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WEIGHTED ESTIMATES FOR A CLASS OF MATRIX OPERATORS

NAZERKE ZHANGABERGENOVA

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Abstract. In this paper, we have obtained criteria for the fulfillment of weighted inequalities for the class of quasilinear discrete operators involving matrix kernels.

1. Introduction

Let $0 < p,q,r < \infty$, $\frac{1}{p}+\frac{1}{p'}=1$ and $f=\{f_i\}_{i=1}^{\infty}$ be an arbitrary sequence of real numbers. Suppose that $u=\{u_i\}_{i=1}^{\infty}$, $v=\{v_i\}_{i=1}^{\infty}$ and $w=\{w_i\}_{i=1}^{\infty}$ are positive sequences of real numbers, which will be called weight sequences. We denote by $l_{p,v}$ the space of sequences f of real numbers such that

$$\|f\|_{p,v}=\left(\sum_{i=1}^{\infty}|v_if_i|^p\right)^{\frac{1}{p}}<\infty.$$

In this work, we consider the following operators

$$(K_1f):=\left(\sum_{k=1}^{n}\left|w_k\sum_{i=k}^{\infty}a_{i,k}f_i\right|^r\right)^{\frac{1}{r}} \tag{1}$$

$$(K_2f):=\left(\sum_{k=n}^{\infty}\left|w_k\sum_{i=1}^{k}a_{k,i}f_i\right|^r\right)^{\frac{1}{r}} \tag{2}$$

where $(a_{i,k})$, $i\geqslant k$ is a matrix, whose non-negative entries satisfy the discrete Oinarov condition: there exists a constant $d\geqslant 1$ such that the inequalities

$$\frac{1}{d}(a_{i,j}+a_{j,k})\leqslant a_{i,k}\leqslant d(a_{i,j}+a_{j,k}) \tag{3}$$

or, equivalently, the relation $a_{i,k}\approx a_{i,j}+a_{j,k}$ hold for all $i\geqslant j\geqslant k\geqslant 1$. In view of (3), one can say that $(a_{i,k})$ is almost non-decreasing in i and almost non-increasing in k. We will study the following iterated discrete Hardy-type inequalities

$$\left(\sum_{n=1}^{\infty}u_n^q(K_1f)^q\right)^{\frac{1}{q}}\leqslant C\left(\sum_{i=1}^{\infty}|v_if_i|^p\right)^{\frac{1}{p}}, \tag{4}$$

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$$\left(\sum_{n=1}^{\infty} u_n^q (K_2 f)^q\right)^{\frac{1}{q}} \leqslant C' \left(\sum_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{1}{p}}, \tag{5}$$

where C and C' are positive finite constants independent of $f \in l_{p,v}$.

The purpose of the work is to obtain weighted estimates for the operators (1) and (2) from $l_{p,v}$ to $l_{q,u}$ in the cases: $1 < p \leqslant q < \infty$ and $1 < r < \infty$; $p \leqslant q < \infty$, $0 < p < 1$ and $1 < r < \infty$. Nowadays, these operators and

$$(K_3 f) := \left(\sum_{k=1}^{n} \left| w_k \sum_{i=1}^{k} a_{k,i} f_i \right|^r\right)^{\frac{1}{r}}, \quad (K_4 f) := \left(\sum_{k=n}^{\infty} \left| w_k \sum_{i=k}^{\infty} a_{i,k} f_i \right|^r\right)^{\frac{1}{r}}$$

are being intensively studied. Recently, characterizations of analogous inequalities with quasilinear operators K_3 and K_4 have been found in the work [12]. When $a_{i,k} = 1$ for all $i \geqslant k \geqslant 1$, weighted estimates of the quasilinear Hardy operator

$$(H_w f)_k := w_k \sum_{i=1}^{k} f_i, \quad k \in \mathbb{N}$$

were obtained in the papers [16] and [18] for the different relations of parameters. Also, the work [6] includes weighted estimates for the following discrete iterated Hardy-type inequality

$$\left(\sum_{n \in Z} u_n \left(\sup_{i \geqslant n} w_i \sum_{k \leqslant i} f_k\right)^q\right)^{\frac{1}{q}} \leqslant C \left(\sum_{n \in Z} f_n^p v_n\right)^{\frac{1}{p}}.$$

Moreover, the discrete inequalities involving operators that combine both the kernel and the supremum were discussed in [7].

Characterizations of the continuous analogues of inequalities (4) and (5) have been better studied than discrete versions. For instance, the following weighted integral Hardy-type inequality initially was considered in the works [4] and [13]:

$$\left(\int_0^{\infty} u^q(x) \left((Kf)(t)\right)^q dx\right)^{\frac{1}{q}} \leqslant C \left(\int_0^{\infty} |v(x) f(x)|^p dx\right)^{\frac{1}{p}}, \quad f \in L_{p,v}(0,\infty), \tag{6}$$

for $0 < p,q,r < \infty$, where $u(\cdot)$, $v(\cdot)$ and $w(\cdot)$ are positive functions locally integrable on the interval $(0;\infty)$, $L_{p,v}(0,\infty)$ is a weighted Lebesgue space of functions for which the right side of the inequality (6) is finite and K is a quasilinear operator defined as follows

$$(Kf)(t) := \left(\int_0^x \left| w(t) \int_0^t f(s) ds \right|^r dt\right)^{\frac{1}{r}}. \tag{7}$$

V. Burenkov and R. Oinarov showed the equivalence of the inequality (6) to the inequality, which determines the boundedness of the multidimensional Hardy operator from the Lebesgue space to the local Morrey-type space (see [3]). After this work, weighted estimates of quasilinear Hardy-type operators came into use, and there was a great interest in the study of related inequalities, which began to be studied intensively (see [5], [19], [21]). Necessity and sufficient conditions for the fulfillment of the inequality (6) with kernel were obtained in papers [10], [11] and [14]. The next step was to investigate bilinear Hardy inequalities. Characteristics of bilinear Hardy inequalities follow from the characteristics of iterated inequalities (see [8], [9]).

For the case $0 < p \leqslant 1$ continuous Hardy-type inequalities hold in trivial case only (see [19]), but in this case discrete Hardy-type inequalities can be explored. Therefore, it is important to note that in this paper we consider the case $0 < p \leqslant 1$.

2. Auxiliary statements

To establish inequality (4), we use known theorems and classical inequalities. Let us present them.

THEOREM A. *Let* $0 < p \leqslant 1$, $p \leqslant q < \infty$. *The inequality*

$$\left(\sum_{k=1}^{\infty} u_k^q \Big|\sum_{i=k}^{\infty} f_i\Big|^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{1}{p}}, \quad f \in l_{p,v}, \tag{8}$$

holds if and only if $A < \infty$, *where*

$$A = \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} u_i^q\right)^{\frac{1}{q}} v_j^{-1}.$$

Moreover, $C \approx A$, *where* C *is the best constant in* (8).

