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THE ε–MAXIMAL OPERATOR AND HAAR MULTIPLIERS ON VARIABLE LEBESGUE SPACES

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Abstract. Recently, Stockdale, Villarroya, and Wick introduced the ε-maximal operator to prove the Haar multiplier is bounded on the weighted spaces $L^p(w)$ for a class of weights larger than A_p. We prove the ε-maximal operator and Haar multiplier are bounded on variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ for a larger collection of exponent functions than the log-Hölder continuous functions used to prove the boundedness of the maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$. We also prove that the Haar multiplier is compact when restricted to a dyadic cube Q_0.

1. Introduction

In [2], the authors prove that the Hardy-Littlewood maximal operator is bounded on variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ for log-Hölder continuous exponent functions $p(\cdot) \in LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < \infty$. In [8], the authors use the ε-maximal operator and ε-sparse operator to establish the boundedness of the Haar multiplier on $L^p(w)$ for a class of weights larger than A_p. Motivated by these two results, in this paper we prove the ε-maximal operator and the Haar multiplier are bounded on variable Lebesgue spaces for a collection of exponent functions larger than $LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n)$. In addition, we prove a local compactness result for the Haar multiplier similar to the result in [8].

Before stating our results, we briefly outline some of the definitions involved. We explain them in more detail in Section 2. An exponent function is a measurable function $p(\cdot) : \mathbb{R}^n \to [1,\infty)$. Denote the essential infimum and essential supremum of $p(\cdot)$ by p_- and p_+, respectively. Here, we only consider exponent functions where $1 < p_- \leqslant p_+ < \infty$. Given an exponent function $p(\cdot)$, define the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ as the collection of Lebesgue measurable functions satisfying $\|f\|_{p(\cdot)} < \infty$, where $\|f\|_{p(\cdot)}$ is the norm given by

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)}\,dx \leqslant 1\right\}.$$

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We now define the ε sequences that appear in the operators we are interested in. Denote the set of all dyadic cubes in $\mathbb{R}^n$ by $\mathscr{D}$. Throughout this paper, $\varepsilon = \{\varepsilon_Q\}_{Q\in\mathscr{D}}$ will be a bounded sequence of real numbers ε_Q indexed by the dyadic cubes. If each $\varepsilon_Q \geqslant 0$, we say that ε is non-negative. If ε is non-negative, we say that it has the domination property if for any $P,Q\in\mathscr{D}$ such that $P\subseteq Q$, then $\varepsilon_P \leqslant \varepsilon_Q$. Given any ε, define the new sequence $\bar{\varepsilon}$ by

$$\bar{\varepsilon}_Q = \sup_{\substack{P\in\mathscr{D}\\ P\subseteq Q}} |\varepsilon_P|.$$

Then $\bar{\varepsilon}$ is a non-negative sequence with the domination property, and $|\varepsilon_Q| \leqslant \bar{\varepsilon}_Q$. Given a non-negative sequence ε and $\alpha > 0$, define the new sequence $\varepsilon^\alpha = \{\varepsilon_Q^\alpha\}_{Q\in\mathscr{D}}$.

Given any sequence ε, define the Haar multiplier T_ε acting on $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$T_\varepsilon f = \sum_{Q\in\mathscr{D}} \varepsilon_Q \langle f, h_Q\rangle h_Q. \tag{1.1}$$

Here, $\langle f, h_Q\rangle = \int_Q f(y)h_Q(y)\,dy$, and h_Q is the Haar function adapted to Q defined by

$$h_Q = |Q|^{-1/2}\left(\chi_Q - \frac{1}{2^n}\chi_{\widehat{Q}}\right),$$

where $\widehat{Q}$ is the dyadic parent of Q. Our goal is to prove that the Haar multiplier is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ with assumptions on $p(\cdot)$ configured to the sequence ε.

To do so, we use the dyadic ε-maximal operator M_ε as a tool to control the Haar multiplier. Given a non-negative sequence ε, define M_ε for $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$M_\varepsilon f(x) = \sup_{Q\in\mathscr{D}} \varepsilon_Q \fint_Q |f(y)|\,dy\,\chi_Q(x).$$

Note that if $\varepsilon_Q = 1$ for all Q, then M_ε becomes the dyadic maximal operator M^d. More generally, we have $M_\varepsilon f \leqslant \|\varepsilon\|_\infty M^d f \leqslant \|\varepsilon\|_\infty Mf$, where M is the Hardy-Littlewood maximal operator.

In [2, Theorem 3.16], the authors proved the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ when $p(\cdot)\in LH_0(\mathbb{R}^n)\cap LH_\infty(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < \infty$. The set $LH_\infty(\mathbb{R}^n)$ denotes the collection of exponent functions $p(\cdot)$ that are log-Hölder continuous at infinity: there exist constants C_∞ and p_∞ such that for all $x\in\mathbb{R}^n$,

$$|p(x) - p_\infty| \leqslant \frac{C_\infty}{\log(e+|x|)}.$$

The set $LH_0(\mathbb{R}^n)$ consists of exponent functions that are locally log-Hölder continuous: there exists a constant C_0 such that for all $x,y\in\mathbb{R}^n$ with $|x-y| < 1/2$,

$$|p(x) - p(y)| \leqslant \frac{C_0}{-\log(|x-y|)}. \tag{1.2}$$

When $p_+ < \infty$, $LH_0(\mathbb{R}^n)$ is equivalent to the Diening condition: there exists a constant C such that given any cube Q,

$$|Q|^{p_-(Q)-p_+(Q)} \leqslant C, \tag{1.3}$$

where $p_-(Q)$ and $p_+(Q)$ are the essential infimum and essential supremum of $p(\cdot)$ on Q.

