (H) hold true, $|f|^p\in L^\phi(\Omega)$ for some Young function $\phi\in\Delta_2\cap\nabla_2$ (see Section 2 below) and $R_0>0$. Then there exists a small number $\delta=\delta(n,p,\phi,\Lambda)>0$ such that if $\mathbf{A}$ is (δ,R_0)-vanishing and Ω is (δ,R_0)-Reifenberg flat, we have the following estimate:*

$$\int_\Omega\phi\left(|Du|^p\right)dx+\int_\Omega\phi\left(\mathbf{V}|u|^p\right)dx\leqslant C\int_\Omega\phi\left(|f|^p\right)dx, \tag{1.5}$$

where C is a positive constant independent of u and f, provided that $u\in W_0^{1,p}(\Omega)$ is a weak solution to (1.1) satisfying $|Du|^p\in L^\phi(\Omega)$.

By using Theorem 1.6 we obtain the existence result as below.

THEOREM 1.7. [Existence] *Under the assumptions of Theorem 1.6, then there exists a small number $\delta=\delta(n,p,\phi,\Lambda)>0$ such that if $\mathbf{A}$ is (δ,R_0)-vanishing and Ω is (δ,R_0)-Reifenberg flat, then the problem (1.1) has a unique weak solution u such that $|Du|^p\in L^\phi(\Omega)$. Moreover, we also have the following estimate*

$$\int_\Omega\phi(|Du|^p)dx+\int_\Omega\phi(\mathbf{V}|u|^p)dx\leqslant C\int_\Omega\phi(|f|^p)dx, \tag{1.6}$$

where C is a constant independent of u and f.

Recently, integrability of the gradient of solutions for elliptic/parabolic problems with discontinuous coefficients of VMO/BMO type have been extensively studied by many authors (see [3, 4, 6, 14, 15, 16]). We would like to point out that if a function satisfies the VMO condition, then it satisfies the (δ, R_0)-vanishing condition which we treat in this paper. On the other hand, Reifenberg flat domains were introduced by Reifenberg in the noteworthy paper [21] where he showed that they are locally topological disks if δ is sufficiently small. A good example of Reifenberg flat domains is a flat version of the well-known Van Koch snowflake when the angle of the spike with respect to the horizontal is sufficiently small (see [22]). These domains arise naturally in many areas such as applied mathematics, harmonic analysis and geometric measure theory. They look like coast lines, zigzag functions, atomic clusters. For a further discussion of Reifenberg flat domains we refer the reader to [3, 6, 8, 11, 12, 22].

We summarize here some remarks of the regularity of the solution to the equations of the form (1.1) which have been established recently.

- ◇ Theorem 1.7 was claimed in [7] when $\mathbf{V}=0$.
- ◇ Theorem 1.6 was proved in [24] when $\mathbf{A}=I$, $p=2$ and $\mathbf{V}\in RH_\infty$, i.e. $\mathbf{V}\in L^\infty_{\text{loc}}(\mathbb{R}^n)$ and

$$\sup_{B_r(x_0)} \mathbf{V} \leqslant C \fint_{B_r(x_0)} \mathbf{V}dx,$$

  for all balls $B_r(x_0)$.
- ◇ Theorem 1.6 was established in [25] when $\mathbf{A}\in C^1(\mathbb{R}^n,\mathbb{R}^n)$, $p=2$ and $\mathbf{V}\in RH_\infty$.
- ◇ When $\phi(t)=t^{q/p}$ with $q>p$, the authors in [17] proved Theorem 1.7 for a general vector valued function $\mathbf{A}$ that satisfies growth and ellipticity conditions. However, when $q\geqslant p\gamma$, they do not obtain any estimate for $\left\|\mathbf{V}^{\frac{1}{p}}u\right\|_{L^q}$.

It should be emphasized, as pointed out, that our results are developed as a continuation to the aforementioned works. In particular, these results generalize those of [7], [24], [25]. Furthermore, our work improves the one of [17] when providing an estimate of $\left\|\mathbf{V}^{\frac{1}{p}}u\right\|_{L^q}$ when $q\geqslant p\gamma$ in the sense that $\mathbf{A}$ is a matrix of discontinuous entries.

This paper will be organized as follows. In Section 2, we recall some basic definitions and facts about Orlicz spaces. In Section 3, we prove Theorem 1.6. Our main ingredients to the proof of Theorem 1.6 are Lemma 3.1 and Lemma 3.2. A proof of Lemma 3.1 will be given in Section 4. Finally, Section 5 is devoted to the proof of Theorem 1.7 using approximation.

We end this section by noting that the following notations will be used throughout this paper:

- Positive constants are signified as C although they may be different even on the same line.

- We write $A \lesssim B$ and $A \sim B$ if there exist positive constants C and C' such that

$$A \leqslant CB \quad \text{and} \quad C'A \leqslant B \leqslant CA,$$

respectively.

- If B is a ball of center x and radius R then, for every $\lambda > 0$, we denote by λB the ball of the same center and radius λR.

- We write B_R for any ball of radius R.

2. Orlicz space

Orlicz spaces were first introduced by Orlicz as a generalization of L^p spaces. Since then, the theory of Orlicz spaces has played a crucial role in a very wide spectrum (see [20]). In this section, for the convenience of the readers, we will recall some definitions and basic facts about Orlicz spaces which will be needed in the following. For further properties, we refer the readers to [1] and [26].

DEFINITION 2.1. Denote by Φ the function class that consists of all functions $\phi : [0,\infty) \mapsto [0,\infty)$, which are increasing and convex. Then a function $\phi \in \Phi$ is said to be a Young function if $\phi(0) = 0$ and

$$\lim_{t\to\infty} \phi(t) = \infty, \quad \lim_{t\to 0^+} \frac{\phi(t)}{t} = \lim_{t\to+\infty} \frac{t}{\phi(t)} = 0.$$

It is well-known that a Young function ϕ is differentiable a.e. and can be written as:

$$\phi(x) = \int_0^x \phi'(t)dt, x \geqslant 0,$$

where $\phi'(0) = 0, \phi' : [0,\infty) \to [0,\infty)$ is non-decreasing and left-continuous.

DEFINITION 2.2. Given a Young function ϕ, the function $\phi^* : [0,\infty) \mapsto [0,\infty)$, defined by

$$\phi^*(s) = \sup\{st - \phi(t) : t \geqslant 0\},$$

is called the complementary function of ϕ.

It is a fact that if ϕ is a Young function then so is ϕ^*, and the complementary function of ϕ^* is ϕ. In addition, from the definition of ϕ^* we have Young's inequality: for $s,t \geqslant 0$,

$$st \leqslant \phi(s) + \phi^*(t).$$

DEFINITION 2.3. A function $\phi \in \Phi$ is said to satisfy the global Δ_2 condition, denoted by $\phi \in \Delta_2$, if ϕ is a Young function and there exists a positive constant $K > 0$ such that for every $t > 0$,

$$\phi(2t) \leqslant K\phi(t).$$

DEFINITION 2.4. A function $\phi \in \Phi$ is said to satisfy the global ∇_2 condition, denoted by $\phi \in \nabla_2$, if ϕ is a Young function and there exists a constant $a > 1$ such that for every $t > 0$,

$$\phi(t) \leqslant \frac{\phi(at)}{2a}.$$

Note that $\phi \in \nabla_2$ if and only if $\phi^* \in \Delta_2$.

LEMMA 2.5. ([7]) *$\phi \in \Delta_2 \cap \nabla_2$ if and only if ϕ is a Young function and there exist constants $A_2 \geqslant A_1 > 0$ and $\alpha_1 \geqslant \alpha_2 > 1$ such that for any $0 < s \leqslant t$,*

$$A_1 \left(\frac{s}{t}\right)^{\alpha_1} \leqslant \frac{\phi(s)}{\phi(t)} \leqslant A_2 \left(\frac{s}{t}\right)^{\alpha_2}. \tag{2.7}$$

Moreover, the condition (2.7) implies that for $0 < \theta_1 \leqslant 1 \leqslant \theta_2 < \infty$,

$$\phi(\theta_1 t) \leqslant A_2 \theta_1^{\alpha_2} \phi(t) \text{ and } \phi(\theta_2 t) \leqslant A_1^{-1} \theta_2^{\alpha_1} \phi(t). \tag{2.8}$$

LEMMA 2.6. *Let ϕ be a Young function satisfying $\phi \in \Delta_2 \cap \nabla_2$. Then there exists a positive constant $K > 0$ such that for every $t > 0$ and $0 < \theta_1 \leqslant 1 \leqslant \theta_2 < \infty$,*

$$\phi'(\theta_1 t) \leqslant K A_2 \theta_1^{\alpha_2 - 1} \phi'(t) \text{ and } \phi'(\theta_2 t) \leqslant K A_1^{-1} \theta_2^{\alpha_1 - 1} \phi'(t),$$

where $A_2 \geqslant A_1 > 0$ and $\alpha_1 \geqslant \alpha_2 > 1$ are constants defined in Lemma 2.5.