THEOREM B. *Let* $1 < p \leqslant q < \infty$. *Then the inequality* (8) *holds if and only if*

$$D = \sup_{j\geqslant 1}\left(\sum_{k=1}^{j} u_k^q\right)^{\frac{1}{q}}\left(\sum_{i=j}^{\infty} v_i^{-p'}\right)^{\frac{1}{p'}} < \infty.$$

Moreover, $C \approx D$, *where* C *is the best constant in* (8).

Theorem A is proved in [1] (Theorem 1(iv)). Theorem B is considered in [2] (Theorem 2)).

Weighted discrete Hardy-type inequality for one class of matrix operators has the following form

$$\left(\sum_{k=\alpha}^{\beta} u_k^q \Big|\sum_{i=k}^{\beta} a_{i,k} f_i\Big|^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{i=\alpha}^{\beta} |v_i f_i|^p\right)^{\frac{1}{p}}, \quad f \in l_{p,v}, \tag{9}$$

where $[\alpha,\beta] \subset \mathbb{N}$ and the entries of the matrix $(a_{i,k})$, $i \geqslant k$, satisfy discrete Oinarov condition. The related to the inequality (9) boundedness of these matrix operators was studied in [15], [17], [20] and [22].

THEOREM C. *Let $p \leqslant q < \infty$ and $0 < p \leqslant 1$. Let entries $a_{i,k}$ of a matrix $(a_{i,k})$ be non-decreasing in the first index. Then inequality (9) holds if and only if*

$$E = \sup_{\alpha \leqslant j \leqslant \beta} \left(\sum_{k=1}^{j} a_{j,k}^{q} u_k^{q} \right)^{\frac{1}{q}} v_j^{-1} < \infty$$

holds. Moreover, $C \approx E$, where C is the best constant in (9).

Theorem C follows from Corollary 3.2 of [20].

THEOREM D. (see [17]) *Let $1 < p \leqslant q < \infty$ and the entries of the matrix $(a_{i,k})$ satisfy condition (3). Then the inequality (9) holds if and only if $F = \max\{F_1, F_2\} < \infty$, where*

$$F_1 = \sup_{\alpha \leqslant j \leqslant \beta} \left(\sum_{i=j}^{\beta} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{k=\alpha}^{j} a_{j,k}^{q} u_k^{q} \right)^{\frac{1}{q}},$$

$$F_2 = \sup_{\alpha \leqslant j \leqslant \beta} \left(\sum_{i=j}^{\beta} a_{i,j}^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{k=\alpha}^{j} u_k^{q} \right)^{\frac{1}{q}}.$$

Moreover, $C \approx F$, where C is the best constant in (9).

THEOREM E. (see [15]) *Let $1 < q < p < \infty$ and the entries of the matrix $(a_{i,k})$ satisfy condition (3). Then the inequality (9) holds if and only if $M = \max\{M_1, M_2\} < \infty$, where*

$$M_1 = \left(\sum_{k=\alpha}^{\beta} \left(\sum_{i=k}^{\beta} a_{i,k}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{j=\alpha}^{k} u_j^{q} \right)^{\frac{q}{p-q}} u_k^{q} \right)^{\frac{p-q}{pq}},$$

$$M_2 = \left(\sum_{k=\alpha}^{\beta} \left(\sum_{j=\alpha}^{k} a_{k,j}^{q} u_j^{q} \right)^{\frac{p}{p-q}} \left(\sum_{i=k}^{\beta} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} v_k^{-p'} \right)^{\frac{p-q}{pq}}.$$

Moreover, $C \approx M$, where C is the best constant in (9).

We provide proofs only for the inequality (4), since the proofs for the inequality (5) are similar. Therefore, we have decided not to give analogues of Theorems A, B, C, D and E for inequalities (8) and (9) with corresponding conjugate operators. Theorems A and C are applied for the case $p \leqslant q < \infty$, $p \in (0,1]$, other three theorems are applied for the case $1 < p \leqslant q < \infty$. For our proofs we also need the following Lemma.

LEMMA 1. *Let $r > 0$, $1 \leqslant n < N \leqslant \infty$. Then*

$$\sum_{k=n}^{N} a_k \left(\sum_{j=k}^{N} a_j \right)^{r-1} \approx \left(\sum_{i=n}^{N} a_i \right)^{r} \approx \sum_{k=n}^{N} a_k \left(\sum_{j=n}^{k} a_j \right)^{r-1}. \tag{10}$$

We also use the following elementary inequalities in our estimates: if $a_i > 0$, $i = 1,2,...,k$, then

$$\left(\sum_{i=1}^{k} a_i\right)^{\alpha} \leqslant \sum_{i=1}^{k} a_i^{\alpha}, \ 0 < \alpha < 1, \tag{11}$$

and

$$\left(\sum_{i=1}^{k} a_i\right)^{\alpha} \geqslant \sum_{i=1}^{k} a_i^{\alpha}, \ \alpha \geqslant 1. \tag{12}$$

We also need the following quantities:

$$J^{-}_{r,p}(\alpha,\beta) = \sup_{f\neq 0} \frac{\left(\sum\limits_{k=\alpha}^{\beta} \left|w_k \sum\limits_{i=k}^{\beta} a_{i,k} f_i\right|^r\right)^{\frac{1}{r}}}{\left(\sum\limits_{i=\alpha}^{\beta} |v_i f_i|^p\right)^{\frac{1}{p}}}, \ f \in l_{p,v},$$

$$J^{+}_{r,p}(\alpha,\beta) = \sup_{f\neq 0} \frac{\left(\sum\limits_{k=\alpha}^{\beta} \left|w_k \sum\limits_{i=\alpha}^{k} a_{k,i} f_i\right|^r\right)^{\frac{1}{r}}}{\left(\sum\limits_{i=\alpha}^{\beta} |v_i f_i|^p\right)^{\frac{1}{p}}}, \ f \in l_{p,v}.$$

Convention: The symbol $E \ll F$ means $E \leqslant CF$ with some constant C, depending on the parameters p, q and r. Moreover, the notation $E \approx F$ means $E \ll F \ll E$.

3. Main results for $0 < p \leqslant 1$, $p \leqslant q < \infty$.

THEOREM 1. *Let $0 < p \leqslant 1$, $p \leqslant q < \infty$ and $1 < r < \infty$. Let the entries of the matrix $(a_{i,k})$ satisfy condition (3). Then inequality (4) holds if and only if $B^+ = \max\{B_1^+, B_2^+\} < \infty$, where*

$$B_1^+ = \sup_{j\geqslant 1} \left(\sum_{n=j}^{\infty} u_n^q\right)^{\frac{1}{q}} J^{-}_{r,p}(1,j),$$

$$B_2^+ = \sup_{j\geqslant 1} \left(\sum_{n=1}^{j} u_n^q \left(\sum_{k=1}^{n} a_{j,k}^r w_k^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} v_j^{-1}.$$

Moreover, $C \approx B^+$, where C is the best constant in (4).

