Mathematical
Inequalities
& Applications
Volume 26, Number 3 (2023), 645–653
doi:10.7153/mia-2023-26-39

# A NOTE ON THE CONVOLUTION IN ORLICZ SPACES

IBRAHIM AKBARBAGLU AND SAEID MAGHSOUDI*

(*Communicated by I. Perić*)

*Abstract.* Let $G$ be a locally compact group. In this paper, for given concave Orlicz functions $\Phi$ and $\Psi$ with $\limsup_{t\to\infty}\Phi(t)/t=0$, we prove that the convolution $f*g$ exists, for $f\in L^{\Phi}(G)$ and $g\in L^{\Psi}(G)$, if and only if $G$ is discrete. This extends and completes some recent results concerning the determination of when an Orlicz space on a locally compact group is closed under convolution multiplication.

## 1. Introduction

Throughout this paper, let $G$ be a locally compact group with a fixed left Haar measure $\lambda$ and let $L^0(G)$ be the space of all equivalent classes of $\lambda$-measurable complex-valued functions defined on $G$. For measurable functions $f$ and $g$ on $G$, the convolution

$$(f*g)(x)=\int_G f(y)g(y^{-1}x)\,d\lambda(y)$$

is defined at each point $x$ belongs to $G$ for which the function $y\mapsto f(y)g(y^{-1}x)$ is Haar integrable.

We deal with a problem which has its origin in the 1960's. O'Neil generalizes in [17] the celebrated Young Theorem and gives sufficient condition for the inclusion $L^{\Phi_1}*L^{\Phi_2}\subseteq L^{\Phi_3}$ among Orlicz spaces. More specifically, suppose $\Phi_i$, $i=1,2,3$, are Young functions and $L^{\Phi_i}(G)$ are corresponding Orlicz spaces (for definition see below). A natural question is that: is there any relation among $\Phi_i$'s to be sufficient for the fact that if $f\in L^{\Phi_1}(G)$ and $g\in L^{\Phi_2}(G)$, then $f*g\in L^{\Phi_3}(G)$? O'Neil proves that: let $G$ be a unimodular locally compact group, $\Phi_i$, $i=1,2,3$ be Young functions satisfying

$$\Phi_1^{-1}(x)\Phi_2^{-1}(x)\leqslant x\Phi_3^{-1}(x)$$

for $x\geqslant 0$. Then for any $f_i\in L^{\Phi_i}(G)$, $i=1,2$, the convolution $f_1*f_2$ belongs to $L^{\Phi_3}(G)$.

From the other hand, there are known necessary and sufficient conditions for the space $L^p(G)$ to be closed under convolution multiplication (known as $L^p$-conjecture

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*Mathematics subject classification* (2020): 46E30, 43A15, 54E52.
*Keywords and phrases*: Orlicz space, Orlicz function, porosity, locally compact group, convolution, Young function.
This work is based upon research funded by Iran National Science Foundation (INSF) under project no. 4020699.
* Corresponding author.


Paper MIA-26-39

or Żelazko conjecture [24] which was finally solved by Saeki [21] in 1990), demanding generalization to Orlicz spaces; see for example [13, 14, 19]. Recently, this problem has attracted some attention; see for instance [8, 18].

In 2010, Głąb and Strobin in [10] started the study of quantitative version of the problem for $L^p$ spaces. Specifically, they proved that the set of pairs $(f,g) \in L^p(G) \times L^q(G)$ such that $f*g$ exists is small, namely $\sigma$-porous, if $1/p+1/q<1$; see also [4], and for a similar study on the pointwise product in $L^p$ spaces see [11]. A generalization of their results for Orlicz spaces presented in [2, 3] and for more general setting of Calderón-Lozanowskiĭ spaces in [5]. For details on the diverse notions of porosity see [22].

Our aim in this note is to investigate an overlooked case of Orlicz space $L^{\Phi}(G)$ in which $\Phi$ is concave. This case has not been covered in the above mentioned works in spite of its importance in applications; see for example [6, 7]. In fact, we show the meagerness of the set of all pairs $(f,g) \in L^{\Phi}(G) \times L^{\Psi}(G)$ for which $f*g$ is $\lambda$-a.e. finite on $G$ for given concave functions $\Phi$ and $\Psi$ under a mild condition.