Since the dyadic maximal operator is pointwise smaller than the Hardy-Littlewood maximal operator, $LH_0(\mathbb{R}^n)\cap LH_\infty(\mathbb{R}^n)$ is a sufficient condition for M^d to be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. For the dyadic maximal operator, however, we can replace the $LH_0(\mathbb{R}^n)$ condition with a weaker dyadic condition that is still sufficient (along with the $LH_\infty(\mathbb{R}^n)$ condition) for it to be bounded: more precisely, we can assume that the Diening condition (1.3) holds for dyadic cubes. This fact is implicit in the proof of the boundedness of the Hardy-Littlewood maximal operator: see, for instance, [2, Section 3.4]. This dyadic Diening condition was explicitly introduced and studied in [5, 9].

However, we show that we can replace the dyadic Diening condition with an even weaker local condition that depends on the sequence ε. Given an exponent function $p(\cdot)$ with $p_+<\infty$, we say that $p(\cdot)\in\varepsilon LH_0(\mathbb{R}^n)$ if there exists a constant C depending only on n, $p(\cdot)$ and ε such that given any cube $Q\in\mathscr{D}$ with $\varepsilon_Q\neq 0$,

$$\left(\frac{|Q|}{\varepsilon_Q}\right)^{p_-(Q)-p_+(Q)}\leqslant C. \tag{1.4}$$

Note that if $\varepsilon_Q=1$ for all $Q\in\mathscr{D}$, we obtain the Diening condition (1.3).

REMARK 1.1. Unfortunately, we cannot replace $LH_\infty(\mathbb{R}^n)$ with a condition involving ε in the same way. This is due to the fact that the ε-maximal operator is pointwise equivalent to the dyadic maximal operator M^d near infinity if ε has the domination property. For in this case, given a bounded function f that is supported on a dyadic cube Q_0, we have that for almost every $x\notin Q_0$,

$$\varepsilon_{Q_0}M^d f(x)\leqslant M_\varepsilon f(x)\leqslant\|\varepsilon\|_\infty M^d f(x).$$

Since the constants ε_{Q_0} and $\|\varepsilon\|_\infty$ do not depend on any information about ε_Q for $Q\neq Q_0$, any condition near infinity that we use to bound $M_\varepsilon f$ outside of Q_0 will have to be the same condition we use to bound $M^d f$ outside of Q_0, and not a condition based on the properties of the sequence ε. However, it would still be of interest to find a dyadic version of the $LH_\infty(\mathbb{R}^n)$ that could be used to prove the boundedness of the dyadic maximal operator. The very recent results by Lerner [6] may be applicable to this problem.

We can now state our main results.

THEOREM 1.2. *Fix a non-negative sequence $\varepsilon=\{\varepsilon_Q\}_{Q\in\mathscr{D}}$. Given an exponent function $p(\cdot)$ with $1<p_-\leqslant p_+<\infty$, suppose $p(\cdot)\in LH_\infty(\mathbb{R}^n)\cap\varepsilon LH_0(\mathbb{R}^n)$. Then M_ε is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$: there exists a constant $C=C(n,p(\cdot),\varepsilon)$ such that for any $f\in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|M_\varepsilon f\|_{p(\cdot)}\leqslant C\|f\|_{p(\cdot)}.$$

THEOREM 1.3. *Fix a sequence $\varepsilon=\{\varepsilon_Q\}_{Q\in\mathscr{D}}$. Given an exponent function $p(\cdot)$ with $1<p_-\leqslant p_+<\infty$, suppose $p(\cdot)\in\bar{\varepsilon}^{1/2}LH_0(\mathbb{R}^n)\cap LH_\infty(\mathbb{R}^n)$. Then the Haar multiplier T_ε is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

In our final result we consider the compactness properties of the Haar multiplier if we restrict the domain from $\mathbb{R}^n$ to $Q_0 \in \mathscr{D}$. Let $\mathscr{D}(Q_0)$ be the collection of dyadic cubes contained in Q_0, and define the $\varepsilon LH_0(Q_0)$ condition exactly as in (1.4), but for $Q \in \mathscr{D}(Q_0)$.

THEOREM 1.4. *Fix a cube $Q_0 \in \mathscr{D}$ and a sequence $\varepsilon = \{\varepsilon_Q\}_{Q\in\mathscr{D}(Q_0)}$ such that*

$$\lim_{N\to\infty} \sup\{\bar{\varepsilon}_Q : \ell(Q) < 2^{-N}\} = 0.$$

Given an exponent $p(\cdot)$ with $1 < p_- \leqslant p_+ < \infty$, suppose $p(\cdot) \in \bar{\varepsilon}^{\alpha} LH_0(Q_0)$ for some $0 < \alpha < 1/2$. Then the Haar multiplier is compact on $L^{p(\cdot)}(Q_0)$.

REMARK 1.5. In [8, Section 2.4], the authors give a compactness result for weighted spaces on all of $\mathbb{R}^n$. However, their proof requires that $\bar{\varepsilon}_Q \to 0$ as $\ell(Q) \to \infty$. This is impossible since $\bar{\varepsilon}$ has the domination property unless ε is the zero sequence. But implicit in their proof is a local compactness result, and our proof is modeled on theirs.

In [4, Section 5], the authors give a different proof of the compactness result for weighted spaces $L^p(w)$ on $\mathbb{R}^n$ using a version of Rubio de Francia extrapolation for compactness. We conjecture that the corresponding compactness result is true on $L^{p(\cdot)}(\mathbb{R}^n)$ with the additional assumption that $p(\cdot) \in LH_\infty(\mathbb{R}^n)$.