Proof. Necessity. Suppose that inequality (4) holds with the best constant $C > 0$. Let us show that $B_2^+ < \infty$. We choose $j \geqslant 1$ arbitrary and we take a test sequence

$\widetilde{f}_j = \{\widetilde{f}_{j,i}\}_{i=1}^{\infty}$ defined by $\widetilde{f}_{j,i} = v_i^{-1}$ for $i = j$ and $\widetilde{f}_{j,i} = 0$ for $i \neq j$. Then

$$\|\widetilde{f}_j\|_{v,p} = \left(\sum_{i=1}^{\infty} |\widetilde{f}_{j,i} \cdot v_i|^p\right)^{\frac{1}{p}} = 1. \tag{13}$$

Substituting $\widetilde{f}_j$ in the left-hand side of inequality (4), we deduce that

$$\begin{aligned} I(\widetilde{f}) &:= \left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{s=1}^{n} \left|w_s \sum_{i=s}^{\infty} a_{i,s}\widetilde{f}_{j,i}\right|^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \\ &\geqslant \left(\sum_{n=1}^{j} u_n^q \left(\sum_{s=1}^{n} w_s^r \left(\sum_{i=j}^{\infty} a_{i,s}\widetilde{f}_{j,i}\right)^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}. \end{aligned}$$

By taking into account that $a_{i,s} \geqslant a_{j,s}$ for $i \geqslant j$, we have

$$\begin{aligned} I(\widetilde{f}) &\geqslant \left(\sum_{n=1}^{j} u_n^q \left(\sum_{s=1}^{n} a_{j,s}^r w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \sum_{i=j}^{\infty} \widetilde{f}_{j,i} \\ &\geqslant \left(\sum_{n=1}^{j} u_n^q \left(\sum_{s=1}^{n} a_{j,s}^r w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} v_j^{-1}. \end{aligned} \tag{14}$$

From (13), (14) and (4) it follows that

$$\left(\sum_{n=1}^{j} u_n^q \left(\sum_{s=1}^{n} a_{j,s}^r w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} v_j^{-1} \leqslant C, \ \forall j \geqslant 1.$$

Since $j \geqslant 1$ is arbitrary, we have

$$B_2^+ = \sup_{j\geqslant 1} \left(\sum_{n=1}^{j} u_n^q \left(\sum_{s=1}^{n} w_s^r a_{j,s}^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} v_j^{-1} \leqslant C < \infty. \tag{15}$$

Suppose that

$$f_{j,i} = \begin{cases} f_i, & 1 \leqslant i \leqslant j, \\ 0, & j < i < \infty. \end{cases}$$

$$I(f) \geqslant \left(\sum_{n=j}^{\infty} u_n^q \left(\sum_{s=1}^{n} \left|w_s \sum_{i=s}^{\infty} a_{i,s} f_{j,i}\right|^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \geqslant \left(\sum_{n=j}^{\infty} u_n^q\right)^{\frac{1}{q}} \left(\sum_{s=1}^{j} \left|w_s \sum_{i=s}^{j} a_{i,s} f_i\right|^r\right)^{\frac{1}{r}}.$$

From the validity of (4) we have

$$C\left(\sum_{i=1}^{j}|v_if_i|^p\right)^{\frac{1}{p}} \geqslant \left(\sum_{n=j}^{\infty}u_n^q\right)^{\frac{1}{q}}\left(\sum_{s=1}^{j}\left|w_s\sum_{i=s}^{j}a_{i,s}f_i\right|^r\right)^{\frac{1}{r}}.$$

Hence,

$$C \geqslant \left(\sum_{n=j}^{\infty}u_n^q\right)^{\frac{1}{q}}\sup_{f\geqslant 0}\frac{\left(\sum_{s=1}^{j}\left|w_s\sum_{i=s}^{j}a_{i,s}f_i\right|^r\right)^{\frac{1}{r}}}{\left(\sum_{i=1}^{j}|v_if_i|^p\right)^{\frac{1}{p}}} = \left(\sum_{n=j}^{\infty}u_n^q\right)^{\frac{1}{q}}J_{r,p}^{-}(1,j),$$

$$B_1^+ \leqslant C. \tag{16}$$

The inequalities (15) and (16) give that

$$B^+ \leqslant C < \infty. \tag{17}$$

Sufficiency. Suppose that $B^+ < \infty$. Now we prove that inequality (4) holds for a finite constant C. Without loss of generality we assume that $0 \leqslant f \in l_{p,v}$.

If $0 < q \leqslant 1$, we have

$$\begin{aligned}
I(f) &= \left(\sum_{n=1}^{\infty}u_n^q\left(\sum_{s=1}^{n}w_s^r\left(\sum_{i=s}^{\infty}a_{i,s}f_i\right)^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \\
&\leqslant \left(\sum_{n=1}^{\infty}u_n^q\left(\left(\sum_{s=1}^{n}w_s^r\left(\sum_{i=s}^{n}a_{i,s}f_i\right)^r\right)^{\frac{1}{r}}+\left(\sum_{s=1}^{n}w_s^r\left(\sum_{i=n}^{\infty}a_{i,s}f_i\right)^r\right)^{\frac{1}{r}}\right)^q\right)^{\frac{1}{q}} \\
&\leqslant 2^{\frac{1}{q}-1}\left[\left(\sum_{n=1}^{\infty}u_n^q\left(\sum_{s=1}^{n}w_s^r\left(\sum_{i=s}^{n}a_{i,s}f_i\right)^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\sum_{n=1}^{\infty}u_n^q\left(\sum_{s=1}^{n}w_s^r\left(\sum_{i=n}^{\infty}a_{i,s}f_i\right)^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