Before going further, let us gather some necessary information about Orlicz spaces. We refer the interested reader to [15] and [20] for more details about Orlicz spaces.

The function $\Delta : G \to (0,\infty)$ defined by $\lambda(Ax) = \Delta(x)\lambda(A)$ is called the modular function of $G$. It is clear that $\Delta$ is a continuous homomorphism on $G$. Moreover, for every measurable subset $A$ of $G$,

$$\lambda(A^{-1}) = \int_A \Delta(x^{-1})\,d\lambda(x);$$

for more details see [9]. For $1 \leqslant p \leqslant \infty$, classical Lebesgue spaces on $G$ with respect to the Haar measure $\lambda$ will be denoted by $L^p(G)$ with the norm $\|\cdot\|_p$ as defined in [9].

An even function $\Phi : \mathbb{R} \to [0,+\infty]$ is called an Orlicz function if $\Phi$ is non-decreasing and $\Phi(t) > 0$ for all $t \neq 0$ with

$$\lim_{t\to\infty} \Phi(t) = +\infty, \qquad \lim_{t\to 0^+} \Phi(t) = \Phi(0) = 0.$$

Any Orlicz function $\Phi$ determines a functional $\rho_\Phi : L^0(G) \to [0,+\infty]$, called a modular, defined by the formula

$$\rho_\Phi(f) = \int_G \Phi\big(|f(t)|\big)\,d\lambda(t).$$

The subset

$$L^{\Phi}(G) = \big\{ f \in L^0(G) : \rho_\Phi(\alpha f) < +\infty \quad \text{for some} \quad \alpha > 0 \big\}, \tag{1}$$

in $L^0(G)$ is called Orlicz space.

Defining the functional $\|f\|_\Phi$ on $L^{\Phi}(G)$ by

$$\|f\|_\Phi = \inf\big\{ k > 0 : \rho_\Phi(f/k) \leqslant k \big\}, \tag{2}$$

it satisfies the following conditions:

1) For all $f$, $0 \leqslant \|f\|_\Phi < \infty$ and $\|f\|_\Phi = 0$ if and only if $f = 0$,

2) For all $f,g \in L^{\Phi}(G)$, if $|f| \leqslant |g|$ then $\|f\|_{\Phi} \leqslant \|g\|_{\Phi}$,

3) $0 \leqslant \|f+g\|_{\Phi} \leqslant \|f\|_{\Phi} + \|g\|_{\Phi}$ for all $f$ and $g$,

4) $\|\lambda_n f\|_{\Phi} \to 0$ for every $f$ and every sequence $(\lambda_n)$ of scalars with $\lim_{n\to\infty} \lambda_n = 0$.

Then $\|\cdot\|_{\Phi}$ is called (the Mazur-Orlicz) $F$-norm and the space $L^{\Phi}(G)$ is a complete metric linear space with respect to this $F$-norm [16].

A convex function $\Phi : \mathbb{R} \to [0,\infty]$ is called a Young function if $\Phi$ is even and left continuous with $\Phi(0) = 0$; it is assumed that $\Phi$ is neither identically zero nor identically infinite on $\mathbb{R}$. For a Young function $\Phi$ and $y \in [0,\infty)$ let

$$\Phi^{-1}(y) = \sup\{x \geqslant 0 : \Phi(x) \leqslant y\}.$$

In this case, the Orlicz space $L^{\Phi}(G)$ is a Banach space under the norm $N_{\Phi}(\cdot)$ (called the Luxemburg-Nakano norm) defined for $f \in L^{\Phi}(G)$ by

$$N_{\Phi}(f) = \inf\{k > 0 : \rho_{\Phi}(f/k) \leqslant 1\}.$$

## 2. Main results

We begin by recalling a fact on Borel measures due to R. E. Jamison proved in [12, 11.44].