The remainder of this paper is organized as follows. In Section 2, we state the necessary definitions and lemmas for variable Lebesgue spaces. We prove Theorem 1.2 in Section 3. In Section 4, we prove Theorems 1.3 and 1.4. Lastly, in Section 5 we show that the $\varepsilon LH_0(\mathbb{R}^n)$ hypothesis of Theorem 1.2 is weaker than the local log-Hölder continuity condition defined in inequality (1.2). We do this by showing there are exponent functions that are not in $LH_0(\mathbb{R}^n)$, but are in $\varepsilon LH_0(\mathbb{R}^n)$ for some ε.

Throughout this paper, n will denote the dimension of the underlying space $\mathbb{R}^n$, and C will denote a constant that may vary in value from line to line and which will depend on underlying parameters. If we want to specify the dependence, we will write, for instance, $C(n,\varepsilon)$. If the value of the constant is not important, we will often write $A \lesssim B$ instead of $A \leqslant cB$ for some constant c. We will also use the convention that $1/\infty = 0$.

2. Preliminaries

We begin with the necessary definitions related to variable Lebesgue spaces. We refer the reader to [2] for more information.

DEFINITION 2.1. An exponent function on a set $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable function $p(\cdot) : \Omega \to [1,\infty)$. Denote the collection of exponent functions on Ω by $\mathscr{P}(\Omega)$. Denote the essential infimum and essential supremum of $p(\cdot)$ on a set E by $p_-(E)$ and $p_+(E)$, respectively. Denote $p_+(\Omega)$ by p_+ and $p_-(\Omega)$ by p_-.

DEFINITION 2.2. Given $p(\cdot) \in \mathscr{P}(\Omega)$ with $p_+ < \infty$, and a Lebesgue measurable function f, define the modular associated with $p(\cdot)$ by

$$\rho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx.$$

In situations where there is no ambiguity we will simply write $\rho(f)$.

DEFINITION 2.3. Given $p(\cdot) \in \mathscr{P}(\Omega)$, define the space $L^{p(\cdot)}(\Omega)$ as the set of Lebesgue measurable functions f satifying $\|f\|_{L^{p(\cdot)}(\Omega)} < \infty$, where the norm $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ is defined as

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leqslant 1\}.$$

In situations where there is no ambiguity, we will write $\|f\|_{p(\cdot)}$ instead of $\|f\|_{L^{p(\cdot)}(\Omega)}$.

The following propositions relate the modular and the norm and will be used to prove Theorem 1.2. The first proposition allows us to conclude a norm is finite when the modular is finite.

PROPOSITION 2.4. [2, Proposition 2.12] *Given $p(\cdot) \in \mathscr{P}(\Omega)$ with $p_+ < \infty$, $f \in L^{p(\cdot)}(\Omega)$ if and only if $\rho(f) < \infty$.*

PROPOSITION 2.5. [2, Corollary 2.22] *Let $p(\cdot) \in \mathscr{P}(\Omega)$. If $\|f\|_{p(\cdot)} \leqslant 1$, then $\rho(f) \leqslant \|f\|_{p(\cdot)}$.*

We will use our assumption that $p(\cdot) \in LH_\infty(\mathbb{R}^n)$ to apply the following lemma when we prove Theorem 1.2.

LEMMA 2.6. [2, Lemma 3.26] *Let $p(\cdot) \in LH_\infty(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < \infty$. Let $R(x) = (e+|x|)^{-n}$. Then there exists a constant C, depending on n and the LH_∞ constants of $p(\cdot)$, such that given any set E and any function F with $0 \leqslant F(x) \leqslant 1$, for $x \in E$,*

$$\int_E F(x)^{p(x)}\,dx \leqslant C\int_E F(x)^{p_\infty}\,dx + \int_E R(x)^{p_-}\,dx, \tag{2.1}$$

$$\int_E F(x)^{p_\infty}\,dx \leqslant C\int_E F(x)^{p(x)}\,dx + \int_E R(x)^{p_-}\,dx. \tag{2.2}$$

We now recall the definition and basic properties of dyadic cubes. These are well-known and can be found in [2, Section 3.2].

DEFINITION 2.7. Let $Q_0 = [0,1)^n$, and let $\mathscr{D}_0$ be the set of all translates of Q_0 whose vertices are on the lattice $\mathbb{Z}^n$. More generally, for each $k \in \mathbb{Z}$, let $Q_k = 2^{-k}Q_0 = [0,2^{-k})^n$, and let $\mathscr{D}_k$ be the set of all translates of Q_k whose vertices are on the lattice $2^{-k}\mathbb{Z}^n$. Define the set of dyadic cubes $\mathscr{D}$ by

$$\mathscr{D} = \bigcup_{k\in\mathbb{Z}} \mathscr{D}_k.$$

PROPOSITION 2.8. *Dyadic cubes have the following properties:*

1. *For each* $k \in \mathbb{Z}$, *if* $Q \in \mathscr{D}_k$, *then* $\ell(Q) = 2^{-k}$, *where* $\ell(Q)$ *is the side length of* Q.
2. *For each* $x \in \mathbb{R}^n$ *and* $k \in \mathbb{Z}$, *there exists a unique cube* $Q \in \mathscr{D}_k$ *such that* $x \in Q$.
3. *Given any two cubes* $Q_1, Q_2 \in \mathscr{D}$, *either* $Q_1 \cap Q_2 = \emptyset$, $Q_1 \subset Q_2$, *or* $Q_2 \subset Q_1$.
4. *For each* $k \in \mathbb{Z}$, *if* $Q \in \mathscr{D}_k$, *then there exists a unique cube* $\widehat{Q} \in \mathscr{D}_{k-1}$ *such that* $Q \subset \widehat{Q}$. *(*$\widehat{Q}$ *is referred to as the dyadic parent of* Q.*)*
5. *For each* $k \in \mathbb{Z}$, *if* $Q \in \mathscr{D}_k$, *then there exist* 2^n *cubes* $P_i \in \mathscr{D}_{k+1}$ *such that* $P_i \subset Q$.