Proof. Since $\phi' : [0,\infty) \to [0,\infty)$ is non-decreasing, we have

$$\phi(t) = \int_0^t \phi'(x)dx \leqslant t\phi'(t),$$

and

$$\phi(2t) = \int_0^{2t} \phi'(x)dx \geqslant \int_t^{2t} \phi'(x)dx \geqslant t\phi'(t). \tag{2.9}$$

On the other hand, by $\phi \in \Delta_2$, there exists a positive constant K such that $\phi(2t) \leqslant K\phi(t)$ for every $t > 0$. Therefore, for every $t > 0$

$$\phi(t) \leqslant t\phi'(t) \leqslant K\phi(t), \tag{2.10}$$

which is equivalent to

$$\frac{\phi(t)}{t} \leqslant \phi'(t) \leqslant K \frac{\phi(t)}{t}. \tag{2.11}$$

Finally, it follows from (2.11) and Lemma 2.5 that for $0 < \theta_1 \leqslant 1 \leqslant \theta_2 < \infty$ and $t > 0$, we have

$$\phi'(\theta_1 t) \leqslant K \frac{\phi(\theta_1 t)}{\theta_1 t} \leqslant K A_2 \theta_1^{\alpha_2 - 1} \frac{\phi(t)}{t} \leqslant K A_2 \theta_1^{\alpha_2 - 1} \phi'(t),$$
$$\phi'(\theta_2 t) \leqslant K \frac{\phi(\theta_2 t)}{\theta_2 t} \leqslant K A_1^{-1} \theta_2^{\alpha_1 - 1} \frac{\phi(t)}{t} \leqslant K A_1^{-1} \theta_2^{\alpha_1 - 1} \phi'(t). \quad \square$$

LEMMA 2.7. *Let ϕ be a Young function satisfying $\phi \in \Delta_2 \cap \nabla_2$ and $b > 0$. Then there exists a positive constant C such that for any $t > 0$, the following holds true*

$$\int_0^{bt} \frac{\phi'(\lambda)}{\lambda} d\lambda \leqslant C\phi'(t). \tag{2.12}$$

Proof. Take an arbitrary $\alpha \in (0, bt)$ and choose $n \in \mathbb{N}$ such that $\frac{bt}{2^n} < \alpha$. Since ϕ' is non-decreasing, we have

$$\begin{aligned}
\int_\alpha^{bt} \frac{\phi'(\lambda)}{\lambda} d\lambda &\leqslant \int_{\frac{bt}{2^n}}^{bt} \frac{\phi'(\lambda)}{\lambda} d\lambda \\
&= \int_{\frac{bt}{2^n}}^{\frac{bt}{2^{n-1}}} \frac{\phi'(\lambda)}{\lambda} d\lambda + \int_{\frac{bt}{2^{n-1}}}^{\frac{bt}{2^{n-2}}} \frac{\phi'(\lambda)}{\lambda} d\lambda + \cdots + \int_{\frac{bt}{2}}^{bt} \frac{\phi'(\lambda)}{\lambda} d\lambda \\
&\leqslant \int_{\frac{bt}{2^n}}^{\frac{bt}{2^{n-1}}} \frac{\phi'(\frac{bt}{2^{n-1}})}{\lambda} d\lambda + \int_{\frac{bt}{2^{n-1}}}^{\frac{bt}{2^{n-2}}} \frac{\phi'(\frac{bt}{2^{n-2}})}{\lambda} d\lambda + \cdots + \int_{\frac{bt}{2}}^{bt} \frac{\phi'(bt)}{\lambda} d\lambda \\
&= \ln 2 \left(\phi'\left(\frac{bt}{2^{n-1}}\right) + \phi'\left(\frac{bt}{2^{n-2}}\right) + \cdots + \phi'(bt) \right).
\end{aligned}$$

Then in view of Lemma 2.6, we derive

$$\int_\alpha^{bt} \frac{\phi'(\lambda)}{\lambda} d\lambda \leqslant \left(\frac{1}{(2^{\alpha_2 - 1})^{n-1}} + \frac{1}{(2^{\alpha_2 - 1})^{n-2}} + \cdots + 1 \right) KA_2 \ln 2 . \phi'(bt) \leqslant C\phi'(t), \tag{2.13}$$

where C is a positive constant depending only on ϕ and b.

Letting $\alpha \to 0^+$ in (2.13) gives (2.12). □

DEFINITION 2.8. Let ϕ be a Young function. Then the Orlicz class $K^\phi(\Omega)$ is defined to be the set of all measurable functions $f : \Omega \to \mathbb{R}$ satisfying the condition $\int_\Omega \phi(|f|)dx < \infty$, and the Orlicz space $L^\phi(\Omega)$ is defined to be the linear hull of $K^\phi(\Omega)$.

In $L^\phi(\Omega)$ we consider the following analog of the Luxemburg norm

$$\|u\|_{L^\phi(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \phi\left(\frac{|u(x)|}{\lambda}\right) dx \leqslant 1 \right\}.$$

It is well-known that if $\phi \in \Delta_2$, then the space $C_c^\infty(\Omega)$ of infinitely differentiable functions with compact support is dense in $L^\phi(\Omega)$ and $(L^\phi)^*(\Omega) = L^{\phi^*}(\Omega)$ (see [1, p. 271]). Furthermore, if ϕ and $\phi^* \in \Delta_2$ then $L^\phi(\Omega)$ is reflexive (see [1, p. 274]).

LEMMA 2.9. ([1]) *Let ϕ be a Young function satisfying $\phi \in \Delta_2 \cap \nabla_2$. Then*

$$L^{\alpha_1}(\Omega) \subset L^\phi(\Omega) \subset L^{\alpha_2}(\Omega) \subset L^1(\Omega),$$

with $\alpha_1 \geqslant \alpha_2 > 1$ as in Lemma 2.5.

REMARK 2.10. ([26]) In general, $K^{\phi}(\Omega) \subset L^{\phi}(\Omega)$. However, if $\phi \in \Delta_2$, then we have $K^{\phi}(\Omega) = L^{\phi}(\Omega)$. Moreover, if $g \in L^{\phi}(\Omega)$, then

$$\int_{\Omega} \phi(|g(x)|)dx = \int_0^{\infty} |\{x \in \Omega : |g| > \lambda\}| d[\phi(\lambda)]. \tag{2.14}$$

LEMMA 2.11. ([19, Theorem 1.5]) *Given $f \in L^{\phi}(\Omega)$ and $g \in L^{\phi}(\Omega)$. If for all $\lambda > 0$, we have*

$$\int_{\Omega} \phi(|\lambda f|)dx \leqslant \int_{\Omega} \phi(|\lambda g|)dx$$

then

$$\|f\|_{L^{\phi}(\Omega)} \leqslant \|g\|_{L^{\phi}(\Omega)}.$$

3. Proof of Theorem 1.6

This section is devoted to proving Theorem 1.6. Our main tools are Lemma 3.1 and Lemma 3.2 below. We will give a proof of Lemma 3.1 in the next section. Note that, in Lemma 3.1, if $B_{2R} \not\subset \Omega$ then we shall use the same notation u for its zero extension on $\mathbb{R} \setminus \Omega$.

LEMMA 3.1. *Assume that the assumptions of Theorem 1.6 hold true. Let $u \in W_0^{1,p}(\Omega)$ be a weak solution to (1.1). Then, for each ball*

$$B_{2R} = B_{2R}(x_0) \ \text{with} \ x_0 \in \overline{\Omega} \ \text{and} \ R \in (0, R_0/20],$$

there exist a function $h \in W^{1,p}(\Omega_R)$ and a constant $\delta > 0$ independent of R and x_0 such that the following hold true

$$\fint_{\Omega_R} (|D(u-h)|^p + \mathbf{V}|u-h|^p)\, dx \lesssim \delta^{\frac{p}{p-1}} \fint_{\Omega_{2R}} (|Du|^p + \mathbf{V}|u|^p)\, dx + \fint_{\Omega_{2R}} |f|^p dx, \tag{3.15}$$

$$\fint_{\Omega_R} (|Dh|^p + \mathbf{V}|h|^p)\, dx \lesssim \fint_{\Omega_{2R}} (|Du|^p + \mathbf{V}|u|^p)\, dx + \fint_{\Omega_{2R}} |f|^p dx. \tag{3.16}$$

Here $\Omega_{2R} = B_{2R} \cap \Omega$ and $\Omega_R = B_R \cap \Omega$ with $B_R = B_R(x_0)$.

LEMMA 3.2. *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution to (1.1) and*

$$\lambda_0 := \fint_{\Omega} \left(|Du|^p + \mathbf{V}|u|^p\right) dx + \frac{1}{\delta^p} \fint_{\Omega} |f|^p dx.$$

Then we can find a constant $M_0 > 0$ so that if $\lambda > M_0 \lambda_0$ and

$$E(\lambda) := \{x \in \Omega : |Du(x)|^p + \mathbf{V}(x)|u(x)|^p > \lambda\} \neq \emptyset,$$

then there exists a disjoint family of balls

$$B_{R_j} = B_{R_j}(x_j) \ \text{with} \ 0 < R_j < R_0/20 \ \text{and} \ x_j \in E(\lambda)$$

such that

◇ $E(\lambda) \subset \bigcup_{j=1}^{\infty} \left(\Omega_{5R_j} \cap E(\lambda)\right)$, *with* $\Omega_{5R_j} = B_{5R_j}(x_j) \cap \Omega$.

◇ $\rho_{x_j}(R_j) = \lambda$ *and* $\rho_{x_j}(R) < \lambda$ *for any* $R_j < R < R_0$, *where*

$$\rho_{x_j}(R) := \fint_{B_R(x_j)\cap\Omega} \left(|Du|^p + \mathbf{V}|u|^p\right) dx + \frac{1}{\delta^p} \fint_{B_R(x_j)\cap\Omega} |f|^p dx.$$

Proof. Let $\lambda > 0$. For every $x \in E(\lambda)$, we define the function $\rho_x : (0, R_0] \to \mathbb{R}^+$ by

$$\rho_x(R) = \fint_{B_R(x)\cap\Omega} \left(|Du|^p + \mathbf{V}|u|^p\right) dx + \frac{1}{\delta^p} \fint_{B_R(x)\cap\Omega} |f|^p dx.$$

By elementary calculation we can show that there exists a constant $M_0 > 0$ such that

$$\rho_x(R) \leqslant \frac{|\Omega|}{|B_R(x)\cap\Omega|} \lambda_0 \leqslant M_0 \lambda_0,$$

for any $R_0/20 \leqslant R \leqslant R_0$. So we have $\rho_x(R) < \lambda$ provided that $\lambda > M_0\lambda_0$ and $R_0/20 \leqslant R \leqslant R_0$. On the other hand, it follows from Lebesgue's differentiation theorem that

$$\lim_{R\to 0^+} \rho_x(R) > \lambda.$$

Due to the continuity of the function $R \mapsto \rho_x(R)$, we deduce that for any $\lambda > M_0\lambda_0$, there is a constant $R_x \in (0, R_0/20)$ such that $\rho_x(R_x) = \lambda$ and $\rho_x(R) < \lambda$ for $R_x < R \leqslant R_0$.

Finally, applying Vitali's covering lemma for the family of balls

$$\{B_{R_x}(x) : x \in E(\lambda)\}$$

completes the proof of this lemma. □

Proof of Theorem 1.6. The proof includes three steps.