LEMMA 1. *Let $X$ be a locally compact Hausdorff space and $\mu$ be a Borel measure on $X$ such that $\mu(\{x\}) = 0$ for all $x \in X$ and $\mu$ is inner regular. Suppose that $A$ is a measurable set and $0 < \alpha < \mu(A)$ is a real number. Then there is a compact subset $K$ of $A$ such that $\mu(K) = \alpha$.*

Let $\Phi$ and $\Psi$ be Orlicz functions. In the sequel, the space $L^{\Phi}(G) \times L^{\Psi}(G)$ will be equipped with the $F$-norm

$$\|(f,g)\|_{\Phi,\Psi} = \max\left\{\|f\|_{\Phi}, \|g\|_{\Psi}\right\} \quad \left(f \in L^{\Phi}(G), g \in L^{\Psi}(G)\right).$$

Also the function $\chi_A$ denotes the characteristic function of a subset $A$.

Let us note a simple fact that will be used in the following theorem: if $\Phi$ is a concave function on $[0,\infty)$ then $\Phi(\lambda t) \geqslant \lambda \Phi(t)$ for all $0 \leqslant \lambda \leqslant 1$ and $t \geqslant 0$. Also, we have

$$\int_G \Phi(|f|)d\lambda \leqslant \|f\|_{\Phi} \qquad (f \in L^{\Phi}(G), \|f\|_{\Phi} \leqslant 1).$$

Let us remark that our method for the proof of the following theorem is based on that used essentially in [10]. However, several technical problems in the case of concave Orlicz functions arise had to be figured out to get our result.

THEOREM 1. *Let $G$ be a non-discrete locally compact group with a fixed left Haar measure $\lambda$. Also let $\Phi$ and $\Psi$ be concave Orlicz functions with*

$$\limsup_{t\to\infty} \frac{\Phi(t)}{t} = 0.$$

*Then for any compact symmetric neighborhood $V$ of the identity element of $G$, the set*

$$E = \{(f,g) \in L^{\Phi}(G) \times L^{\Psi}(G) : \exists x \in V, |f| * |g|(x) < \infty\}.$$

*is of first category in $L^{\Phi}(G) \times L^{\Psi}(G)$.*

*Proof.* Take a compact symmetric neighborhood $V$ of the identity element of $G$. For any natural number $n$, put

$$E_n = \{(f,g) \in L^{\Phi}(G) \times L^{\Psi}(G) : \exists x \in V, |f| * |g|(x) \leqslant n\}$$

So, $E = \bigcup_{n\in\mathbb{N}} E_n$. We will show that for each $n \in \mathbb{N}$, $E_n$ is a nowhere dense set in $L^{\Phi}(G) \times L^{\Psi}(G)$. This will complete the proof.

Let

$$\eta = \sup\{\Delta(x) : x \in V\}$$

and $c = 1/9$. Then $8c = 1-c$. Fix $0 < \alpha < c$, then $8\alpha < 1-\alpha$. By continuity of the map $x \mapsto 8\alpha/x$ on $(0,1)$, we conclude that there exist $0 < \beta < 1-\alpha$ and $d < 1$ such that $8\alpha < \beta(1-d)$.

Fix $n \in \mathbb{N}$, $0 < r < \alpha$ and $(f,g) \in E_n$. Since $G$ is not discrete, $\inf\{\lambda(U) : \lambda(U) > 0\} = 0$. Hence, we can choose a compact symmetric neighborhood $F$ in $V$ with $\lambda(FV) \leqslant 2\lambda(V)$ such that

$$\beta^2 r^4 d^2 \lambda(F)\Phi^{-1}\left(\frac{1}{\lambda(F)}\right)\Psi^{-1}\left(\frac{r^2}{8\lambda(V)}\right) > 16n. \tag{3}$$

Also, by Lemma 1, there exists a compact subset $U_0$ of $F$ such that $\lambda\left(U_0^{-1}\right) = \dfrac{r^2}{4\eta}\lambda(F)$.