The next proposition gives an equivalent characterization of $\varepsilon LH_0(\mathbb{R}^n)$ which will be used in the proof of Theorem 1.2.

PROPOSITION 2.9. *Given a non-negative sequence* $\varepsilon = \{\varepsilon_Q\}_{Q\in\mathscr{D}}$, $p(\cdot) \in \varepsilon LH_0(\mathbb{R}^n)$ *if and only if there exists* $C > 0$ *such that for all* $Q \in \mathscr{D}$ *with* $\varepsilon_Q \neq 0$ *and almost every* $x \in Q$,

$$\left(\frac{|Q|}{\varepsilon_Q}\right)^{p_-(Q)-p(x)} \leqslant C. \tag{2.3}$$

Proof. Assume $p(\cdot) \in \varepsilon LH_0(\mathbb{R}^n)$. Fix $Q \in \mathscr{D}$ with $\varepsilon_Q \neq 0$. Observe that if $|Q| > \varepsilon_Q$, then for any $x \in Q$, (2.3) holds with $C = 1$. Suppose $|Q| \leqslant \varepsilon_Q$. Then for any $x \in Q$, we have

$$\left(\frac{|Q|}{\varepsilon_Q}\right)^{p_-(Q)-p(x)} \leqslant \left(\frac{|Q|}{\varepsilon_Q}\right)^{p_-(Q)-p_+(Q)} \leqslant C.$$

To prove the converse, observe that if $|Q| > \varepsilon_Q$, then (1.4) holds with $C = 1$. Suppose $|Q| \leqslant \varepsilon_Q$. Let $\delta > 0$ be arbitrarily small and choose $x_0 \in Q$ such that $p(x_0) + \delta > p_+(Q)$. Then by the definition of $\varepsilon LH_0(\mathbb{R}^n)$, we have

$$\left(\frac{|Q|}{\varepsilon_Q}\right)^{p_-(Q)-p_+(Q)} \leqslant \left(\frac{|Q|}{\varepsilon_Q}\right)^{p_-(Q)-p(x_0)-\delta} \leqslant C\left(\frac{|Q|}{\varepsilon_Q}\right)^{-\delta}.$$

If we let δ tend to 0, we see that $p(\cdot) \in \varepsilon LH_0(\mathbb{R}^n)$. $\square$

In order to prove Theorem 1.2, we need the following Calderon-Zygmund decomposition for the ε-maximal operator. This is very similar to the classical Calderon-Zygmund decomposition for the dyadic maximal operator [2, Lemma 3.9]. For the convenience of the reader we include the short proof.

LEMMA 2.10. *Fix a non-negative sequence* $\varepsilon = \{\varepsilon_Q\}_{Q\in\mathscr{D}}$. *Let* $f \in L^1_{loc}(\mathbb{R}^n)$ *be such that* $⨍_Q |f(y)|\,dy \to 0$ *as* $|Q| \to \infty$. *Given* $\lambda > 0$, *there exists a (possibly empty) collection of disjoint dyadic cubes* $\{Q_j^\lambda\}_j$ *such that*

$$\Omega_\lambda = \{x \in \mathbb{R}^n : M_\varepsilon f(x) > \lambda\} = \bigcup_j Q_j^\lambda, \tag{2.4}$$

and for each Q_j^λ,

$$\lambda < \varepsilon_{Q_j^\lambda} \fint_{Q_j^\lambda} |f(y)|\,dy. \tag{2.5}$$

If, in addition, ε *has the domination property, then*

$$\varepsilon_{Q_j^\lambda} \fint_{Q_j^\lambda} |f(y)|\,dy \leqslant 2^n \lambda. \tag{2.6}$$

Proof. If Ω_λ is empty, then we choose an empty collection and the conclusions hold trivially. Suppose Ω_λ is nonempty and let $x \in \Omega_\lambda$. Then there exists $Q \in \mathscr{D}$ containing x such that

$$\varepsilon_Q \fint_Q |f(y)|\,dy > \lambda.$$

Since $\{\varepsilon_Q\}_{Q \in \mathscr{D}}$ is bounded and $\fint_Q |f(y)|\,dy \to 0$ as $|Q| \to \infty$, there is a maximal dyadic cube with this property. Denote it by Q_x. Clearly, $\Omega_\lambda \subseteq \bigcup_{x \in \Omega_\lambda} Q_x$. The reverse inclusion holds as well. To see this, consider any Q_x and let $z \in Q_x$. Then

$$M_\varepsilon f(z) \geqslant \varepsilon_{Q_x} \fint_{Q_x} |f(y)|\,dy\, \chi_{Q_x}(z) > \lambda,$$

and so $z \in \Omega_\lambda$. By the properties of dyadic cubes, any two cubes in $\{Q_x\}_{x \in \Omega_\lambda}$ are equal or disjoint. Since $\mathscr{D}$ is countable, there are at most countably many such cubes Q_x. Enumerate these cubes by $\{Q_j^\lambda\}_j$. Clearly these cubes satisfy (2.4).