Step 1. Let $\lambda > M_0\lambda_0$ and $E(\lambda) \neq \emptyset$, where M_0 is defined as in Lemma 3.2. We first estimate the size of

$$\Omega_{R_j} = B_{R_j} \cap \Omega.$$

It follows from Lemma 3.2 that either

$$\fint_{\Omega_{R_j}} |f|^p dx \geqslant \frac{\delta^p \lambda}{2} \quad \text{or} \quad \fint_{\Omega_{R_j}} (|Du|^p + \mathbf{V}|u|^p) dx \geqslant \frac{\lambda}{2}.$$

If the former holds then, for any $T > 0$, we have

$$\begin{aligned} |\Omega_{R_j}| &\leqslant \frac{2}{\delta^p \lambda} \int_0^\infty \left|\{x \in \Omega_{R_j} : |f(x)|^p > t\}\right| dt \\ &\leqslant \frac{2}{\delta^p \lambda} \left(\int_0^T \left|\{x \in \Omega_{R_j} : |f(x)|^p > t\}\right| dt + \int_T^\infty \left|\{x \in \Omega_{R_j} : |f(x)|^p > t\}\right| dt \right). \end{aligned}$$

Then by choosing $T = \delta^p\lambda/4$, we obtain

$$|\Omega_{R_j}| \leqslant \frac{4}{\delta^p\lambda}\int_{\delta^p\lambda/4}^{\infty}\left|\{x\in\Omega_{R_j} : |f(x)|^p > t\}\right|dt.$$

Similarly, if the latter holds, that is,

$$\fint_{\Omega_{R_j}}(|Du|^p + \mathbf{V}|u|^p)dx \geqslant \frac{\lambda}{2}$$

then we also have

$$|\Omega_{R_j}| \leqslant \frac{4}{\lambda}\int_{\lambda/4}^{\infty}\left|\{x\in\Omega_{R_j} : |Du|^p + \mathbf{V}|u|^p > t\}\right|dt.$$

So by combining the estimates above, we derive

$$\begin{aligned}|\Omega_{R_j}| \leqslant &\frac{4}{\lambda}\int_{\lambda/4}^{\infty}\left|\{x\in\Omega_{R_j} : |Du|^p + \mathbf{V}|u|^p > t\}\right|dt\\ &+\frac{4}{\delta^p\lambda}\int_{\delta^p\lambda/4}^{\infty}\left|\{x\in\Omega_{R_j} : |f(x)|^p > t\}\right|dt. \end{aligned}\tag{3.17}$$

Step 2. Let $\varepsilon > 0$, $\lambda > M_0\lambda_0$ and $E(\lambda) \neq \emptyset$. We then show that there exists a constant $\Upsilon > 1$ such that

$$\begin{aligned}|E(\Upsilon\lambda)| \lesssim \varepsilon &\left[\frac{1}{\lambda}\int_{\lambda/4}^{\infty}\left|\{x\in\Omega : |Du|^p + \mathbf{V}|u|^p > t\}\right|dt\right.\\ &\left.+\frac{1}{\delta^p\lambda}\int_{\delta^p\lambda/4}^{\infty}\left|\{x\in\Omega : |f(x)|^p > t\}\right|dt\right]. \end{aligned}\tag{3.18}$$

Indeed, for $\Upsilon > 1$, it follows from Lemma 3.2 that

$$\begin{aligned}\left|E(\Upsilon\lambda)\right| &\leqslant \sum_{j=1}^{\infty}\left|\left\{x\in\Omega_{5R_j} : |Du(x)|^p + \mathbf{V}(x)|u(x)|^p > \Upsilon\lambda\right\}\right|\\ &\leqslant \sum_{j=1}^{\infty}\left|\left\{x\in\Omega_{5R_j} : |D(u(x)-h_j(x))|^p + \mathbf{V}(x)|u(x)-h_j(x)|^p > \frac{\Upsilon\lambda}{2^{p+1}}\right\}\right|\\ &\quad+\sum_{j=1}^{\infty}\left|\left\{x\in\Omega_{5R_j} : |Dh_j(x)|^p + \mathbf{V}(x)|h_j(x)|^p > \frac{\Upsilon\lambda}{2^{p+1}}\right\}\right|,\end{aligned}$$

where h_j is defined by applying Lemma 3.1 for the case $R = 5R_j$ and $x_0 = x_j$.

For $j\in\mathbb{N}$, we set

$$\begin{aligned}\mathscr{O}_j^1 &:= \left\{x\in\Omega_{5R_j} : |D(u(x)-h_j(x))|^p + \mathbf{V}(x)|u(x)-h_j(x)|^p > \frac{\Upsilon\lambda}{2^{p+1}}\right\},\\ \mathscr{O}_j^2 &:= \left\{x\in\Omega_{5R_j} : |Dh_j(x)|^p + \mathbf{V}(x)|h_j(x)|^p > \frac{\Upsilon\lambda}{2^{p+1}}\right\}.\end{aligned}$$

Then it follows from Lemma 3.1 and Lemma 3.2 that

$$|\mathscr{O}_j^2| \leqslant \frac{2^{p+1}}{\Upsilon\lambda}\int_{\Omega_{5R_j}} \big(|Dh_j|^p + \mathbf{V}|h_j|^p\big)dx \lesssim \Upsilon^{-1}|\Omega_{5R_j}|$$

and

$$|\mathscr{O}_j^1| \leqslant \frac{2^{p+1}}{\Upsilon\lambda}\int_{\Omega_{5R_j}} \big(|Du-Dh_j|^p + \mathbf{V}|u-h_j|^p\big)dx \lesssim \varepsilon\Upsilon^{-1}|\Omega_{5R_j}|.$$

Therefore, by choosing Υ sufficiently large and using the following fact

$$\frac{|B_R(x)|}{|B_R(x)\cap\Omega|} \leqslant \left(\frac{2}{1-\delta}\right)^n \leqslant 4^n,$$

for $x\in\Omega$ and $0<R<R_0$, we obtain

$$|E(\Upsilon\lambda)| \lesssim \varepsilon\sum_{j=1}^{\infty}|\Omega_{5R_j}| \lesssim \varepsilon\sum_{j=1}^{\infty}|B_{R_j}| \lesssim \varepsilon\sum_{j=1}^{\infty}|\Omega_{R_j}|.$$

Finally, the estimates above together with (3.17) yield (3.18).

Step 3. In light of Remark 2.10, one has

$$\int_\Omega \phi(|Du|^p)dx + \int_\Omega \phi(\mathbf{V}|u|^p)dx \leqslant 2\int_0^\infty |E(\lambda))|d[\phi(\lambda)] \leqslant 2\int_0^\infty |E(\Upsilon\lambda)|d[\phi(\Upsilon\lambda)],$$

where the constant Υ is chosen in Step 2. On the other hand, we can write

$$\int_0^\infty |E(\Upsilon\lambda)|d[\phi(\Upsilon\lambda)] = \int_0^{\hat\lambda} |E(\Upsilon\lambda)|d[\phi(\Upsilon\lambda)] + \int_{\hat\lambda}^\infty |E(\Upsilon\lambda)|d[\phi(\Upsilon\lambda)] = \mathscr{I}_1 + \mathscr{I}_2,$$

where $\hat\lambda > M_0\lambda_0$ is defined by

$$\hat\lambda := (M_0+N_0)\lambda_0 = (M_0+N_0)\left(\fint_\Omega \big(|Du|^p + \mathbf{V}|u|^p\big)dx + \frac{1}{\delta^p}\fint_\Omega |f|^p dx\right),$$

for some $N_0>0$. Now we estimate the terms $\mathscr{I}_1$ and $\mathscr{I}_2$.

Estimate $\mathscr{I}_1 = \int_0^{\hat\lambda} |E(\Upsilon\lambda)|d[\phi(\Upsilon\lambda)]$:

Since $\hat\lambda \leqslant C\fint_\Omega |f|^p dx$, it follows from the properties of ϕ and Jensen's inequality that

$$\begin{aligned}\mathscr{I}_1 &\leqslant \phi(\Upsilon\hat\lambda)|\Omega| \leqslant C|\Omega|\,\phi\left(\fint_\Omega |f|^p dx\right)\\ &\leqslant C(n,p,\delta,\phi,\Omega)\int_\Omega \phi\left(|f|^p\right)dx.\end{aligned}$$

Estimate $\mathscr{I}_2 = \int_{\hat{\lambda}}^{\infty} \left|E(\Upsilon\lambda)\right| d[\phi(\Upsilon\lambda)]$:

It follows from (3.18) that

$$\mathscr{I}_2 \lesssim \varepsilon \int_{\hat{\lambda}}^{\infty} \frac{1}{\lambda} \int_{\lambda/4}^{\infty} \left|\{x \in \Omega_{R_j} : |Du(x)|^p + \mathbf{V}(x)|u(x)|^p > t\}\right| dt\, d[\phi(\Upsilon\lambda)]$$
$$+ \frac{\varepsilon}{\delta^p} \int_{\hat{\lambda}}^{\infty} \frac{1}{\lambda} \int_{\delta^p\lambda/4}^{\infty} \left|\{x \in \Omega_{R_j} : |f(x)|^p > t\}\right| dt\, d[\phi(\Upsilon\lambda)].$$

By interchanging the order of integration and using Lemma 2.7, we obtain

$$\int_{\hat{\lambda}}^{\infty} \frac{1}{\lambda} \int_{\delta^p\lambda/4}^{\infty} \left|\{x \in \Omega_{R_j} : |f(x)|^p > t\}\right| dt\, d[\phi(\Upsilon\lambda)]$$
$$\leqslant \int_0^{\infty} \left|\{x \in \Omega_{R_j} : |f(x)|^p > t\}\right| dt \int_{\hat{\lambda}}^{4t\delta^{-p}} \frac{1}{\lambda} d[\phi(\Upsilon\lambda)]$$
$$\lesssim \int_0^{\infty} \left|\{x \in \Omega_{R_j} : |f(x)|^p > t\}\right| \phi'(t) dt \lesssim \int_{\Omega} \phi\left(|f|^p\right) dx.$$

Similarly, we also have

$$\int_{\hat{\lambda}}^{\infty} \frac{1}{\lambda} \int_{\lambda/4}^{\infty} \left|\{x \in \Omega_{R_j} : |Du(x)|^p + \mathbf{V}(x)|u(x)|^p > t\}\right| dt\, d[\phi(\Upsilon\lambda)]$$
$$\lesssim \int_{\Omega} \phi\left(|Du|^p\right) dx + \int_{\Omega} \phi\left(\mathbf{V}|u|^p\right) dx.$$

Combining the estimates above and taking ε small enough yield

$$\mathscr{I}_2 \leqslant C(n,p,\delta,\phi,\Omega). \int_{\Omega} \phi\left(|f|^p\right) dx.$$

Therefore, we complete the proof of Theorem 1.6. □

4. Proof of Lemma 3.1

In order to prove Lemma 3.1, we need to establish some key results related to the following homogeneous equation

$$-\operatorname{div}\left((\mathbf{A}Dw.Dw)^{\frac{p-2}{2}} \mathbf{A}Dw\right) + \mathbf{V}|w|^{p-2}w = 0 \quad \text{in} \quad B_\rho, \tag{4.19}$$

for some ball B_ρ of $\mathbb{R}^n$. We begin with the following Caccioppoli-type estimate.