Now, define functions $\widetilde{f}$ and $\widetilde{g}$ on $G$ by setting

$$\widetilde{f}(y) := \begin{cases} f(y) & y \notin U_0 \\ f(y) + \beta r \Phi^{-1}\left(\frac{\eta}{\lambda(F)\Delta(y)}\right) & \mathrm{Re}(f(y)) \geqslant 0,\ y \in U_0 \\ f(y) - \beta r \Phi^{-1}\left(\frac{\eta}{\lambda(F)\Delta(y)}\right) & \mathrm{Re}(f(y)) < 0,\ y \in U_0. \end{cases}$$

$$\widetilde{g}(y) := \begin{cases} g(y) & y \notin U_0^{-1}V \\ g(y) + \beta r \Psi^{-1}\left(\frac{r^2}{4\lambda(FV)}\right) & \mathrm{Re}(g(y)) \geqslant 0,\ y \in U_0^{-1}V \\ g(y) - \beta r \Psi^{-1}\left(\frac{r^2}{4\lambda(FV)}\right) & \mathrm{Re}(g(y)) < 0,\ y \in U_0^{-1}V. \end{cases}$$

It follows that

$$\begin{aligned}\int_G \Phi\left(\frac{2}{r}\Phi^{-1}\left(\frac{\eta}{\lambda(F)\Delta(x)}\chi_{U_0}(x)\right)\right)d\lambda(x) &< \int_{U_0}\frac{2\eta}{r\lambda(F)\Delta(x)}d\lambda(x)\\ &= \frac{2\eta}{r\lambda(F)}\lambda\left(U_0^{-1}\right)\\ &= \frac{2\eta}{r\lambda(F)}\frac{r^2}{4\eta}\lambda(F)\\ &= \frac{r}{2},\end{aligned}$$

and analogously

$$\int_G \Psi\left(\frac{2}{r}\Psi^{-1}\left(\frac{r^2}{4\lambda(FV)}\chi_{U_0^{-1}V}(x)\right)\right)d\lambda(x) < \frac{r}{2}.$$

The above inequalities imply that

$$\left\|\Phi^{-1}\left(\frac{\eta}{\lambda(F)\Delta}\right)\chi_{U_0}\right\|_\Phi < \frac{r}{2}$$

and

$$\left\|\Psi^{-1}\left(\frac{r^2}{4\lambda(FV)}\right)\chi_{U_0^{-1}V}\right\|_\Psi < \frac{r}{2}.$$

Consequently,

$$\|\widetilde{f}-f\|_\Phi < \frac{r}{2} \quad \text{and} \quad \|\widetilde{g}-g\|_\Phi < \frac{r}{2}.$$

Thus it remains only to show that for any positive real number $\delta$ with

$$\delta < \frac{r^3\lambda\left(U_0^{-1}\right)}{4\eta\lambda(FV)},$$

we get $B\big((\widetilde{f},\widetilde{g}),\delta r\big) \subseteq B\big((\widetilde{f},\widetilde{g}),\alpha r\big)\setminus E_n$. To prove this, take $(h,k)\in B\big((\widetilde{f},\widetilde{g}),\delta r\big)$. Put

$$\begin{aligned}A_1 &:= \left\{x\in U_0 : |h(x)| < \beta r d\,\Phi^{-1}\left(\frac{\eta}{\lambda(F)\Delta(x)}\right)\right\},\\ B_1 &:= \left\{x\in U_0^{-1}V : |k(x)| < \beta r d\,\Psi^{-1}\left(\frac{r^2}{4\lambda(FV)}\right)\right\}.\end{aligned}$$

Therefore we have

$$\begin{aligned}
\delta r > \|\widetilde{f}-h\|_{\Phi} &\geqslant \left\|\left(\widetilde{f}-h\right)\chi_{A_1}\right\|_{\Phi} \\
&\geqslant \left\|\left(|\widetilde{f}|-|h|\right)\chi_{A_1}\right\|_{\Phi} \\
&\geqslant \left\|\beta r\Phi^{-1}\left(\frac{\eta\chi_{A_1}}{\lambda(F)\Delta}\right)-\beta rd\Phi^{-1}\left(\frac{\eta\chi_{A_1}}{\lambda(F)\Delta}\right)\right\|_{\Phi} \\
&= \left\|\beta r(1-d)\Phi^{-1}\left(\frac{\eta\chi_{A_1}}{\lambda(F)\Delta}\right)\right\|_{\Phi} \\
&\geqslant \left\|\Phi^{-1}\left(\frac{\eta\beta r(1-d)\chi_{A_1}}{\lambda(F)\Delta}\right)\right\|_{\Phi} \\
&\geqslant \int_{A_1}\frac{\eta\beta r(1-d)}{\lambda(F)\Delta(x)}d\lambda(x).
\end{aligned}$$