By condition (3), we obtain

$$\begin{aligned}
I(f) &\leqslant 2^{\frac{1}{q}-1}\left(\sum_{n=1}^{\infty}u_n^q\left(\sum_{s=1}^{n}w_s^r\left(\sum_{i=s}^{n}a_{i,s}f_i\right)^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \\
&\quad+2^{\frac{1}{q}-1}\left(\sum_{n=1}^{\infty}u_n^q\left(\left(\sum_{s=1}^{n}w_s^r\left(\sum_{i=n}^{\infty}a_{i,n}f_i\right)^r\right)^{\frac{1}{r}}+\left(\sum_{s=1}^{n}w_s^r\left(\sum_{i=n}^{\infty}a_{n,s}f_i\right)^r\right)^{\frac{1}{r}}\right)^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 4^{\frac{1}{q}-1}\left[\left(\sum_{n=1}^{\infty} u_n^q\left(\sum_{s=1}^{n} w_s^r\right)^{\frac{q}{r}}\left(\sum_{i=n}^{\infty} a_{i,n} f_i\right)^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\sum_{n=1}^{\infty} u_n^q\left(\sum_{s=1}^{n} a_{n,s}^r w_s^r\right)^{\frac{q}{r}}\left(\sum_{i=n}^{\infty} f_i\right)^q\right)^{\frac{1}{q}}\right]\\
&\quad+2^{\frac{1}{q}-1}\left(\sum_{n=1}^{\infty} u_n^q\left(\sum_{s=1}^{n} w_s^r\left(\sum_{i=s}^{n} a_{i,s} f_i\right)^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\\
&= 4^{\frac{1}{q}-1}\left[I_1+I_2\right]+2^{\frac{1}{q}-1} I_3. \qquad (18)
\end{aligned}
$$

If $q>1$, by using discrete Minkowski inequality, we obtain

$$
I(f) \leqslant I_1+I_2+I_3. \qquad (19)
$$

This means that we need separately estimate I_1, I_2 and I_3. Let us start with I_1. By Theorem C, we have

$$
I_1 \leqslant\left\{\sup_{j \geqslant 1}\left(\sum_{n=1}^{j} a_{j,n}^q u_n^q\left(\sum_{k=1}^{n} w_k^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} v_j^{-1}\right\}\|f\|_{p,v}. \qquad (20)
$$

Taking into account that $a_{j,n} \leqslant a_{j,k}$ for $n \geqslant k$, we get

$$
I_1 \leqslant\left\{\sup_{j \geqslant 1}\left(\sum_{n=1}^{j} u_n^q\left(\sum_{k=1}^{n} a_{j,k}^r w_k^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} v_j^{-1}\right\}\|f\|_{p,v} \leqslant B_2^{+}\|f\|_{p,v}. \qquad (21)
$$

Let us estimate I_2. By Theorem A, we have

$$
I_2 \leqslant\left\{\sup_{j \geqslant 1}\left(\sum_{n=1}^{j} u_n^q\left(\sum_{k=1}^{n} a_{n,k}^r w_k^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} v_j^{-1}\right\}\|f\|_{p,v}. \qquad (22)
$$

Since $a_{n,k} \leqslant a_{j,k}$ for $j \geqslant n$ and from (22) we get

$$
I_2 \leqslant B_2^{+}\|f\|_{p,v}. \qquad (23)
$$

For all $n \geqslant 1$ we consider the following set:

$$
L_n=\max\left\{l \in \mathbb{Z}: \sum_{s=1}^{n}\left(w_s \sum_{i=s}^{n} a_{i,s} f_i\right)^r \geqslant 2^{rl}\right\}
$$

Then for any $n \geqslant 1$:

$$2^{rL_n} \leqslant \sum_{s=1}^{n}\left(w_s\sum_{i=s}^{n}a_{i,s}f_i\right)^r < 2^{r(L_n+1)} \tag{24}$$

Let $n_1 = 1$ and $M_1 = \{n \in \mathbb{N} : L_n = L_{n_1} = L_1\}$. We will define the value n_2 as $n_2 = \sup M_1 + 1$. Obviously $n_2 > n_1$.If the set M_1 is bounded from above, then $n_2 < \infty$ and $n_2 = \max M_1 + 1$. Let $1 = n_1 < n_2 < ... < n_k < \infty$ be inductively determined for $k \geqslant 1$. Then assume that $n_{k+1} = \sup M_k + 1$ to determine the value n_{k+1}, where $M_k = \{n \in \mathbb{N} : L_n = L_{n_k}\}$. Let $\mathbb{N}_0 = \{k \in \mathbb{N} : n_k < \infty\}$. For convenience, we introduce the notation $L_{n_k} = m_k$. Then from the definition of n_k and by (24) for $k \in \mathbb{N}_0$, we have

$$2^{rm_k} \leqslant \sum_{s=1}^{n}\left(w_s\sum_{i=s}^{n}a_{i,s}f_i\right)^r < 2^{r(m_k+1)}, \ \ n_k \leqslant n \leqslant n_{k+1}-1, \tag{25}$$

and

$$\mathbb{N} = \bigcup_{k\in\mathbb{N}_0}[n_k,n_{k+1}), \ \ [n_k,n_{k+1})\bigcap_{k\neq l}[n_l,n_{l+1}) = \oslash.$$

Therefore, we can write I_3 as follows:

$$I_3 = \left(\sum_{k\in\mathbb{N}_0}\sum_{n=n_k}^{n_{k+1}-1}u_n^q\left(\sum_{s=1}^{n}\left(w_s\sum_{i=s}^{n}a_{i,s}f_i\right)^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \leqslant 4\left(\sum_{k=1}^{k_\infty}2^{q(m_k-1)}\sum_{n=n_k}^{n_{k+1}-1}u_n^q\right)^{\frac{1}{q}}. \tag{26}$$

Using that $m_{k-2}+1 \leqslant m_k - 1$, which is derived from $m_{k-2} < m_{k-1} < m_k$ and from (25) and (3), we have

$$\begin{aligned}
2^{m_k-1} &= 2^{m_k} - 2^{m_k-1} \leqslant 2^{m_k} - 2^{m_{k-2}+1}\\
&\leqslant \left(\sum_{s=1}^{n_k}w_s^r\left(\sum_{i=s}^{n_k}a_{i,s}f_i\right)^r\right)^{\frac{1}{r}} - \left(\sum_{s=1}^{n_{k-1}-1}w_s^r\left(\sum_{i=s}^{n_{k-1}-1}a_{i,s}f_i\right)^r\right)^{\frac{1}{r}}\\
&\leqslant \left(\sum_{s=1}^{n_{k-1}-1}w_s^r\left(\sum_{i=s}^{n_{k-1}-1}a_{i,s}f_i + \sum_{i=n_{k-1}}^{n_k}a_{i,s}f_i\right)^r\right)^{\frac{1}{r}} + \left(\sum_{s=n_{k-1}}^{n_k}w_s^r\left(\sum_{i=s}^{n_k}a_{i,s}f_i\right)^r\right)^{\frac{1}{r}}\\
&\quad - \left(\sum_{s=1}^{n_{k-1}-1}w_s^r\left(\sum_{i=s}^{n_{k-1}-1}a_{i,s}f_i\right)^r\right)^{\frac{1}{r}} \leqslant \left(\sum_{s=1}^{n_{k-1}-1}w_s^r\left(\sum_{i=n_{k-1}}^{n_k}a_{i,s}f_i\right)^r\right)^{\frac{1}{r}}\\
&\quad + \left(\sum_{s=n_{k-1}}^{n_k}w_s^r\left(\sum_{i=s}^{n_k}a_{i,s}f_i\right)^r\right)^{\frac{1}{r}} \ll \left(\sum_{s=1}^{n_{k-1}}w_s^r\right)^{\frac{1}{r}}\sum_{i=n_{k-1}}^{n_k}a_{i,n_{k-1}}f_i\\
&\quad + \left(\sum_{s=1}^{n_{k-1}}a_{n_{k-1},s}^r w_s^r\right)^{\frac{1}{r}}\sum_{i=n_{k-1}}^{n_k}f_i + \left(\sum_{s=n_{k-1}}^{n_k}w_s^r\left(\sum_{i=s}^{n_k}a_{i,s}f_i\right)^r\right)^{\frac{1}{r}}.
\end{aligned} \tag{27}$$