PROPOSITION 4.1. *Let w be a solution to the equation* (4.19). *Then there exists a positive constant $C = C(n,p,\Lambda)$ such that*

$$\int_{B_\tau} \left(|Dw|^p + \mathbf{V}|w|^p\right) dx \leqslant C \int_{B_t} \frac{|w|^p}{(t-\tau)^p} dx,$$

for all balls $B_\tau \subset B_t \subset B_\rho$, where B_ρ is mentioned in (4.19).

Proof. Let $\eta \in C_c^\infty(B_t)$ be a smooth function satisfying the following conditions

$$0 \leqslant \eta \leqslant 1, \quad \eta \equiv 1 \text{ on } B_\tau, \quad \text{and} \quad |D\eta| \lesssim \frac{1}{t-\tau}.$$

By using $\phi = w\eta^p$ as a test function we have

$$\int_{B_t} \eta^p \left((\mathbf{A}Dw.Dw)^{\frac{p}{2}} + \mathbf{V}.|w|^p \right) dx = -p \int_{B_t} w\eta^{p-1} (\mathbf{A}Dw.Dw)^{\frac{p-2}{2}} \mathbf{A}Dw.D\eta dx.$$

In addition, it follows from the property of uniform ellipticity that

$$\int_{B_t} \eta^p \left((\mathbf{A}Dw.Dw)^{\frac{p}{2}} + \mathbf{V}.|w|^p \right) dx \geqslant \Lambda^{-1} \int_{B_t} \eta^p \left(|Dw|^p + \mathbf{V}|w|^p\right) dx.$$

Since A is uniformly bounded, we obtain

$$-p \int_{B_t} \eta^{p-1} w.(\mathbf{A}Dw.Dw)^{\frac{p-2}{2}} \mathbf{A}Dw.D\eta dx \leqslant p\Lambda \int_{B_t} \eta^{p-1} |Dw|^{p-1} |w| |D\eta| dx.$$

By Young's inequality, it is clear to see that for any $\varepsilon > 0$,

$$p\Lambda \int_{B_t} \eta^{p-1} |Dw|^{p-1} |w| |D\eta| dx \leqslant \varepsilon \int_{B_t} \eta^p |Dw|^p dx + C(\varepsilon, p, \Lambda) \int_{B_t} \frac{|w|^p}{(t-\tau)^p} dx.$$

Finally, combining the estimates above and taking ε small enough lead to

$$\int_{B_\tau} \left(|Dw|^p + \mathbf{V}|w|^p\right) dx \leqslant C \int_{B_t} \frac{|w|^p}{(t-\tau)^p} dx,$$

which completes the proof of this proposition. □

Next, in light of Proposition 4.1 and the improved Fefferman-Phong inequality (see [2, Lemma 4.1]), we obtain the following result.

PROPOSITION 4.2. *Let w be a solution to the equation* (4.19). *Then there exist positive constants*

$$C(n,p,\Lambda,[\mathbf{V}]_{A_\infty}) \quad \text{and} \quad \beta \in (0,1)$$

depending only on the A_∞ constant of $\mathbf{V}$, p and n such that for every ball $B_R \subset B_\rho$, the following inequalities hold true

$$\Big(R^p \fint_{B_R} \mathbf{V} dx\Big)^{k\beta} \fint_{B_R} |w|^p dx \leqslant C \fint_{B_{2R}} |w|^p dx, \tag{4.20}$$

and

$$\Big(R^p \fint_{B_R} \mathbf{V} dx\Big)^{k\beta} \fint_{B_R} \big(|Dw|^p + \mathbf{V}|w|^p\big) dx \leqslant C \fint_{B_{2R}} \big(|Dw|^p + \mathbf{V}|w|^p\big) dx, \tag{4.21}$$

where $k = \left[\frac{1}{\beta}\right]$.

Proof. If $R^p \fint_{B_R} \mathbf{V} dx \leqslant 1$ then (4.20) and (4.21) are obvious. If $R^p \fint_{B_R} \mathbf{V} dx > 1$ then it follows from the improved Fefferman-Phong inequality ([2, Lemma 4.1]) that there exist positive constants

$$C(n, p, \Lambda, [\mathbf{V}]_{A_\infty}) \quad \text{and} \quad \beta \in (0,1)$$

depending only on the A_∞ constant of $\mathbf{V}$, p and n such that for any $\kappa > 0$,

$$\frac{1}{R^p}\Big(R^p \fint_{B_{\kappa R}} \mathbf{V} dx\Big)^{\beta} \int_{B_{\kappa R}} |w|^p dx \leqslant c \int_{B_{\kappa R}} \big(|Dw|^p + \mathbf{V}|w|^p\big) dx. \tag{4.22}$$

Set $k = \left[\dfrac{1}{\beta}\right]$, and take $1 = s_0 < s_1 < \cdots < s_k = 2$, with

$$s_i - s_{i-1} = \frac{1}{k}, \quad i \in \{1, 2, \ldots, k\}.$$

Then in view of Proposition 4.1 we have

$$\int\limits_{B_{s_{i-1}R}} \big(|Dw|^p + \mathbf{V}|w|^p\big) dx \leqslant C \int\limits_{B_{s_i R}} \frac{|w|^p}{R^p} dx. \tag{4.23}$$

Therefore, it follows from (4.22) and (4.23) that

$$\Big(R^p \fint_{B_{\kappa R}} \mathbf{V} dx\Big)^{\beta} \int\limits_{B_{s_{i-1}R}} \big[|Dw|^p + \mathbf{V}|w|^p\big] dx \leqslant C \int\limits_{B_{s_i R}} \big(|Dw|^p + \mathbf{V}|w|^p\big) dx, \tag{4.24}$$

and

$$\Big(R^p \fint_{B_{\kappa R}} \mathbf{V} dx\Big)^{\beta} \int\limits_{B_{s_{i-1}R}} |w|^p dx \leqslant C \int\limits_{B_{s_i R}} |w|^p dx. \tag{4.25}$$

Hence iterating (4.24) yields

$$\left(R^p \fint_{B_R} \mathbf{V} dx\right)^{k\beta} \int_{B_R} (|Dw|^p + \mathbf{V}|w|^p)\, dx \leqslant C \int_{B_{2R}} (|Dw|^p + \mathbf{V}|w|^p)\, dx,$$

which leads to (4.21). Similarly, iterating (4.25) gives (4.20). □

We shall now prove the following lemma of Gehring-type inequality.

LEMMA 4.3. *Assume that the assumptions of Theorem 1.6 hold true. Let w be a solution to the equation*

$$-\operatorname{div}\left((\mathbf{A}Dw.Dw)^{\frac{p-2}{2}}\mathbf{A}Dw\right) + \mathbf{V}|w|^{p-2}w = 0 \quad \text{in } \Omega_R = B_R(x_0) \cap \Omega, \tag{4.26}$$

where $0 < R < R_0/20$ and either $x_0 \in \Omega$, $B_R(x_0) \subset \Omega$ or $x_0 \in \partial\Omega$. Then there exists a positive constant ε_0 such that for $0 < \mu \leqslant \varepsilon_0$,

$$\begin{aligned}&\left(\fint_{\Omega_{R/2}} \left(|Dw|^{p(1+\mu)} + \mathbf{V}^{1+\mu}|w|^{p(1+\mu)}\right) dx\right)^{\frac{1}{1+\mu}} \\ &\leqslant C(n,p,\Lambda) \fint_{\Omega_R} (|Dw|^p + \mathbf{V}|w|^p)\, dx. \end{aligned}\tag{4.27}$$

Proof. We consider the following cases.

Case 1*:* $\Omega_R = B_R(x_0) \subset \Omega$.

First we note that by using [10, Proposition V.1.1], it suffices to prove that there exist constants

$$C(n,p,\Lambda) > 0, \quad 1 < \nu \leqslant p \quad \text{and} \quad \theta \in (0,1)$$

such that

$$\begin{aligned}\fint_{B_{r/4}} (|Dw|^p + \mathbf{V}|w|^p)\, dx \leqslant& \theta \fint_{B_r} (|Dw|^p + \mathbf{V}|w|^p) dx \\ &+ C(n,p,\Lambda)\left(\fint_{B_r} |Dw|^{\frac{p}{\nu}} dx + \fint_{B_r} \mathbf{V}^{\frac{1}{\nu}} |w|^{\frac{p}{\nu}} dx\right)^{\nu},\end{aligned}\tag{4.28}$$

for any ball $B_r \subset \Omega_R$.