Accordingly,

$$\lambda(A_1^{-1}) < \frac{\delta r\lambda(F)}{\eta\beta r(1-d)} < \frac{\delta\lambda(F)}{8\alpha\eta} < \frac{\delta\lambda(U_0^{-1})}{2r^3} < \frac{\lambda(U_0^{-1})}{8}.$$

Furthermore,

$$\begin{aligned}
\delta r > \|\widetilde{g}-k\|_{\Psi} \geqslant \|(\widetilde{g}-k)\chi_{B_1}\|_{\Psi} &\\
&\geqslant \left\|\Psi^{-1}\left(\frac{\beta r^3(1-d)\chi_{B_1}}{4\lambda(FV)}\right)\right\|_{\Psi} \\
&\geqslant \int_{B_1}\frac{\beta r^3(1-d)}{4\lambda(FV)}d\lambda(x).
\end{aligned}$$

that implies

$$\lambda(B_1) < \frac{4\delta r\lambda(FV)}{\beta r^3(1-d)} < \frac{r\lambda\left(U_0^{-1}\right)}{8\alpha\eta} < \frac{\lambda(U_0^{-1})}{8\eta}.$$

The above inequalities also show that the sets

$$A_2 := U_0\setminus A_1 \quad \text{and} \quad B_2 := U_0^{-1}V\setminus B_1$$

are of positive measure and so non-empty.

Now, let $z\in V$ be an arbitrary element, and define the sets

$$H = (A_2^{-1}z)\cap B_2 \quad \text{and} \quad S = zH^{-1}.$$

Since $A_2^{-1}z\subseteq U_0^{-1}V$, we obtain

$$\begin{aligned}
\lambda(S^{-1}) = \lambda(Hz^{-1}) &= \lambda(A_2^{-1}) - \lambda\left(A_2^{-1}\setminus(B_2z^{-1})\right) \\
&\geqslant \lambda(A_2^{-1}) - \lambda\left((U_0^{-1}V\setminus B_2)z^{-1}\right) \\
&= \lambda(A_2^{-1}) - \lambda(B_1z^{-1}) \\
&= \lambda(U_0^{-1}) - \lambda(A_1^{-1}) - \Delta(z^{-1})\lambda(B_1) \\
&> \frac{\lambda(U_0^{-1})}{4}.
\end{aligned}$$

Also, $S \subseteq A_2$ and $S^{-1}z \subseteq B_2$.

Finally, we conclude that

$$\begin{aligned}\int_S |h(y)||k(y^{-1}z)|d\lambda(y) &\geqslant \int_S \beta^2 r^2 d^2 \Phi^{-1}\left(\frac{\eta}{\lambda(F)\Delta(y)}\right)\Psi^{-1}\left(\frac{r^2}{4\lambda(FV)}\right)d\lambda(y)\\ &\geqslant \int_S \beta^2 r^2 d^2 \frac{\eta}{\Delta(y)}\Phi^{-1}\left(\frac{1}{\lambda(F)}\right)\Psi^{-1}\left(\frac{r^2}{4\lambda(FV)}\right)d\lambda(y)\\ &\geqslant \beta^2 r^2 d^2 \eta\lambda(S^{-1})\Phi^{-1}\left(\frac{1}{\lambda(F)}\right)\Psi^{-1}\left(\frac{r^2}{8\lambda(V)}\right)\\ &= \frac{\beta^2 r^4 d^2}{16}\lambda(F)\Phi^{-1}\left(\frac{1}{\lambda(F)}\right)\Psi^{-1}\left(\frac{r^2}{8\lambda(V)}\right)\\ &> n\end{aligned}$$

This means that $(h,k) \notin E_n$. $\square$

REMARK 1. By using a similar argument as used in [1, 2, 10], it can be seen that the assertion of Theorem 1 holds if we assume that $\Psi$ is a Young function.

The following is a consequence of our main result.