Combining (26) with (27), for $q>1$ we have

$$
\begin{aligned}
I_3 \ll &\left(\sum_{k\in\mathbb{N}_0}\sum_{n=n_k}^{n_{k+1}-1} u_n^q\left(\sum_{s=1}^{n_{k-1}} w_s^r\right)^{\frac{q}{r}}\left(\sum_{i=n_{k-1}}^{n_k} a_{i,n_{k-1}} f_i\right)^q\right)^{\frac{1}{q}}\\
&+\left(\sum_{k\in\mathbb{N}_0}\sum_{n=n_k}^{n_{k+1}-1} u_n^q\left(\sum_{s=1}^{n_{k-1}} a^r_{n_{k-1},s} w_s^r\right)^{\frac{q}{r}}\left(\sum_{i=n_{k-1}}^{n_k} f_i\right)^q\right)^{\frac{1}{q}}\\
&+\left(\sum_{k\in\mathbb{N}_0}\sum_{n=n_k}^{n_{k+1}-1} u_n^q\left(\sum_{s=n_{k-1}}^{n_k} w_s^r\left(\sum_{i=s}^{n_k} a_{i,s} f_i\right)^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\\
= &\; I_{31}+I_{32}+I_{33}.
\end{aligned}\tag{28}
$$

If $0<q\leqslant 1$, we get

$$
I_3 \leqslant 4\cdot 3^{\frac{1}{q}-1}[I_{31}+I_{32}+I_{33}].
$$

Let's start estimate I_{31}. Since $0<p\leqslant 1$, using (11) and (3) we have

$$
\begin{aligned}
I_{31} = &\left(\sum_{k\in\mathbb{N}_0}\sum_{n=n_k}^{n_{k+1}-1} u_n^q\left(\sum_{s=1}^{n_{k-1}} w_s^r\right)^{\frac{q}{r}}\left(\sum_{i=n_{k-1}}^{n_k} a_{i,n_{k-1}} f_i v_i v_i^{-1}\right)^{p\frac{q}{p}}\right)^{\frac{1}{q}}\\
\leqslant &\left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_i f_i|^p\right)^{\frac{q}{p}}\sup_{n_{k-1}\leqslant z\leqslant n_k} a^q_{z,n_{k-1}} v_z^{-q}\sum_{n=n_k}^{n_{k+1}-1} u_n^q\left(\sum_{s=1}^{n_{k-1}} w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\\
\leqslant &\left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_i f_i|^p\right)^{\frac{q}{p}}\sum_{n=n_k}^{n_{k+1}-1} u_n^q\sup_{n_{k-1}\leqslant z\leqslant n_k} v_z^{-q}\left(\sum_{s=1}^{n_{k-1}} a^r_{z,s} w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\\
\leqslant &\left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_i f_i|^p\right)^{\frac{q}{p}}\sum_{n=n_k}^{n_{k+1}-1} u_n^q\sup_{n_{k-1}\leqslant z\leqslant n_k} v_z^{-q}\left(\sum_{s=1}^{z} a^r_{z,s} w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\\
\leqslant &\left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_i f_i|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\sup_{n_k\geqslant 1}\left(\sum_{n=n_k}^{\infty} u_n^q\right)^{\frac{1}{q}}\sup_{1\leqslant z\leqslant n_k} v_z^{-1}\left(\sum_{s=1}^{z} a^r_{z,s} w_s^r\right)^{\frac{1}{r}}\\
\leqslant &\; 2^{\frac{1}{p}}\left(\sum_{i=1}^{\infty}|v_i f_i|^p\right)^{\frac{1}{p}}\sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty} u_n^q\right)^{\frac{1}{q}}\sup_{1\leqslant z\leqslant j} v_z^{-1}\left(\sum_{s=1}^{z} a^r_{z,s} w_s^r\right)^{\frac{1}{r}}.
\end{aligned}
$$

In this case the ratio of the parameters p and r is $p<r<\infty$, $p\in(0,1]$. Therefore, by

Theorem C we have

$$J^{-}_{r,p}(1,j) = \sup_{f\neq 0} \frac{\left(\sum\limits_{k=1}^{j}\left|w_k \sum\limits_{i=k}^{j} a_{i,k} f_i\right|^r\right)^{\frac{1}{r}}}{\left(\sum\limits_{i=1}^{j}|v_i f_i|^p\right)^{\frac{1}{p}}} \approx \sup_{1\leqslant i\leqslant j}\left(\sum_{s=1}^{i} a^r_{i,s} w^r_s\right)^{\frac{1}{r}} v_i^{-1}. \tag{29}$$

Thus,

$$I_{31} \ll B^{+}_1 \|f\|_{p,v}. \tag{30}$$

In the same way, we evaluate I_{32}.

$$\begin{aligned}
I_{32} &\leqslant \left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_i f_i|^p\right)^{\frac{q}{p}} \sup_{n_{k-1}\leqslant z\leqslant n_k} v_z^{-q}\sum_{n=n_k}^{n_{k+1}-1} u^q_n\left(\sum_{s=1}^{n_{k-1}} a^r_{n_{k-1},s} w^r_s\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\\
&\leqslant \left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_i f_i|^p\right)^{\frac{q}{p}} \sum_{n=n_k}^{n_{k+1}-1} u^q_n \sup_{n_{k-1}\leqslant z\leqslant n_k} v_z^{-q}\left(\sum_{s=1}^{n_{k-1}} a^r_{z,s} w^r_s\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\\
&\leqslant \left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_i f_i|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \sup_{n_k\geqslant 1}\left(\sum_{n=n_k}^{\infty} u^q_n\right)^{\frac{1}{q}} \sup_{1\leqslant z\leqslant n_k} v_z^{-1}\left(\sum_{s=1}^{z} a^r_{z,s} w^r_s\right)^{\frac{1}{r}}\\
&\leqslant 2^{\frac{1}{p}} \sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty} u^q_n\right)^{\frac{1}{q}} \sup_{1\leqslant z\leqslant j} v_z^{-1}\left(\sum_{s=1}^{z} a^r_{z,s} w^r_s\right)^{\frac{1}{r}} \|f\|_{p,v}\\
&\ll \sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty} u^q_n\right)^{\frac{1}{q}} J^{-}_{r,p}(1,j)\|f\|_{p,v}.
\end{aligned}$$