Let $\eta \in C_c^\infty\left(B_{r/2}\right)$ be a smooth function satisfying the conditions

$$0 \leqslant \eta \leqslant 1, \quad \eta \equiv 1 \text{ on } B_{r/4}, \quad \text{and} \quad |D\eta| \lesssim \frac{1}{r}.$$

By choosing $\phi = \eta^p(w - (w)_{B_{r/2}})$ as a test function, we have

$$\begin{aligned}\fint_{B_{r/2}} \eta^p\big((\mathbf{A}Dw.Dw)^{\frac{p}{2}} + \mathbf{V}|w|^p\big)dx \\ = -p\fint_{B_{r/2}} \eta^{p-1}(w-(w)_{B_{r/2}})(\mathbf{A}Dw.Dw)^{\frac{p-2}{2}}(\mathbf{A}Dw.D\eta)dx \\ + \fint_{B_{r/2}} \eta^p\mathbf{V}|w|^{p-2}w(w)_{B_{r/2}}dx = J_1 + J_2, \quad (4.29)\end{aligned}$$

where

$$J_1 := -p\fint_{B_{r/2}} \eta^{p-1}(w-(w)_{B_{r/2}})(\mathbf{A}Dw.Dw)^{\frac{p-2}{2}}(\mathbf{A}Dw.D\eta)dx,$$

$$J_2 := \fint_{B_{r/2}} \eta^p\mathbf{V}|w|^{p-2}w(w)_{B_{r/2}}dx.$$

Using similar arguments as in the proof of Proposition 4.1, we deduce that

$$\text{LHS of (4.29)} \geqslant \Lambda^{-1}\fint_{B_{r/2}} \eta^p\left(|Dw|^p + \mathbf{V}|w|^p\right)dx. \quad (4.30)$$

Next we estimate two terms in the right-hand side of (4.29). It follows from the uniform boundedness of A and Young's inequality with $\varepsilon > 0$ that

$$\begin{aligned}J_1 &\leqslant p\Lambda\fint_{B_{r/2}} \eta^{p-1}|w-(w)_{B_{r/2}}||Dw|^{p-1}|D\eta|dx \\ &\leqslant \varepsilon\fint_{B_{r/2}} \eta^p|Dw|^p dx + C(\varepsilon,p,\Lambda)\fint_{B_{r/2}} \frac{|w-(w)_{B_{r/2}}|^p}{r^p}dx.\end{aligned}$$

Then it follows from Sobolev-Poincaré's inequality

$$\fint_{B_{r/2}} \frac{|w-(w)_{B_{r/2}}|^p}{r^p}dx \leqslant C(n,p)\left(\fint_{B_{r/2}} |Dw|^{\frac{p}{\nu}}dx\right)^{\nu},$$

where $1 < \nu \leqslant \dfrac{n+p}{n}$, that

$$J_1 \leqslant \varepsilon\fint_{B_{r/2}} \eta^p|Dw|^p dx + C(\varepsilon,n,p,\Lambda)\Big(\fint_{B_{r/2}} |Dw|^{\frac{p}{\nu}}dx\Big)^{\nu}. \quad (4.31)$$

Next, we estimate the second term of the right-hand side of (4.29) as follows:

$$\begin{aligned}J_2 &\leqslant \fint_{B_{r/2}} \eta^p\mathbf{V}|w|^{p-1}|(w)_{B_{r/2}}|dx \\ &\leqslant |(w)_{B_{r/2}}|\left(\fint_{B_{r/2}} \mathbf{V}^{\frac{1}{\nu}}|w|^{\frac{p}{\nu}}dx\right)^{\frac{(p-1)\nu}{p}}\left(\fint_{B_{r/2}} \mathbf{V}^{\frac{1}{p-\nu(p-1)}}dx\right)^{\frac{p-\nu(p-1)}{p}}.\end{aligned}$$

Then in light of Young's inequality, it is clear to see that

$$J_2 \leqslant C(\theta)\left(\fint_{B_{r/2}} \mathbf{V}^{\frac{1}{\nu}}|w|^{\frac{p}{\nu}}dx\right)^{\nu} + \theta|(w)_{B_{r/2}}|^p\left(\fint_{B_{r/2}} \mathbf{V}^{\frac{1}{p-\nu(p-1)}}dx\right)^{p-\nu(p-1)}.$$

Since $\mathbf{V} \in RH_\gamma$ with $\gamma > 1$, we can take $\nu > 1$ such that $\mathbf{V} \in RH_{\frac{1}{p-\nu(p-1)}}$. As a result of that, we obtain

$$J_2 \leqslant C(\theta)\left(\fint_{B_{r/2}} \mathbf{V}^{\frac{1}{\nu}}|w|^{\frac{p}{\nu}}dx\right)^{\nu} + \theta|(w)_{B_{r/2}}|^p\fint_{B_{r/2}} \mathbf{V}dx. \tag{4.32}$$

Observe that the improved Fefferman-Phong's inequality ([2, Lemma 4.1]) gives

$$|(w)_{B_{r/2}}|^p\fint_{B_{r/2}} \mathbf{V}dx \lesssim \frac{r^p \fint_{B_{r/2}} \mathbf{V}dx}{m_\beta\left(r^p \fint_{B_{r/2}} \mathbf{V}dx\right)}\fint_{B_{r/2}} (|Dw|^p + \mathbf{V}|w|^p)dx, \tag{4.33}$$

where $m_\beta(z) = 1$ if $z \leqslant 1$ and $m_\beta(z) = z^\beta$ if $z > 1$.

We now combine the estimates from (4.29) to (4.33) and choose ε small enough to get

$$\begin{aligned}\fint_{B_{r/2}} \eta^p (|Dw|^p + \mathbf{V}|w|^p)\,dx &\leqslant C(\theta,n,p,\Lambda)\left(\fint_{B_{r/2}} |Dw|^{\frac{p}{\nu}}dx + \fint_{B_{r/2}} \mathbf{V}^{\frac{1}{\nu}}|w|^{\frac{p}{\nu}}dx\right)^{\nu} \\ &\quad + \theta\underbrace{\frac{r^p \fint_{B_{r/2}} \mathbf{V}dx}{m_\beta\left(r^p \fint_{B_{r/2}} \mathbf{V}dx\right)}\fint_{B_{r/2}} (|Dw|^p + \mathbf{V}|w|^p)dx}_{K}.\end{aligned} \tag{4.34}$$

At this stage, if $r^p \fint_{B_{r/2}} \mathbf{V}dx \leqslant 1$, then it is clear to see that

$$K \leqslant \fint_{B_{r/2}} (|Dw|^p + \mathbf{V}|w|^p)dx \lesssim \fint_{B_r} (|Dw|^p + \mathbf{V}|w|^p)dx. \tag{4.35}$$

Otherwise, if $r^p \fint_{B_{r/2}} \mathbf{V}dx > 1$ then by applying Proposition 4.2, we derive

$$K \lesssim \frac{\left(r^p \fint_{B_{r/2}} \mathbf{V}dx\right)^{1-\beta}}{\left(r^p \fint_{B_{r/2}} \mathbf{V}\right)^{[\frac{1}{\beta}]\beta}}\fint_{B_r} \left(|Dw|^p + \mathbf{V}|w|^p\right)dx \lesssim \fint_{B_r} (|Dw|^p + \mathbf{V}|w|^p)dx, \tag{4.36}$$

which together with (4.34)–(4.35) yields

$$\fint_{B_{r/4}} (|Dw|^p + \mathbf{V}|w|^p)\,dx \leqslant \theta \fint_{B_r} (|Dw|^p + \mathbf{V}|w|^p)dx \\ + C(n,p,\Lambda)\left(\fint_{B_r} |Dw|^{\frac{p}{\nu}}dx + \fint_{B_r} \mathbf{V}^{\frac{1}{\nu}}|w|^{\frac{p}{\nu}}dx\right)^{\nu},$$

for some $\theta \in (0,1)$.

Case 2: $x_0 \in \partial\Omega$.

We extend w by zero in $\mathbb{R}^n \setminus \Omega$. In view of [10, Proposition V.1.1] again, it suffices to prove that

$$\fint_{B_r(y)} (|Dw|^p + \mathbf{V}|w|^p)\,dx \leqslant \theta \fint_{B_{4r}(y)} (|Dw|^p + \mathbf{V}|w|^p)dx \\ + C(n,p,\Lambda)\left(\fint_{B_{4r}(y)} |Dw|^{\frac{p}{\nu}}dx + \fint_{B_{4r}(y)} \mathbf{V}^{\frac{1}{\nu}}|w|^{\frac{p}{\nu}}dx\right)^{\nu}, \tag{4.37}$$

for all balls $B_{4r}(y) \subset B_R(x_0)$, and for some $\nu \in (1,p)$.

Since Ω is a (δ,R_0)-Reifenberg flat domain with $\delta \in (0,1/2]$, there exists a coordinate system $\{y_1,y_2,\ldots,y_n\}$ such that in this coordinate system, the origin is an interior point of Ω and

$$x_0 = \left(0,\ldots,0,\frac{-\delta r}{1-\delta}\right),$$

$$B_r^+(0) \subset B_r(0)\cap\Omega \subset B_r(0)\cap\{y_n > -4\delta r\},$$

where $B_r^+(0) = B_r(0)\cap\{y_n > 0\}$. Hence, if we restrict $\delta < 1/5$ then

$$B_r(x_0) \subset B_{\frac{5r}{4}}(0) \subset B_{\frac{5r}{2}}(0) \subset B_{2r}(0) \subset B_{4r}(x_0).$$

Now notice that for the balls centered at 0, we can use the following Sobolev-Poincaré's inequality near the boundary of a Reifenberg domain.

LEMMA 4.4. [13, Lemma 3.1] *Let $\Omega \subset \mathbb{R}^n$ is a (δ,R_0)-Reifenberg flat domain and let $x_0 \in \partial\Omega$, $r < R_0$. Suppose that u is a p-quasi continuous function in $W^{1,p}(\Omega_r(x_0))$ with $p \in (1,\infty)$. Then we have*

$$\left(\fint_{\Omega_r(x_0)} |u|^{\kappa p}dx\right)^{\frac{1}{\kappa p}} \leqslant cr\left(\fint_{B_r(x_0)} |D\bar{u}|^p dx\right)^{1/p}, \tag{4.38}$$

where $c = c(n,p) > 0$, $\bar{u}$ is the zero extension of u from $\Omega_r(x_0)$ to $B_r(x_0)$, and

$$\kappa = \begin{cases} 2 & \text{if } p \geqslant n, \\ \frac{n}{n-p} & \text{if } 1 < p < n. \end{cases}$$

In particular, we have

$$\left(\fint_{\Omega_r(x_0)} |u|^p dx\right)^{\frac{1}{p}} \leqslant cr \left(\fint_{B_r(x_0)} |D\bar{u}|^p dx\right)^{1/p}. \tag{4.39}$$

At this stage, by using the similar arguments as in Case 1, we can obtain

$$\begin{aligned}\fint_{B_r(y)} (|Dw|^p + \mathbf{V}|w|^p)\,dx &\leqslant C \fint_{B_{5r/4}(0)} (|Dw|^p + \mathbf{V}|w|^p)\,dx \\ &\leqslant \theta \fint_{B_{5r/2}} (|Dw|^p + \mathbf{V}|w|^p) dx \\ &\quad + C(n,p,\Lambda)\left(\fint_{B_{5r/2}} |Dw|^{\frac{p}{\nu}} dx + \fint_{B_{5r/2}} \mathbf{V}^{\frac{1}{\nu}} |w|^{\frac{p}{\nu}} dx\right)^{\nu},\end{aligned}$$

which implies (4.37). The proof of Lemma 4.3 is thus completed. □

Proof of Lemma 3.1. *Step* 1. Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of (1.1) and $w \in u + W_0^{1,p}(B_{2R})$ be a weak solution of the following localized homogeneous equation:

$$\begin{cases} -\operatorname{div} \mathscr{A}(x, Dw) + \mathbf{V}|w|^{p-2}w = 0 & \text{in } \Omega_{2R}, \\ w = u & \text{on } \partial\Omega_{2R}, \end{cases} \tag{4.40}$$

where $\mathscr{A}(x,\xi) = (\mathbf{A}\xi.\xi)^{\frac{p-2}{2}} \mathbf{A}\xi, \xi \in \mathbb{R}^n$.