COROLLARY 1. *Let $G$ be a locally compact group and let $\Phi$ be a concave Orlicz function with*

$$\limsup_{t\to\infty} \frac{\Phi(t)}{t} = 0.$$

*Then the following hold:*

1) *The convolution of any two functions in $L^\Phi(G)$ exists if and only if $G$ is discrete.*
2) *If $\Phi$ is sub-multiplicative (i.e., $\Phi(st) \leqslant C\Phi(s)\Phi(t)$ for some $C \geqslant 1$ and each positive reals $s,t$ ) then $L^\Phi(G)$ is a topological algebra under convolution if and only if $G$ is discrete.*

*Proof.*

1) Assume that the convolution of any two functions in $L^\Phi(G)$ exists and, by way of contradiction, $G$ is non-discrete. Since $L^\Phi(G) \times L^\Phi(G)$ is complete with respect to its metric, by the Baire category theorem, it can not be meager and, in particular, can not coincide with the set $E$ in Theorem 1. This contradicts our assumption and it follows that $G$ is discrete.

   For the converse, assume that $G$ is discrete, then $L^\Phi(G) \subseteq L^1(G)$. Because, given a function $f \in L^\Phi(G)$, we can find $\alpha > 0$ and an at most countable set $H \subseteq G$ such that for every $x \in H$, $f(x) \neq 0$. Also we have

$$\sum_{x\in H} \Phi\big(\alpha|f(x)|\big) < \infty.$$

Now since $\Phi$ is concave, for any $0 < s < t$,

$$\frac{\Phi(t)}{t} \leqslant \frac{\Phi(s)}{s}.$$

Hence for all except finitely many elements of $H$, we have $0 < \Phi(\alpha|f|) < \Phi(1)$. It follows that $0 < \alpha|f| < 1$, and consequently $\alpha\Phi(1)|f| \leqslant \Phi(\alpha|f|)$.

2) In view of Part 1), we only need to consider when $G$ is discrete. By the fact that for any non-negative reals $a$ and $b$, $\Phi(a+b) \leqslant \Phi(a)+\Phi(b)$ we obtain, for every $f$ and $g$ in $L^{\Phi}(G)$,

$$\begin{aligned}\sum_{x\in G}\Phi\left(\frac{|f*g|(x)}{C\|f\|_{\Phi}\|g\|_{\Phi}}\right) &\leqslant \sum_{x\in G}\Phi\left(\sum_{y\in G}\frac{|f(y)||g(y^{-1}x)|}{\|f\|_{\Phi}\|g\|_{\Phi}}\right)\\ &\leqslant \sum_{x\in G}\sum_{y\in G}C\Phi\left(\frac{|f(y)|}{\|f\|_{\Phi}}\right)\Phi\left(\frac{|g(y^{-1}x)|}{\|g\|_{\Phi}}\right)\\ &\leqslant C\|f\|_{\Phi}\|g\|_{\Phi}.\end{aligned}$$

Therefore $\|f*g\|_{\Phi} \leqslant C\|f\|_{\Phi}\|g\|_{\Phi}$. □

Let us give an example.

EXAMPLE 1. Define $\Phi(x) = \ln(1+x+\sqrt{(1+x)^2-1})$ and note that

$$\lim_{t\to 0}\Phi(t)/\sqrt{t} = \sqrt{2}$$

and

$$\lim_{t\to\infty}\Phi(t)/t = 0.$$

By Corollary 1, the Orlicz space $L^{\Phi}(G)$ is an algebra under convolution if and only if $G$ is discrete.

Finally, as an another corollary of our main theorem, we recover a result due to W. Żelazko [23]; see also [4]. Note that in the special case $\Phi(t) = t^p$ for some $0 < p < 1$ we have $L^{\Phi}(G) = L^p(G)$ and

$$\|f\|_{\Phi} = \left(\int_G |f(t)|^p \, d\lambda(t)\right)^{\frac{1}{1+p}}.$$

COROLLARY 2. *For a locally compact group $G$ and for $0 < p < 1$, the space $L^p(G)$ is an algebra under convolution multiplication if and only if $G$ is discrete.*

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(Received July 17, 2022)

*Ibrahim Akbarbaglu*
*Department of Mathematics education*
*Farhangian University*
*P.O. Box 14665-889, Tehran, Iran*
*e-mail:* i.akbarbaglu@cfu.ac.ir
ibrahim.akbarbaglu@gmail.com

*Saeid Maghsoudi*
*Department of Mathematics*
*University of Zanjan*
*Zanjan 45195-313, Iran*
*e-mail:* s_maghsodi@znu.ac.ir