Inequality (2.5) is immediate by our choice of $\{Q_j^\lambda\}_j$. If we assume the domination property holds, to show (2.6), note that we have $\varepsilon_{\widehat{Q}_j^\lambda} \geqslant \varepsilon_{Q_j^\lambda}$. If we combine this with the maximality of Q_j^k, we get

$$\lambda \geqslant \varepsilon_{\widehat{Q}_j^\lambda} \fint_{\widehat{Q}_j^\lambda} |f(y)|\,dy \geqslant \varepsilon_{Q_j^\lambda} \fint_{\widehat{Q}_j^\lambda} |f(y)|\,dy \geqslant 2^{-n} \varepsilon_{Q_j^\lambda} \fint_{Q_j^\lambda} |f(y)|\,dy.$$

If we multiply by 2^n, we get the desired upper bound. □

In order to prove Theorem 1.4, we need a local version of Lemma 2.10. We state it and briefly outline how to adapt the proof of Lemma 2.10 to prove it.

LEMMA 2.11. *Given* $Q_0 \in \mathscr{D}$, *a sequence* $\varepsilon = \{\varepsilon_Q\}_{Q \in \mathscr{D}(Q_0)}$, *and* $f \in L^1(Q_0)$, *for any* $\lambda > \varepsilon_{Q_0} \fint_{Q_0} |f(y)|\,dy$, *there exists a (possibly empty) collection of disjoint cubes* $\{Q_j^\lambda\}_j$ *such that*

$$\Omega_\lambda = \{x \in Q_0 : M_\varepsilon f(x) > \lambda\} = \bigcup_j Q_j^\lambda,$$

and for each Q_j^λ, *inequality* (2.5) *holds. If, in addition,* ε *has the domination property, then inequality* (2.6) *holds.*

Proof. Choose the collection $\{Q_j^\lambda\}_j$ as in the proof of Lemma 2.10. The lower bound in inequality (2.5) is immediate. The proof of the upper bound depends on every Q_j^λ having a dyadic parent $\widehat{Q}_j^\lambda$ in Q_0, which will hold if and only if Q_0 is not in the collection $\{Q_j^\lambda\}_j$. Recall that we chose the cubes Q_j^λ as the maximal cubes satisfying $\varepsilon_{Q_j^\lambda} \fint_{Q_j^\lambda} |f(y)|\,dy > \lambda$. Since we only consider $\lambda > \varepsilon_{Q_0} \fint_{Q_0} |f(y)|\,dy$, Q_0 is not in $\{Q_j^\lambda\}_j$. Hence, every cube in $\{Q_j^\lambda\}_j$ has a dyadic parent in Q_0, and so the proof of inequality (2.6) is the same as in the proof of Lemma 2.10. $\square$

The following lemma allows us to apply the Calderon-Zygmund decomposition to any function in $L^{p(\cdot)}(\mathbb{R}^n)$ when $p_+ < \infty$.

LEMMA 2.12. [2, Lemma 3.29] *Given $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, suppose $p_+ < \infty$. Then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $\fint_Q |f(y)|\,dy \to 0$ as $|Q| \to \infty$.*

To prove Theorem 1.3, we need some lemmas about the conjugate exponent function $p'(\cdot)$, defined pointwise by

$$\frac{1}{p'(x)} = 1 - \frac{1}{p(x)}.$$

The first two lemmas will allow us to transfer properties of $p(\cdot)$ to $p'(\cdot)$. The first is well-known and is an immediate consequence of the definition. See [2].

LEMMA 2.13. *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < \infty$. Then $p(\cdot) \in LH_\infty(\mathbb{R}^n)$ if and only if $p'(\cdot) \in LH_\infty(\mathbb{R}^n)$.*

LEMMA 2.14. *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < \infty$. Then $p(\cdot) \in \varepsilon LH_0(\mathbb{R}^n)$ if and only if $p'(\cdot) \in \varepsilon LH_0(\mathbb{R}^n)$.*

Proof. Since $1 < p_- \leqslant p_+ < \infty$, we have that $p_+(Q) - p_-(Q)$ and $(p')_+(Q) - (p')_-(Q)$ are finite for all $Q \in \mathscr{D}$. Assume first that $p(\cdot) \in \varepsilon LH_0(\mathbb{R}^n)$. Let $Q \in \mathscr{D}$. Since $(p')_-(Q) - (p')_+(Q) \leqslant 0$, we have that if $|Q| > \varepsilon_Q$, then inequality (1.4) holds with $C = 1$. Suppose that $|Q| \leqslant \varepsilon_Q$. To show $p'(\cdot) \in \varepsilon LH_0(\mathbb{R}^n)$, it suffices to show that there is a constant $C_1 > 0$ depending only on $p(\cdot)$ such that for any $Q \in \mathscr{D}$, we have

$$(p')_+(Q) - (p')_-(Q) \leqslant C_1(p_+(Q) - p_-(Q)). \tag{2.7}$$

For if this is the case, then, since $|Q| \leqslant \varepsilon_Q$, if (2.7) holds, we have

$$\left(\frac{|Q|}{\varepsilon_Q}\right)^{(p')_-(Q)-(p')_+(Q)} \leqslant \left(\frac{|Q|}{\varepsilon_Q}\right)^{C_1(p_-(Q)-p_+(Q))}.$$

Since $p(\cdot) \in \varepsilon LH_0(\mathbb{R}^n)$, the right hand side is bounded by a constant depending only on n, $p(\cdot)$, and C_1 and so $p'(\cdot) \in \varepsilon LH_0(\mathbb{R}^n)$.