We extend u by zero outside Ω and extend w from Ω_{2R} to $\mathbb{R}^n$ by the extension of u. These extensions are still denoted by u and w respectively.

First, by using $\varphi = w - u$ as a test function, we have

$$\begin{aligned}&\int_{\Omega_{2R}} \big(\mathscr{A}(x,Du) - \mathscr{A}(x,Dw)\big)(Du - Dw) dx \\ &\quad + \int_{\Omega_{2R}} \mathbf{V}\big(|u|^{p-2}u - |w|^{p-2}w\big)(u-w) dx = \int_{\Omega_{2R}} |f|^{p-2} f.(Du - Dw) dx.\end{aligned}$$

From Lemma 2 in [27], it follows that

$$\begin{aligned}&\Lambda^{-1} \int_{\Omega_{2R}} \big(|Du|^2 + |Dw|^2\big)^{\frac{p-2}{2}} |Du - Dw|^2 dx \\ &\quad + \int_{\Omega_{2R}} \mathbf{V}\big(|u|^{p-2}u - |w|^{p-2}w\big)(u-w) dx \leqslant \int_{\Omega_{2R}} |f|^{p-1}.|Du - Dw| dx.\end{aligned}$$

If $p \geqslant 2$, by applying the following elementary inequality

$$|a-b|^2 \big(|a|^2 + |b|^2\big)^{\frac{p-2}{2}} \geqslant C_p |a-b|^p,$$

for all $a,b \in \mathbb{R}^n$, one gets

$$\fint_{\Omega_{2R}} (|D(u-w)|^p + \mathbf{V}|u-w|^p)\,dx \leqslant C(n,p,\Lambda) \fint_{\Omega_{2R}} |f|^{p-1}|Du-Dw|dx. \tag{4.41}$$

If $1<p<2$ then it follows from the elementary inequalities

$$|a-b|^p \leqslant \varepsilon(|a|+|b|)^p + C(p,\varepsilon)(|a|+|b|)^{p-2}|a-b|^2,$$
$$(|a|^{p-2}a-|b|^{p-2}b).(a-b) \geqslant C(p)(|a|^2+|b|^2)^{\frac{p-2}{2}}|a-b|^2,$$

for all $a,b \in \mathbb{R}^n$ and $\varepsilon > 0$, that

$$\fint_{\Omega_{2R}} (|D(u-w)|^p + \mathbf{V}|u-w|^p)\,dx \leqslant C(\varepsilon,n,p,\Lambda) \fint_{\Omega_{2R}} |f|^{p-1}|Du-Dw|dx + \varepsilon \fint_{\Omega_{2R}} \left(|Du|^2+|Dw|^2\right)^{\frac{p}{2}} dx + \varepsilon \fint_{\Omega_{2R}} \mathbf{V}\left(|u|^2+|w|^2\right)^{\frac{p}{2}} dx. \tag{4.42}$$

Then in light of Young's inequality, (4.41) and (4.42), we derive

$$\fint_{\Omega_{2R}} (|D(u-w)|^p + \mathbf{V}|u-w|^p)\,dx \leqslant C(\varepsilon,\kappa,n,p,\Lambda) \fint_{\Omega_{2R}} |f|^p + \varepsilon \fint_{\Omega_{2R}} \left(|Du|^2+|Dw|^2\right)^{\frac{p}{2}} dx + \varepsilon \fint_{\Omega_{2R}} \mathbf{V}\left(|u|^2+|w|^2\right)^{\frac{p}{2}} dx, \tag{4.43}$$

for some small $\kappa > 0$ and for any small $\varepsilon > 0$. As a result of that, for any $\varepsilon > 0$, the following estimate holds true

$$\fint_{\Omega_{2R}} (|D(u-w)|^p + \mathbf{V}|u-w|^p)\,dx \lesssim \varepsilon \fint_{\Omega_{2R}} \left(|Du|^p + \mathbf{V}|u|^p\right) dx + \fint_{\Omega_{2R}} |f|^p dx. \tag{4.44}$$

Step 2. Now we consider a weak solution h of the following problem

$$\begin{cases} -\operatorname{div} \overline{\mathscr{A}}_{\Omega_R}(Dh) + \mathbf{V}|h|^{p-2}h = 0 & \text{in } \Omega_R, \\ h = w & \text{on } \partial\Omega_R, \end{cases} \tag{4.45}$$

with $\overline{\mathscr{A}}_{\Omega_R} : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\overline{\mathscr{A}}_{\Omega_R}(\xi) = \left(\bar{\mathbf{A}}_{\Omega_R}\xi.\xi\right)^{\frac{p-2}{2}} \bar{\mathbf{A}}_{\Omega_R}\xi, \quad \bar{\mathbf{A}}_{\Omega_R} = \fint_{\Omega_R} \mathbf{A}(x)dx, \xi \in \mathbb{R}^n.$$

By taking $\eta = w-h$ as a test function for (4.40) and (4.45), we derive

$$\fint_{\Omega_R} \left(\overline{\mathscr{A}}_{\Omega_R}(Dw) - \overline{\mathscr{A}}_{\Omega_R}(Dh)\right).D(w-h)dx + \fint_{\Omega_R} \mathbf{V}(|w|^{p-2}w - |h|^{p-2}h)(w-h)dx = -\fint_{\Omega_R} \left(\mathscr{A}(x,Dw) - \overline{\mathscr{A}}_{\Omega_R}(Dw)\right).D(w-h)dx.$$

Using similar arguments as in Step 1 yields

$$\fint_{\Omega_R} (|D(w-h)|^p + \mathbf{V}|w-h|^p)\,dx \lesssim \varepsilon \fint_{\Omega_R} \left(|Dw|^p + \mathbf{V}|w|^p\right) dx + \fint_{\Omega_R} \left|\mathscr{A}(x,Dw) - \bar{\mathscr{A}}_{B_R}(Dw)\right| . |D(w-h)|\,dx,$$

for any $\varepsilon > 0$. Put

$$\Gamma(\mathbf{A},\Omega_R) = \frac{\left|\mathscr{A}(x,Dw) - \bar{\mathscr{A}}_{B_R}(Dw)\right|}{|Dw|^{p-1}}.$$

Then, in view of (1.4), we obtain

$$\begin{aligned}\fint_{\Omega_R} \left|\mathscr{A}(x,Dw) - \overline{\mathscr{A}}_{\Omega_R}(Dw)\right| |D(w-h)|\,dx &\leqslant \fint_{\Omega_R} \Gamma(\mathbf{A},\Omega_R)|Dw|^{p-1}|Dw-Dh|\,dx \\ &\leqslant \left(\fint_{\Omega_R} |Dw-Dh|^p dx\right)^{\frac{1}{p}} \left(\fint_{\Omega_R} |Dw|^{p(1+\mu)} dx\right)^{\frac{p-1}{p(1+\mu)}} \left(\fint_{\Omega_R} (\Gamma(\mathbf{A},\Omega_R))^{s_0} dx\right)^{\frac{1}{s_0}} \\ &\leqslant \delta \left(\fint_{\Omega_R} |Dw-Dh|^p dx\right)^{\frac{1}{p}} \left(\fint_{\Omega_R} |Dw|^{p(1+\mu)} dx\right)^{\frac{p-1}{p(1+\mu)}},\end{aligned}$$

where $\mu < \varepsilon_0$ as in Lemma 4.3 and $s_0 = \dfrac{p(1+\mu)}{\mu(p-1)}$. This leads to

$$\fint_{\Omega_R} \left|\mathscr{A}(x,Dw) - \overline{\mathscr{A}}_{\Omega_R}(Dw)\right| |D(w-h)|\,dx \leqslant \delta \left(\fint_{\Omega_R} |Dw-Dh|^p dx\right)^{\frac{1}{p}} \left(\fint_{\Omega_{2R}} \left(|Dw|^p + \mathbf{V}|w|^p dx\right)\right)^{\frac{p-1}{p}}.$$

Hence

$$\fint_{\Omega_R} (|D(w-h)|^p + \mathbf{V}|w-h|^p)\,dx \lesssim \varepsilon \fint_{\Omega_R} \left(|Dw|^p + \mathbf{V}|w|^p\right) dx + \delta \left(\fint_{\Omega_R} |Dw-Dh|^p dx\right)^{\frac{1}{p}} \left(\fint_{\Omega_{2R}} \left(|Dw|^p + \mathbf{V}|w|^p\right) dx\right)^{\frac{p-1}{p}}.$$

Finally, by using Young's inequality and choosing a suitable ε, we derive

$$\fint_{\Omega_R} (|D(w-h)|^p + \mathbf{V}|w-h|^p)\,dx \lesssim \delta^{\frac{p}{p-1}} \fint_{\Omega_{2R}} \left(|Dw|^p + \mathbf{V}|w|^p\right) dx. \tag{4.46}$$

Finally, combining (4.44) and (4.46) gives

$$\begin{aligned}&\fint_{\Omega_R} (|D(u-h)|^p + \mathbf{V}|u-h|^p)\,dx \\ &\quad\lesssim \fint_{\Omega_R} (|D(u-w)|^p + \mathbf{V}|u-w|^p)\,dx + \fint_{\Omega_R} (|D(w-h)|^p + \mathbf{V}|w-h|^p)\,dx \\ &\quad\leqslant \delta^{\frac{p}{p-1}} \fint_{\Omega_{2R}} \left(|Du|^p + \mathbf{V}|u|^p\right) dx + \fint_{\Omega_{2R}} |f|^p dx.\end{aligned}$$

This means that (3.15) is proved. Then (3.16) is only a consequence of (3.15). □

5. Proof of Theorem 1.7 by approximation

Recall that the given bounded, open domain Ω is (δ,R) is Reifenberg flat. Now for each small $\delta>0$, we write

$$\Omega_\varepsilon=\{x\in\Omega: d(x,\partial\Omega)>\varepsilon\},$$

where d is the standard distance function defined by

$$d(x,y)=|x-y|$$

and

$$d(x,\partial\Omega)=\inf\{d(x,y): y\in\partial\Omega\}\ (x\in\Omega).$$

It is well-known that an ε inner neighborhood of the (δ,R)-Reifenberg flat domain is a Lipschitz domain with the (δ,R)-Reifenberg flat property for δ small; that is, Ω_ε is a Lipschitz domain with the uniform (δ,R)-Reifenberg flat (see [5, Lemma 4.2]). Therefore, according to a standard approximation of a Lipschitz domain by smooth domains, we can construct a further approximation of Ω_ε for any fixed small $\delta>0$ by smooth domains $\Omega_\varepsilon^\eta\subset\Omega_\varepsilon$ with the uniform (δ,R)-Reifenberg flat property for a properly chosen $\eta=\eta(\varepsilon)>0$.