Therefore, also

$$I_{32} \ll B^{+}_1 \|f\|_{p,v}. \tag{31}$$

Estimate the last remaining I_{33}

$$\begin{aligned}
I_{33} &= \left(\sum_{k\in\mathbb{N}_0}\sum_{n=n_k}^{n_{k+1}-1} u^q_n \frac{\left(\sum\limits_{s=n_{k-1}}^{n_k} w^r_s\left(\sum\limits_{i=s}^{n_k} a_{i,s} f_i\right)^r\right)^{\frac{q}{r}}}{\left(\sum\limits_{i=n_{k-1}}^{n_k}|v_i f_i|^p\right)^{\frac{q}{p}}}\left(\sum_{i=n_{k-1}}^{n_k}|v_i f_i|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\\
&\leqslant \left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_i f_i|^p\right)^{\frac{q}{p}} \sum_{n=n_k}^{n_{k+1}-1} u^q_n [J^{-}_{r,p}(n_{k-1},n_k)]^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leqslant \left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_if_i|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\sup_{n_k\geqslant 1}\left(\sum_{n=n_k}^{\infty}u_n^q\right)^{\frac{1}{q}}J_{r,p}^{-}(1,n_k)$$

$$\leqslant 2^{\frac{1}{p}}\sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty}u_n^q\right)^{\frac{1}{q}}J_{r,p}^{-}(1,j)\|f\|_{p,v}\ll B_1^+\|f\|_{p,v}. \tag{32}$$

From the inequalities (21), (23), (30), (31) and (32) we have that

$$\left(\sum_{n=1}^{\infty}u_n^q\left(\sum_{s=1}^{n}w_s^r\left(\sum_{i=s}^{\infty}a_{i,s}f_i\right)^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\ll B^+\|f\|_{p,v}. \tag{33}$$

and $C\ll B^+$, where C is the best constant in (4). The latter together with (17), gives $C\approx B^+$. The proof is complete. $\square$

THEOREM 2. *Let $0<p\leqslant 1$, $p\leqslant q<\infty$ and $1<r<\infty$. Let the entries of the matrix $(a_{k,i})$ satisfy condition (3). Then inequality (5) holds if and only if $B^-=\max\{B_1^-,B_2^-\}<\infty$, where*

$$B_1^-=\sup_{j\geqslant 1}\left(\sum_{n=1}^{j}u_n^q\right)^{\frac{1}{q}}J_{r,p}^{+}(j,\infty),$$

$$B_2^-=\sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty}u_n^q\left(\sum_{k=n}^{\infty}a_{k,j}^rw_k^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}v_j^{-1}.$$

Moreover, $C'\approx B^-$, where C' is the best constant in (5).

The proof of Theorem 2 is similar to the proof of Theorem 1.

4. Main results for $1<p\leqslant q<\infty$.

THEOREM 3. *Let $1<p\leqslant q<\infty$ and $1<r<\infty$. Let the entries of the matrix $(a_{i,k})$ satisfy condition (3). Then inequality (4) holds if and only if*

$$M^+=\max\{M_1^+,M_2^+,M_3^+\}<\infty,$$

where

$$M_1^+=\sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty}u_n^q\right)^{\frac{1}{q}}J_{r,p}^{-}(1,j),$$

$$M_2^+=\sup_{j\geqslant 1}\left(\sum_{n=1}^{j}u_n^q\left(\sum_{k=1}^{n}a_{j,k}^rw_k^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\left(\sum_{i=j}^{\infty}v_i^{-p'}\right)^{\frac{1}{p'}},$$

$$M_3^+ = \sup_{j\geqslant 1}\left(\sum_{n=1}^{j} u_n^q\left(\sum_{k=1}^{n} w_k^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\left(\sum_{i=j}^{\infty} a_{i,j}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}}.$$

Moreover, $C \approx M^+$, where C is the best constant in (4).

Proof. Necessity. Suppose that the inequality (4) holds with the best constant $C > 0$. Let us show that $M^+ < \infty$. In the same way as we obtain the estimate $C \geqslant B_1^+$ in the proof of Theorem 1, we get that

$$M_1^+ \leqslant C. \tag{34}$$

Let $1 \leqslant j < N < \infty$ and we take a test sequence $\widetilde{f}_j = \{\widetilde{f}_{j,i}\}_{i=1}^{\infty}$ such that $\widetilde{f}_{j,i} = a_{i,j}^{p'-1} v_i^{-p'}$ for $j \leqslant i \leqslant N$ and $\widetilde{f}_{j,i} = 0$ for $1 \leqslant i < j$ and $N < i$. Then

$$\|\widetilde{f}_j\|_{p,v} = \left(\sum_{i=1}^{\infty} |\widetilde{f}_{j,i} v_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=j}^{N} |a_{i,j}^{p'-1} v_i^{-p'} v_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=j}^{N} a_{i,j}^{p'} v_i^{-p'}\right)^{\frac{1}{p}} < \infty, \tag{35}$$

i.e. $\widetilde{f}_j \in l_{p,v}$. By substituting $\widetilde{f}_j$ in the left-hand side of inequality (4) and taking into account that $a_{i,s} \geqslant a_{i,j}$ for $j \geqslant s$, we have

$$\begin{aligned} I(\widetilde{f}) &\geqslant \left(\sum_{n=1}^{j} u_n^q\left(\sum_{s=1}^{n} w_s^r\left(\sum_{i=j}^{N} a_{i,s}\widetilde{f}_{j,i}\right)^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \\ &\geqslant \left(\sum_{n=1}^{j} u_n^q\left(\sum_{s=1}^{n} w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \sum_{i=j}^{N} a_{i,j}\widetilde{f}_{j,i} \\ &= \left(\sum_{n=1}^{j} u_n^q\left(\sum_{s=1}^{n} w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \sum_{i=j}^{N} a_{i,j}^{p'} v_i^{-p'}. \end{aligned} \tag{36}$$

From (35), (36) and (4) we obtain

$$C \geqslant \left(\sum_{n=1}^{j} u_n^q\left(\sum_{s=1}^{n} w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\left(\sum_{i=j}^{N} a_{i,j}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}}, \quad \text{for all } 1 \leqslant j < N < \infty. \tag{37}$$