We now prove that inequality (2.7) holds. By the definition of conjugate exponent functions,

$$\frac{1}{(p')_+(Q)} = 1 - \frac{1}{p_-(Q)} \qquad \text{and} \qquad \frac{1}{(p')_-(Q)} = 1 - \frac{1}{p_+(Q)}.$$

But then we have that

$$\begin{aligned}
(p')_+(Q) - (p')_-(Q) &= (p')_+(Q)(p')_-(Q)\left[\frac{1}{(p')_-(Q)} - \frac{1}{(p')_+(Q)}\right] \\
&= (p')_+(Q)(p')_-(Q)\left[\frac{1}{p_-(Q)} - \frac{1}{p_+(Q)}\right] \\
&= \frac{(p')_+(Q)(p')_-(Q)}{p_-(Q)p_+(Q)}[p_+(Q) - p_-(Q)] \\
&\leqslant \frac{((p')_+)^2}{(p_-)^2}[p_+(Q) - p_-(Q)].
\end{aligned}$$

This proves inequality (2.7), and so $p'(\cdot) \in \varepsilon LH_0(\mathbb{R}^n)$. The proof of the converse is the same, except we interchange the roles of $p(\cdot)$ and $p'(\cdot)$. $\square$

The next result allows us to apply the previous two lemmas when proving Theorem 1.3 and Theorem 1.4.

LEMMA 2.15. [2, Theorem 2.34] *Given $p(\cdot) \in \mathscr{P}(\Omega)$ with $1 < p_- \leqslant p_+ < \infty$, define the associate norm $\|\cdot\|'_{p(\cdot)}$ by*

$$\|f\|'_{p(\cdot)} = \sup\left\{\int_\Omega f(x)g(x)\,dx : g \in L^{p'(\cdot)}(\Omega), \|g\|_{p'(\cdot)} \leqslant 1\right\}.$$

Then for any $f \in L^{p(\cdot)}(\Omega)$, we have $\|f\|_{p(\cdot)} \leqslant \|f\|'_{p(\cdot)}$.

The final lemma is the variable exponent version of Hölder's inequality.

LEMMA 2.16. [2, Theorem 2.26] *Given $p(\cdot) \in \mathscr{P}(\Omega)$ with $1 < p_- \leqslant p_+ < \infty$, for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, $fg \in L^1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|\,dx \leqslant 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

3. Boundedness of the ε-maximal operator

We now prove Theorem 1.2. The proof is adapted from [2, Theorem 3.16].

Proof. We begin the proof by making some reductions. We may assume f is nonnegative since $M_\varepsilon(f) = M_\varepsilon(|f|)$. By homogeneity, we may further assume that $\|f\|_{p(\cdot)} = 1$. From Proposition 2.5, we get that $\rho(f) \leqslant 1$. Decompose f as $f_1 + f_2$, where

$$f_1 = f\chi_{\{x: f(x) > 1\}}, \text{ and } f_2 = f\chi_{\{x: f(x) \leqslant 1\}}.$$

Then $\rho(f_i) \leqslant \|f_i\|_{p(\cdot)} \leqslant 1$ for $i = 1,2$. Further, since $M_\varepsilon f \leqslant M_\varepsilon f_1 + M_\varepsilon f_2$, it will suffice to show for $i = 1,2$ that $\|M_\varepsilon f_i\|_{p(\cdot)} \leqslant C(n, p(\cdot), \varepsilon)$. Since $p_+ < \infty$, by Proposition 2.4 it will in turn suffice to show that for $i = 1,2$,

$$\rho(M_\varepsilon f_i) = \int_{\mathbb{R}^n} M_\varepsilon f_i(x)^{p(x)}\,dx \leqslant C(n, p(\cdot), \varepsilon).$$

We first consider the estimate for f_1. Let $A = 2^n$. For each $k \in \mathbb{Z}$, define

$$\Omega_k = \{x \in \mathbb{R}^n : M_\varepsilon f_1(x) > A^k\}.$$

Up to a set of measure zero, $\mathbb{R}^n = \bigcup_{k\in\mathbb{Z}} \Omega_k \backslash \Omega_{k+1}$. Since $p_+ < \infty$, by Lemma 2.12, f satisfies the hypotheses of Lemma 2.10. Thus, for each k we may form a collection of pairwise disjoint cubes $\{Q_j^k\}_j$ such that (2.4) and (2.5) hold. For each k, define the sets $E_j^k = Q_j^k \cap (\Omega_k \backslash \Omega_{k+1})$. Then for each k, $\{E_j^k\}_j$ forms a pairwise disjoint collection such that $\Omega_k \backslash \Omega_{k+1} = \bigcup_j E_j^k$.

We can now estimate as follows:

$$\begin{aligned}
\rho(M_\varepsilon f_1) &= \sum_k \int_{\Omega_k \backslash \Omega_{k+1}} M_\varepsilon f_1(x)^{p(x)}\,dx \\
&\leqslant \sum_k \int_{\Omega_k \backslash \Omega_{k+1}} (A^{k+1})^{p(x)}\,dx \\
&\leqslant A^{p_+} \sum_{k,j} \int_{E_j^k} \left(\varepsilon_{Q_j^k} \fint_{Q_j^k} f_1(y)\,dy \right)^{p(x)} dx.
\end{aligned}$$

For each k and j, define $p_{jk} = p_-(Q_j^k)$. Since for any $x \in \mathbb{R}^n$, $f_1(x) > 1$ or $f_1(x) = 0$, we then have

$$\int_{Q_j^k} f_1(y)\,dy \leqslant \int_{Q_j^k} f_1(y)^{p(y)/p_{jk}}\,dy \leqslant \int_{Q_j^k} f_1(y)^{p(y)}\,dy \leqslant 1. \tag{3.1}$$