Next we use a standard diagonal argument to extract a subsequence of smooth domains Ω^k with the uniform (δ,R)-Reifenberg flat property such that

$$\Omega^k\subset\Omega \text{ and } d_H(\partial\Omega^k,\partial\Omega)\to 0 \text{ as } k\to\infty, \tag{5.47}$$

where the Hausdorff distance d_H is defined as follows:

$$d_H(X,Y)=\max\left\{\sup_{x\in X}\inf_{y\in Y}d(x,y),\sup_{y\in Y}\inf_{x\in X}d(x,y)\right\}.$$

Proof of Theorem 1.7. Given a Young function $\phi\in\Delta_2\cap\nabla_2$, we choose $\{\mathbf{A}_k\}$ to be a sequence of smooth functions with the uniform ellipticity and the uniform (δ,R)-vanishing property converging to $\mathbf{A}$ in L^q for any $1<q<\infty$, and $\{f_k\}$ and $\{\mathbf{V}_k\}$ to be sequences of smooth functions in $C_c^\infty(\Omega,\mathbb{R}^n)$ and $C_c^\infty(\Omega,\mathbb{R})$ respectively such that

$$f_k\to f \quad \text{in} \quad L^p(\Omega,\mathbb{R}^n), \quad |f_k|^p\to|f|^p \quad \text{in} \quad L^\phi(\Omega), \tag{5.48}$$

$$\int_\Omega |f_k|^p dx\leqslant C\int_\Omega |f|^p dx,\ \int_\Omega \phi(|f_k|^p)dx\leqslant C\int_\Omega \phi(|f|^p)dx, \tag{5.49}$$

and

$$\mathbf{V}_k\to\mathbf{V} \quad \text{in} \quad L^p(\Omega,\mathbb{R}), \quad \mathbf{V}_k\in RH_\gamma. \tag{5.50}$$

According to the standard theory for nonlinear uniformly elliptic equations of p-Laplacian type with the corresponding smooth data on smooth domains, the following Dirichlet problems

$$\begin{cases} \operatorname{div}\left((\mathbf{A}_k.Du_k)^{\frac{p-2}{2}}\mathbf{A}_k Du_k\right)+\mathbf{V}_k|u_k|^{p-2}u_k=\operatorname{div}(|f_k|^{p-2}f_k) & \text{in} \quad \Omega^k, \\ u_k=0 & \text{on} \quad \partial\Omega^k \end{cases} \tag{5.51}$$

have unique weak solutions $u_k \in W_0^{1,p}(\Omega^k)$ with the regularity $u_k \in C^{1,\alpha}(\bar{\Omega}^k)$ for some $\alpha=\alpha(n,p,k)\in(0,1)$ and $u_k=0$ on $\partial\Omega^k$ in the classical sense. In addition, these solutions satisfy

$$|Du_k|^p, \mathbf{V}_k|u_k|^p \in L^\phi(\Omega^k). \tag{5.52}$$

Then it follow from Theorem 1.6 and (5.52) that these solutions have the uniform gradient estimates in Orlicz space with respect to the above approximation; that is,

$$\int_{\Omega^k}\phi(|Du_k|^p)dx+\int_{\Omega^k}\phi(\mathbf{V}_k|u_k|^p)dx \leqslant C\int_{\Omega^k}\phi(|f_k|^p)dx, \tag{5.53}$$

where the constant C is independent of $k\in\mathbb{N}$. We extend u_k from Ω^k to Ω by the zero extension and denote by $\bar{u}_k$. Then $\bar{u}_k \in W_0^{1,p}(\Omega)$. In addition, (5.53) and (5.49) imply that

$$\int_{\Omega}\phi(|D\bar{u}_k|^p)dx+\int_{\Omega}\phi(\mathbf{V}_k|\bar{u}_k|^p)dx \leqslant C\int_{\Omega}\phi(|f_k|^p)dx \leqslant C\int_{\Omega}\phi(|f|^p)dx. \tag{5.54}$$

From (5.54) with $\phi(t)\equiv t$, we have

$$\int_{\Omega}|D\bar{u}_k|^p dx+\int_{\Omega}\mathbf{V}_k|\bar{u}_k|^p dx \leqslant C\int_{\Omega}|f_k|^p dx \leqslant C\int_{\Omega}|f|^p dx. \tag{5.55}$$

Therefore, there exist a subsequence of $\bar{u}_k$ (still denoted by $\bar{u}_k$) and a function $v\in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} \mathbf{V}_k|\bar{u}_k|^p \to \mathbf{V}|v|^p & \text{strongly in} \quad L^1_{\text{loc}}(\Omega), \\ \bar{u}_k \to v & \text{strongly in} \quad L^p(\Omega), \\ D\bar{u}_k \rightharpoonup Dv & \text{weakly in} \quad L^p(\Omega). \end{cases} \tag{5.56}$$

We claim that

$$D\bar{u}_k \to Dv \quad \text{strongly in} \quad L^p_{\text{loc}}(\Omega). \tag{5.57}$$

The proof of this claim will be given later. Then it follows from (5.57) and the particular selection of $\mathbf{A}_k$ and f_k that $v\in W_0^{1,p}(\Omega)$ is also a weak solution of (1.1). Thanks to the uniqueness of the weak solution to (1.1), we deduce that $v=u$ and

$$D\bar{u}_k \to Du \quad \text{strongly in} \quad L^p_{\text{loc}}(\Omega). \tag{5.58}$$

Consequently, we can use a standard diagonal argument to extract a subsequence of $\bar{u}_k$ (still denoted by $\bar{u}_k$) such that

$$D\bar{u}_k \to Du \quad \text{a.e. in} \quad \Omega. \tag{5.59}$$

Finally, applying Fatou's lemma to the left-hand side of (5.54) yields (1.6):

$$\int_\Omega \phi(|Du|^p)dx + \int_\Omega \phi(\mathbf{V}|u|^p)dx \leqslant C\int_\Omega \phi(|f|^p)dx.$$

We now prove the claim (5.57). We only consider the case that $p \geqslant 2$. The other case $1 < p < 2$ can be addressed in the same way (see [6, 14]). To this end, choose a cut-off function $\zeta \in C_c^\infty(\Omega)$ satisfying

$$p \leqslant \zeta \leqslant 1, \quad \operatorname{supp}\zeta \subset \Omega^2, \quad \zeta = 1 \quad \text{on} \quad \Omega^1.$$

Then function $\varphi = \zeta^p(\bar{u}_m - \bar{u}_n)$ with $m,n \geqslant 2$ is a qualified test function for (5.51) when $k = m$ or $k = n$. Thus we have

$$\int_\Omega (\mathbf{A}_m D\bar{u}_m.D\bar{u}_m)^{\frac{p-2}{2}}\mathbf{A}_m D\bar{u}_m.D[\zeta^p(\bar{u}_m - \bar{u}_n)]dx + \int_\Omega \mathbf{V}_m|\bar{u}_m|^{p-1}\bar{u}_m\varphi dx$$
$$= \int_\Omega |f_m|^{p-2}f_m.D[\zeta^p(\bar{u}_m - \bar{u}_n)]dx,$$

and

$$\int_\Omega (\mathbf{A}_n D\bar{u}_n.D\bar{u}_n)^{\frac{p-2}{2}}\mathbf{A}_n D\bar{u}_n.D[\zeta^p(\bar{u}_m - \bar{u}_n)]dx + \int_\Omega \mathbf{V}_n|\bar{u}_n|^{p-1}\bar{u}_n\varphi dx$$
$$= \int_\Omega |f_n|^{p-2}f_n.D[\zeta^p(\bar{u}_m - \bar{u}_n)]dx.$$

After several simple computations, we derive the following equation

$$I_8 + I_1 = \sum_{i=2}^{7} I_i,$$

where

$$I_1 = \int_\Omega \zeta^p[(\mathbf{A}_m D\bar{u}_m.D\bar{u}_m)^{\frac{p-2}{2}}\mathbf{A}_m D\bar{u}_m - (\mathbf{A}_m D\bar{u}_n.D\bar{u}_n)^{\frac{p-2}{2}}\mathbf{A}_m D\bar{u}_n].D(\bar{u}_m - \bar{u}_n)dx$$
$$+ \int_\Omega \xi^p \mathbf{V}_m[|\overline{u}_m|^{p-2}\overline{u}_m - |\overline{u}_n|^{p-2}\overline{u}_n](\bar{u}_m - \bar{u}_n)dx,$$
$$I_2 = -p\int_\Omega \zeta^{p-1}(\bar{u}_m - \bar{u}_n)(\mathbf{A}_m D\bar{u}_m.D\bar{u}_m)^{\frac{p-2}{2}}\mathbf{A}_m D\bar{u}_m.D\zeta dx,$$
$$I_3 = p\int_\Omega \zeta^{p-1}(\bar{u}_m - \bar{u}_n)(\mathbf{A}_m D\bar{u}_n.D\bar{u}_n)^{\frac{p-2}{2}}\mathbf{A}_m D\bar{u}_n.D\zeta dx,$$
$$I_4 = \int_\Omega p\zeta^{p-1}(\bar{u}_m - \bar{u}_n)\left[|f_m|^{p-2}f_m - |f_n|^{p-2}f_n\right].D\zeta dx,$$

$$I_5 = \int_\Omega \zeta^p \left[|f_m|^{p-2} f_m - |f_n|^{p-2} f_n\right].D(\bar{u}_m - \bar{u}_n)dx,$$

$$I_6 = -\int_\Omega \zeta^p D(\bar{u}_m - \bar{u}_n).\left[(\mathbf{A}_m D\bar{u}_n.D\bar{u}_n)^{\frac{p-2}{2}} \mathbf{A}_m D\bar{u}_n - (\mathbf{A}_n D\bar{u}_n.D\bar{u}_n)^{\frac{p-2}{2}} \mathbf{A}_n D\bar{u}_n\right] dx,$$

$$I_7 = -p\int_\Omega \zeta^{p-1}(\bar{u}_m - \bar{u}_n) D\zeta.\left[(\mathbf{A}_m D\bar{u}_n.D\bar{u}_n)^{\frac{p-2}{2}} \mathbf{A}_m D\bar{u}_n - (\mathbf{A}_n D\bar{u}_n.D\bar{u}_n)^{\frac{p-2}{2}} \mathbf{A}_n D\bar{u}_n\right] dx,$$

$$I_8 = \int_\Omega \xi^p (\mathbf{V}_m - \mathbf{V}_n)|\bar{u}_n|^{p-2}\bar{u}_n(\bar{u}_m - \bar{u}_n)dx.$$