Since $j \geqslant 1$ is arbitrary, taking the supremum over j and passing to the limit as $N \to \infty$, we get that

$$M_3^+ = \sup_{j\geqslant 1}\left(\sum_{n=1}^{j} u_n^q\left(\sum_{s=1}^{n} w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\left(\sum_{i=j}^{\infty} a_{i,j}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \leqslant C. \tag{38}$$

Let us show that $M_2^+ < \infty$. Now for $1 \leqslant j < N < \infty$ we suppose that $\widetilde{f}_{j,i} = v_i^{-p'}$ for $j \leqslant i \leqslant N$ and $\widetilde{f}_{j,i} = 0$ for $1 \leqslant i < j$ and $N < i$. Then

$$\|\widetilde{f}_j\|_{p,v} = \left(\sum_{i=j}^{N} v_i^{-p'}\right)^{\frac{1}{p}}, \quad \widetilde{f}_j \in l_{p,v}. \tag{39}$$

Similarly as above, substituting $\widetilde{f}_j$ in the left-hand side of inequality (4) and taking into account that $a_{i,s} \geqslant a_{j,s}$ for $i \geqslant j$, we can deduce that

$$\begin{aligned} I(\widetilde{f}) &\geqslant \left(\sum_{n=1}^{j} u_n^q \left(\sum_{s=1}^{n} a_{j,s}^r w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \sum_{i=j}^{N} \widetilde{f}_{j,i} \\ &= \left(\sum_{n=1}^{j} u_n^q \left(\sum_{s=1}^{n} a_{j,s}^r w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \sum_{i=j}^{N} v_i^{-p'}. \end{aligned} \tag{40}$$

From (39), (40) and (4) it follows that

$$C \geqslant \left(\sum_{n=1}^{j} u_n^q \left(\sum_{s=1}^{n} a_{j,s}^r w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \left(\sum_{i=j}^{N} v_i^{-p'}\right)^{\frac{1}{p'}}, \quad \text{for all } 1 \leqslant j < N < \infty.$$

Similarly as above, by taking the supremum on j and by passing to the limit on N, we find that

$$M_2^+ = \sup_{j \geqslant 1} \left(\sum_{n=1}^{j} u_n^q \left(\sum_{s=1}^{n} a_{j,s}^r w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \left(\sum_{i=j}^{\infty} v_i^{-p'}\right)^{\frac{1}{p'}} \leqslant C. \tag{41}$$

From (34), (38) and (41) we have that

$$M^+ \leqslant C. \tag{42}$$

Sufficiency. Let $M^+ < \infty$. Now we prove that inequality (4) holds. Let $0 \leqslant f \in l_{p,v}$. The sufficient part of Theorem 3 can be proved in the same way as the sufficient part in Theorem 1. In this case since $q \geqslant 1$, in the same way we get $I(f) \leqslant I_1 + I_2 + I_3$, where I_1, I_2 and I_3 are values from (19). We use Theorem D to estimate I_1, then taking into account the condition (3) we obtain the following inequality

$$I_1 \ll \max\{M_2^+, M_3^+\}\|f\|_{p,v}. \tag{43}$$

For estimating I_2 we use Theorem B and condition (3).

$$I_2 \ll M_2^+ \|f\|_{p,v}. \tag{44}$$

To estimate I_3, we obtain the same values I_{31}, I_{32} and I_{33} as in the proof of Theorem 1. Next, to evaluate them we must consider the cases $p \leqslant r$ and $r < p$ separately.

The case $1<p\leqslant r$. By using Hölder's inequality with powers p and p', we obtain that

$$I_{31}\leqslant\left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_if_i|^p\right)^{\frac{q}{p}}\left(\sum_{z=n_{k-1}}^{n_k}a_{z,n_{k-1}}^{p'}v_z^{-p'}\right)^{\frac{q}{p'}}\sum_{n=n_k}^{n_{k+1}-1}u_n^q\left(\sum_{s=1}^{n_{k-1}}w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}. \tag{45}$$

Hence, we have that

$$I_{31}\leqslant\left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_if_i|^p\right)^{\frac{q}{p}}\sum_{n=n_k}^{n_{k+1}-1}u_n^q\sup_{1\leqslant j\leqslant n_k}\left(\sum_{s=1}^{j}w_s^r\right)^{\frac{q}{r}}\left(\sum_{z=j}^{n_k}a_{z,j}^{p'}v_z^{-p'}\right)^{\frac{q}{p'}}\right)^{\frac{1}{q}}$$

$$\leqslant\left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_if_i|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\sup_{n_k\geqslant 1}\left(\sum_{n=n_k}^{\infty}u_n^q\right)^{\frac{1}{q}}\sup_{1\leqslant j\leqslant n_k}\left(\sum_{s=1}^{j}w_s^r\right)^{\frac{1}{r}}\left(\sum_{z=j}^{n_k}a_{z,j}^{p'}v_z^{-p'}\right)^{\frac{1}{p'}}.$$

By applying (12) with $\frac{q}{p}\geqslant 1$, we obtain that

$$I_{31}\leqslant 2^{\frac{1}{p}}\|f\|_{p,v}\sup_{m\geqslant 1}\left(\sum_{n=m}^{\infty}u_n^q\right)^{\frac{1}{q}}\sup_{1\leqslant j\leqslant m}\left(\sum_{s=1}^{j}w_s^r\right)^{\frac{1}{r}}\left(\sum_{z=j}^{m}a_{z,j}^{p'}v_z^{-p'}\right)^{\frac{1}{p'}}.$$

As $J_{r,p}^-(1,m)\approx C$ when $\alpha=1$, $\beta=m$ and $q=r$, where C is the best constant in (9), by Theorem D we have that

$$J_{r,p}^-(1,m)\approx\sup_{1\leqslant j\leqslant m}\left(\sum_{s=1}^{j}w_s^r\right)^{\frac{1}{r}}\left(\sum_{z=j}^{m}a_{z,j}^{p'}v_z^{-p'}\right)^{\frac{1}{p'}}.$$

This gives

$$I_{31}\ll\sup_{m\geqslant 1}\left(\sum_{n=m}^{\infty}u_n^q\right)^{\frac{1}{q}}J_{r,p}^-(1,m)\|f\|_{p,v}\ll M_1^+\|f\|_{p,v}. \tag{46}$$

Let's estimate I_{32}.

$$I_{32}\leqslant\left(\sum_{k\in\mathbb{N}_0}\left(\sum_{i=n_{k-1}}^{n_k}|v_if_i|^p\right)^{\frac{q}{p}}\left(\sum_{z=n_{k-1}}^{n_k}v_z^{-p'}\right)^{\frac{q}{p'}}\sum_{n=n_k}^{n_{k+1}-1}u_n^q\left(\sum_{s=1}^{n_{k-1}}a_{n_{k-1},s}^{r}w_s^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}. \tag{47}$$