By Proposition 2.9, inequality (3.1), and Hölder's inequality we have

$$\begin{aligned}
&\sum_{k,j} \int_{E_j^k} \left(\varepsilon_{Q_j^k} \fint_{Q_j^k} f_1(y)\,dy \right)^{p(x)} dx \\
&\qquad = \sum_{k,j} \int_{E_j^k} \left(\frac{\varepsilon_{Q_j^k}}{|Q_j^k|} \right)^{p(x)} \left(\int_{Q_j^k} f_1(y)\,dy \right)^{p(x)} dx \\
&\qquad \lesssim \sum_{k,j} \int_{E_j^k} \left(\frac{\varepsilon_{Q_j^k}}{|Q_j^k|} \right)^{p_{jk}} \left(\int_{Q_j^k} f_1(y)\,dy \right)^{p(x)} dx \\
&\qquad \lesssim (1 + \|\varepsilon\|_\infty)^{p_+} \sum_{k,j} \int_{E_j^k} |Q_j^k|^{-p_{jk}} \left(\int_{Q_j^k} f_1(y)\,dy \right)^{p(x)} dx
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (1+\|\varepsilon\|_\infty)^{p_+}\sum_{k,j}\int_{E_j^k}|Q_j^k|^{-p_{jk}}\left(\int_{Q_j^k} f_1(y)^{p(y)/p_{jk}}\,dy\right)^{p(x)} dx\\
&\leqslant (1+\|\varepsilon\|_\infty)^{p_+}\sum_{k,j}\int_{E_j^k}\left(|Q_j^k|^{-1}\int_{Q_j^k} f_1(y)^{p(y)/p_{jk}}\,dy\right)^{p_{jk}} dx\\
&\leqslant (1+\|\varepsilon\|_\infty)^{p_+}\sum_{k,j}\int_{E_j^k}\left(\fint_{Q_j^k} f_1(y)^{p(y)/p_-}\,dy\right)^{p_-} dx\\
&\leqslant (1+\|\varepsilon\|_\infty)^{p_+}\sum_{k,j}\int_{E_j^k} M^d[f_1^{p(\cdot)/p_-}](x)^{p_-}\,dx\\
&= C(p(\cdot),\varepsilon)\int_{\mathbb{R}^n} M^d[f_1^{p(\cdot)/p_-}](x)^{p_-}\,dx.
\end{aligned}
$$

Since $p_->1$, we have $\|M^d f_1\|_{L^{p_-}(\mathbb{R}^n)}\leqslant (p_-)'\|f_1\|_{L^{p_-}(\mathbb{R}^n)}$ (see [7, Theorem 2.3], [3, Exercise 2.1.12]). If we combine this with the fact that $\rho(f_1)\leqslant 1$, we get that

$$
\rho(M_\varepsilon f_1)\leqslant C(n,p(\cdot),\varepsilon)\rho(f_1)\leqslant C(n,p(\cdot),\varepsilon).
$$

We now estimate $\rho(M_\varepsilon f_2)$. Since $f_2\leqslant 1$, we have $\fint_Q f_2(y)\,dy\leqslant 1$ for all $Q\in\mathscr{D}$. Thus, for all $x\in\mathbb{R}^n$,

$$
\frac{\varepsilon_Q}{\|\varepsilon\|_\infty}\fint_Q f_2(y)\,dy\chi_Q(x)\leqslant 1.
$$

Hence, $0\leqslant \|\varepsilon\|_\infty^{-1}M_\varepsilon f_2\leqslant 1$. Let $R(x)=(e+|x|)^{-n}$. Since $p_->1$, we have $p_\infty>1$, and so $\int_{\mathbb{R}^n} M^d f_2(x)^{p_\infty}\,dx\leqslant ((p_\infty)')^{p_\infty}\int_{\mathbb{R}^n} f(x)^{p_\infty}\,dx$. If we combine this with inequalities (2.1), (2.2), and the pointwise bound $M_\varepsilon f_2(x)\leqslant \|\varepsilon\|_\infty M^d f_2(x)$, we have that

$$
\begin{aligned}
\int_{\mathbb{R}^n} M_\varepsilon f_2(x)^{p(x)}\,dx &\leqslant (1+\|\varepsilon\|_\infty)^{p_+}\int_{\mathbb{R}^n}[\|\varepsilon\|_\infty^{-1}M_\varepsilon f_2(x)]^{p(x)}\,dx\\
&\leqslant C(\varepsilon,p(\cdot))\int_{\mathbb{R}^n}[\|\varepsilon\|_\infty^{-1}M_\varepsilon f_2(x)]^{p_\infty}\,dx + C(\varepsilon,p(\cdot))\int_{\mathbb{R}^n} R(x)^{p_-}\,dx\\
&= C(\varepsilon,p(\cdot))\|\varepsilon\|_\infty^{-p_\infty}\int_{\mathbb{R}^n} M_\varepsilon f_2(x)^{p_\infty}\,dx + C(\varepsilon,p(\cdot))\int_{\mathbb{R}^n} R(x)^{p_-}\,dx\\
&\leqslant C(\varepsilon,p(\cdot))\int_{\mathbb{R}^n}\|\varepsilon\|_\infty^{p_\infty}M^d f_2(x)^{p_\infty}\,dx + C(\varepsilon,p(\cdot))\int_{\mathbb{R}^n} R(x)^{p_-}\,dx\\
&\leqslant C(\varepsilon,p(\cdot))((p_\infty)')^{p_\infty}\int_{\mathbb{R}^n} f_2(x)^{p_\infty}\,dx + \|\varepsilon\|_\infty^{p_+}\int_{\mathbb{R}^n} R(x)^{p_-}\,dx\\
&\leqslant C(n,p(\cdot),\varepsilon)\int_{\mathbb{R}^n} f_2(x)^{p(x)}\,dx + C(n,p(\cdot),\varepsilon)\int_{\mathbb{R}^n} R(x)^{p_-}\,dx.
\end{aligned}
$$

Since $\rho(f_2)\leqslant 1$ and $\int_{\mathbb{R}^n} R(x)^{p_-}$ is finite, we have that $\int_{\mathbb{R}^n} M_\varepsilon f_2(x)^{p(x)}\leqslant C(n,p(\cdot),\varepsilon)$. This completes the proof of Theorem 1.2. □

We will need a local version of Theorem 1.2 to prove Theorem 1.4. We state the local version and outline the modifications to the proof. Note that the necessary lemmas and propositions used to prove Theorem 1.2 still hold when replacing $\mathbb{R}^n$ with Q_0.