Estimate I_1. Since A_m is uniformly elliptic, the vector valued function $a(\xi,x) = (\mathbf{A}_m(x) \times \xi.\xi)^{\frac{p-2}{2}} \mathbf{A}_m(x)\xi$ is strictly monotonic; that is, there is a positive constant c_0 such that

$$\left[\left(\mathbf{A}_m(x)\xi.\xi\right)^{\frac{p-2}{2}} \mathbf{A}_m(x)\xi - \left(\mathbf{A}_m(x)\eta.\eta\right)^{\frac{p-2}{2}} \mathbf{A}_m(x)\eta\right].[\xi-\eta] \geqslant c_0|\xi-\eta|^p,$$

for all $\xi,\eta \in \mathbb{R}^n$. This observation, together with the elementary inequality

$$(|x|^{p-2}x - |y|^{p-2}y).(x-y) \geqslant 0,$$

for all $x,y \in \mathbb{R}^n$, implies that

$$I_1 \geqslant c_0 \int_\Omega \zeta^p \left|D(\bar{u}_m - \bar{u}_n)\right|^p dx. \tag{5.60}$$

Estimate I_2, I_3, I_4, I_7, I_8. It follows from the uniform boundedness of A_m and Young's inequality with $\varepsilon > 0$ that

$$I_2 \leqslant \varepsilon \int_\Omega \zeta^p |D\bar{u}_m|^p dx + C(\varepsilon,p)\int_\Omega \left|(\bar{u}_m - \bar{u}_n)\right|^p dx, \tag{5.61}$$

$$I_3 \leqslant \varepsilon \int_\Omega \zeta^p |D\bar{u}_n|^p dx + C(\varepsilon,p)\int_\Omega \left|(\bar{u}_m - \bar{u}_n)\right|^p dx, \tag{5.62}$$

$$I_4 \leqslant \varepsilon \int_\Omega \zeta^p \left(|f_m|^p + |f_n|^p\right) dx + C(\varepsilon,p)\int_\Omega \left|(\bar{u}_m - \bar{u}_n)\right|^p dx, \tag{5.63}$$

$$I_7 \leqslant \varepsilon \int_\Omega \zeta^p |D\bar{u}_n|^p dx + C(\varepsilon,p)\int_\Omega \left|(\bar{u}_m - \bar{u}_n)\right|^p dx, \tag{5.64}$$

$$|I_8| \leqslant \varepsilon \int_\Omega \xi^{\frac{p^2}{p-1}} |\mathbf{V}_m - \mathbf{V}_n|^{\frac{p}{p-1}} |\bar{u}_n|^p dx + C(\varepsilon,p)\int_\Omega |\bar{u}_m - \bar{u}_n|^p dx. \tag{5.65}$$

Estimate I_5. From the following inequality

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right| \leqslant c(p)\left(|\xi| + |\eta|\right)^{p-2}|\xi-\eta|,$$

for all $\xi,\eta \in \mathbb{R}^n$ and for some positive constant $c(p)$, we have

$$I_5 \leqslant c(p)\int_\Omega \zeta^p \left[\left(|f_m| + |f_n|\right)^{p-2}|f_m - f_n|\left|D(\bar{u}_m - \bar{u}_n)\right|\right] dx.$$

Then in view of Young's inequality with $\varepsilon > 0$ and Hölder's inequality, it is clear to see that

$$I_5 \leqslant \varepsilon \int_\Omega \zeta^p \left|D(\bar{u}_m - \bar{u}_n)\right|^p dx + C(p,\varepsilon)\left(\int_\Omega \left(|f_m|^p + |f_n|^p\right) dx\right)^{\frac{p-2}{p-1}} \left(\int_\Omega |f_m - f_n|^p dx\right)^{\frac{1}{p-1}}$$
$$\leqslant \varepsilon \int_\Omega \zeta^p \left|D(\bar{u}_m - \bar{u}_n)\right|^p dx + C(p,\varepsilon)\left(\int_\Omega |f_m - f_n|^p dx\right)^{\frac{1}{p-1}}.$$

Thus, we get

$$I_5 \leqslant \varepsilon \int_\Omega \zeta^p \left|D(\bar{u}_m - \bar{u}_n)\right|^p dx + C(p,\varepsilon)\left(\int_\Omega |f_m - f_n|^p dx\right)^{\frac{1}{p-1}}. \tag{5.66}$$

Estimate I_6. Using the following elementary inequality

$$\left|\left(\mathbf{A}_m \xi.\xi\right)^{\frac{p-2}{2}} \mathbf{A}_m \xi - \left(\mathbf{A}_n \xi.\xi\right)^{\frac{p-2}{2}} \mathbf{A}_n \xi\right| \leqslant C(p)\left|\mathbf{A}_m - \mathbf{A}_n\right| |\xi|^{p-1}.$$

for all $\xi \in \mathbb{R}^n$, we have

$$I_6 \leqslant C(p) \int_\Omega \zeta^p \left[|\mathbf{A}_m - \mathbf{A}_n| \left|D\bar{u}_n\right|^{p-1} \left|D(\bar{u}_m - \bar{u}_n)\right|\right] dx.$$

Now in light of Lemmas 2.5 and 2.9, one observes that for each $v \in L^\phi(\Omega)$, there exist $A_2 > 0$ and $\alpha_2 > 1$ such that

$$\int_\Omega |v|^{\alpha_2} dx \leqslant \int_{\{x \in \Omega : |v| \leqslant 1\}} |v|^{\alpha_2} dx + \int_{\{x \in \Omega : |v| \geqslant 1\}} |v|^{\alpha_2} dx \leqslant |\Omega| + \frac{A_2}{\Omega_1} \int_\Omega \phi(|v|) dx.$$

But then since $|D\bar{u}_n|^p \in L^\phi(\Omega)$ and (5.54), we have

$$\int_\Omega |D\bar{u}_n|^{p\alpha_2} dx \leqslant |\Omega| + \frac{A_2}{\Omega_1} \int_\Omega \phi\left(|D\bar{u}_n|\right) dx < +\infty.$$

Therefore, using Young's inequality with $\varepsilon > 0$ gives

$$\begin{aligned} I_6 \leqslant & \varepsilon \int_\Omega \zeta^p \left|D(\bar{u}_m - \bar{u}_n)\right|^p dx \\ & + C(\varepsilon, p)\left[\int_\Omega |\mathbf{A}_m - \mathbf{A}_n|^{\frac{p\alpha_2}{(p-1)(\alpha_2-1)}} dx\right]^{1-\frac{1}{\alpha_2}} \left[\int_\Omega \zeta^{p\alpha_2} |D\bar{u}_n|^{p\alpha_2} dx\right]^{\frac{1}{\alpha_2}} \\ \leqslant & \varepsilon \int_\Omega \zeta^p \left|D(\bar{u}_m - \bar{u}_n)\right|^p dx + C(\varepsilon, p)\left[\int_\Omega |\mathbf{A}_m - \mathbf{A}_n|^{\frac{p\alpha_2}{(p-1)(\alpha_2-1)}} dx\right]^{1-\frac{1}{\alpha_2}}. \end{aligned}$$

So we can conclude that

$$I_6 \leqslant \varepsilon \int_\Omega \zeta^p \left|D(\bar{u}_m - \bar{u}_n)\right|^p dx + C(\varepsilon, p)\left[\int_\Omega |\mathbf{A}_m - \mathbf{A}_n|^{\frac{p\alpha_2}{(p-1)(\alpha_2-1)}} dx\right]^{1-\frac{1}{\alpha_2}}. \tag{5.67}$$

At this stage, we combine all the estimates from (5.60) to (5.65) to deduce that for every $\varepsilon > 0$,

$$\begin{aligned}
&\int_\Omega \zeta^p \big|D(\bar{u}_m - \bar{u}_n)\big|^p dx + \varepsilon \int_\Omega \xi^{\frac{p^2}{p-1}} |\mathbf{V}_m - \mathbf{V}_n|^{\frac{p}{p-1}} |\bar{u}_n|^p dx \\
&\leqslant C(\varepsilon, p)\Bigg\{ \int_\Omega |\bar{u}_m - \bar{u}_n|^p dx + \left(\int_\Omega |f_m - f_n|^p dx\right)^{\frac{1}{p-1}} \\
&\quad + \left[\int_\Omega |\mathbf{A}_m - \mathbf{A}_n|^{\frac{p\alpha_2}{(p-1)(\alpha_2-1)}} dx\right]^{1-\frac{1}{\alpha_2}} \Bigg\} \\
&\quad + \varepsilon \int_\Omega \zeta^p \left(|D\bar{u}_m|^p + |D\bar{u}_n|^p + |f_m|^p + |f_n|^p\right) dx.
\end{aligned}$$

Finally, due to the strong convergence of $\{\bar{u}_m\}$ in $L^p(\Omega)$, the particular selection of A_k, f_k and the arbitrariness of $\varepsilon > 0$, we conclude that

$$\int_{\Omega^1} \big|D(\bar{u}_m - \bar{u}_n)\big|^p dx \leqslant \int_\Omega \zeta^p \big|D(\bar{u}_m - \bar{u}_n)\big|^p dx \to 0 \quad \text{as } m,n \to \infty.$$

Analogously, for every fixed $k \in \mathbb{N}$ we have

$$\int_{\Omega^k} \big|D(\bar{u}_m - \bar{u}_n)\big|^p dx \to 0 \quad \text{as } m,n \to \infty,$$

which implies the claim (5.57). □

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