In the same way we get

$$\begin{aligned}I_{32}&\leqslant 2^{\frac{1}{p}}\|f\|_{p,v}\sup_{m\geqslant 1}\left(\sum_{n=m}^{\infty}u_n^q\right)^{\frac{1}{q}}\sup_{j\leqslant m}\left(\sum_{s=1}^{j}a_{j,s}^{r}w_s^r\right)^{\frac{1}{r}}\left(\sum_{z=j}^{m}v_z^{-p'}\right)^{\frac{1}{p'}}\\&\ll\sup_{m\geqslant 1}\left(\sum_{n=m}^{\infty}u_n^q\right)^{\frac{1}{q}}J_{r,p}^-(1,m)\|f\|_{p,v},\end{aligned}$$

that yields

$$I_{32} \ll M_1^+ \|f\|_{p,v}. \tag{48}$$

The case $1 < r < p$. To estimate I_{31} we need the relation

$$\left(\sum_{s=1}^{n_{k-1}} w_s^r\right)^{\frac{1}{r}} \approx \left(\sum_{s=1}^{n_{k-1}} w_s^r \left(\sum_{m=1}^{s} w_m^r\right)^{\frac{r}{p-r}}\right)^{\frac{p-r}{pr}}. \tag{49}$$

Now, we put (49) into (45) and find that

$$\begin{aligned}
I_{31} &\leqslant \left(\sum_{k\in\mathbb{N}_0} \left(\sum_{i=n_{k-1}}^{n_k} |v_i f_i|^p\right)^{\frac{q}{p}} \sum_{n=n_k}^{n_{k+1}-1} u_n^q \left(\sum_{s=1}^{n_{k-1}} w_s^r \left(\sum_{m=1}^{s} w_m^r\right)^{\frac{r}{p-r}}\right.\right.\\
&\quad \times \left.\left.\left(\sum_{z=n_{k-1}}^{n_k} a_{z,s}^{p'} v_z^{-p'}\right)^{\frac{pr}{p'(p-r)}}\right)^{\frac{q(p-r)}{pr}}\right)^{\frac{1}{q}}\\
&\leqslant \left(\sum_{k\in\mathbb{N}_0} \left(\sum_{i=n_{k-1}}^{n_k} |v_i f_i|^p\right)^{\frac{q}{p}} \sum_{n=n_k}^{\infty} u_n^q \times\right.\\
&\quad \times \left.\left(\sum_{s=1}^{n_{k-1}} w_s^r \left(\sum_{m=1}^{s} w_m^r\right)^{\frac{r}{p-r}} \left(\sum_{z=s}^{n_k} a_{z,s}^{p'} v_z^{-p'}\right)^{\frac{r(p-1)}{(p-r)}}\right)^{\frac{q(p-r)}{pr}}\right)^{\frac{1}{q}}\\
&\leqslant \left(\sum_{k\in\mathbb{N}_0} \left(\sum_{i=n_{k-1}}^{n_k} |v_i f_i|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \sup_{n_k\geqslant 1} \left(\sum_{n=n_k}^{\infty} u_n^q\right)^{\frac{1}{q}} \left(\sum_{s=1}^{n_k} w_s^r \left(\sum_{m=1}^{s} w_m^r\right)^{\frac{r}{p-r}}\right.\\
&\quad \times \left.\left(\sum_{z=s}^{n_k} a_{z,s}^{p'} v_z^{-p'}\right)^{\frac{r(p-1)}{(p-r)}}\right)^{\frac{p-r}{pr}}.
\end{aligned}$$

From (12) and Theorem E it follows that

$$I_{31} \leqslant 2^{\frac{1}{p}} \|f\|_{p,v} \sup_{j\geqslant 1} \left(\sum_{n=j}^{\infty} u_n^q\right)^{\frac{1}{q}} J_{r,p}^-(1,j) \ll M_1^+ \|f\|_{p,v}. \tag{50}$$

Consider the following value

$$\left(\sum_{z=n_{k-1}}^{n_k} v_z^{-p'}\right)^{\frac{1}{p'}} \approx \left(\sum_{z=n_{k-1}}^{n_k} v_z^{-p'} \left(\sum_{m=z}^{n_k} v_m^{-p'}\right)^{\frac{p(r-1)}{(p-r)}}\right)^{\frac{p-r}{pr}}. \tag{51}$$

By inserting (51) into (47) then in the same way we as above find that

$$I_{32} \ll \|f\|_{p,v} \sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty} u_n^q\right)^{\frac{1}{q}}\left(\sum_{z=1}^{j} v_z^{-p'}\left(\sum_{m=z}^{j} v_m^{-p'}\right)^{\frac{p(r-1)}{(p-r)}}\left(\sum_{s=1}^{z} a_{z,s}^r w_s^r\right)^{\frac{p}{p-r}}\right)^{\frac{p-r}{pr}}$$

$$\ll \|f\|_{p,v} \sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty} u_n^q\right)^{\frac{1}{q}} J_{r,p}^{-}(1,j) \ll M_1^{+}\|f\|_{p,v}. \tag{52}$$

The estimate

$$I_{33} \ll M_1^{+}\|f\|_{p,v} \tag{53}$$

for both cases $1<r<p$ and $1<p\leqslant r$ can be derived as in (32). From (46), (48), (50), (52) and (53), we have that for both cases inequality (4) is correct. Moreover

$$I_3 \ll M_1^{+}\|f\|_{p,v} \tag{54}$$

The inequalities (43), (44) and (54) give that $C \ll M^{+}$. Therefore, from this estimate and (42) we find $C \approx M^{+}$. The proof of Theorem 3 is complete. □

THEOREM 4. *Let $1<p\leqslant q<\infty$ and $1<r<\infty$. Let the entries of the matrix $(a_{k,i})$ satisfy condition* (3). *Then inequality* (5) *holds if and only if*

$$M^{-} = \max\{M_1^{-}, M_2^{-}, M_3^{-}\} < \infty,$$

where

$$M_1^{-} = \sup_{j\geqslant 1}\left(\sum_{n=1}^{j} u_n^q\right)^{\frac{1}{q}} J_{r,p}^{+}(j,\infty),$$

$$M_2^{-} = \sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty} u_n^q\left(\sum_{k=n}^{\infty} a_{k,j}^r w_k^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{1}{p'}},$$

$$M_3^{-} = \sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty} u_n^q\left(\sum_{k=n}^{\infty} w_k^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}}.$$

Moreover, $C' \approx M^{-}$, where C' is the best constant in (5).

The proof of Theorem 4 is similar to the proof of Theorem 3.

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Nazerke Zhangabergenova
Department of Fundamental Mathematics
L. N. Gumilyov Eurasian National University
Satpayev str. 2, Nur-Sultan, 010000 Kazakhstan
e-mail: zhanabergenova.ns@gmail.com