LEMMA 3.1. *Given $Q_0 \in \mathscr{D}$, fix a non-negative sequence $\varepsilon = \{\varepsilon_Q\}_{Q\in\mathscr{D}(Q_0)}$. Given an exponent function $p(\cdot)$ with $1 < p_- \leqslant p_+ < \infty$, suppose $p(\cdot) \in \varepsilon LH_0(Q_0)$. Then there exists a constant $C = C(p(\cdot), \varepsilon, Q_0)$ such that for all $f \in L^{p(\cdot)}(Q_0)$,*

$$\|M_\varepsilon f\|_{L^{p(\cdot)}(Q_0)} \leqslant C\|f\|_{L^{p(\cdot)}(Q_0}.$$

Proof. We make the same reductions as in the proof of Theorem 1.2; thus, $\|f\|_{L^{p(\cdot)}(Q_0)} = 1$ and we must show that $\rho(M_\varepsilon f_i) \leqslant C$ for $i = 1, 2$. Since $|Q_0|$ is finite and $M_\varepsilon f_2 \leqslant 1$, we immediately have that

$$\rho(M_\varepsilon f_2) \leqslant |Q_0|.$$

To estimate $\rho(M_\varepsilon f_1)$, we modify the argument in Theorem 1.2. Define $A = A(\varepsilon, Q_0, p(\cdot)) = 1 + 2\varepsilon_{Q_0}\|\chi_{Q_0}\|_{L^{p'(\cdot)}(Q_0)}$. Then by the generalized Hölder's inequality in Lemma 2.16,

$$A = 1 + 2\varepsilon_{Q_0}\|\chi_{Q_0}\|_{L^{p'(\cdot)}(Q_0)}\|f\|_{L^{p(\cdot)}(Q_0)} \geqslant 1 + \varepsilon_{Q_0} \fint_{Q_0} |f(y)|\,dy.$$

For each $k \in \mathbb{N}$, define $\Omega_k = \{x \in Q_0 : M_\varepsilon f_1(x) > A^k\}$; the above estimate for A shows that we can apply Lemma 2.11 to form a pairwise disjoint collection $\{Q_j^k\}_j$ such that (2.4) and (2.5) hold. Define the sets $E_j^k = Q_j^k \cap (\Omega_k \backslash \Omega_{k+1})$. Then we can repeat the previous argument to get

$$\begin{aligned}
\rho(M_\varepsilon f_1) &\lesssim \int_{Q_0\backslash\Omega_1} M_\varepsilon f_1(x)^{p(x)}\,dx + \sum_{k,j=1}^{\infty} \int_{E_j^k} \left(\fint_{Q_j^k} f_1(y)^{p(y)/p_-}\,dy \right)^{p_-} dx \\
&\lesssim A^{p_+}|Q_0| + \int_{Q_0} M^d[f_1^{p(\cdot)/p_-}\chi_{Q_0}](x)^{p_-}\,dx \\
&\lesssim C(\varepsilon, Q_0, p(\cdot)) + \int_{Q_0} f_1(x)^{p(x)}\,dx \\
&\leqslant C(\varepsilon, Q_0, p(\cdot)).
\end{aligned}$$

This completes the proof. □

4. Haar multipliers

To prove the Haar multiplier defined in (1.1) is bounded on $L^p(w)$, in [8] they proved it was dominated by a sparse operator. To state their result, first recall that a collection of cubes $\mathscr{S} \subset \mathscr{D}$ is sparse if for every $Q \in \mathscr{S}$, there exists a set $E_Q \subset Q$ such that $|Q| \leqslant 2|E_Q|$ and the family $\{E_Q\}_{Q\in\mathscr{S}}$ is pairwise disjoint.

DEFINITION 4.1. Given a sparse collection $\mathscr{S}$ and a sequence $\varepsilon = \{\varepsilon_Q\}_{Q\in\mathscr{D}}$, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the ε-sparse operator S_ε by

$$S_\varepsilon f(x) = \sum_{Q\in\mathscr{S}} \varepsilon_Q \fint_Q f(y)\,dy\,\chi_Q(x).$$

THEOREM 4.2. [8, Theorem 1.2] *Given a sequence* $\varepsilon = \{\varepsilon_Q\}_{Q\in\mathscr{D}}$, *if* f *is bounded with compact support, then there exists a sparse collection* $\mathscr{S}$ *such that for almost every* $x \in \operatorname{supp}(f)$ *the associated* $\bar{\varepsilon}$*-sparse operator* $S_{\bar{\varepsilon}}$ *satisfies*

$$|T_\varepsilon f(x)| \lesssim S_{\bar{\varepsilon}}|f|(x).$$

Proof of Theorem 1.3. We will first prove that $\|T_\varepsilon f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$ for any $f \in L_c^\infty(\mathbb{R}^n)$. Fix such an f. By Theorem 4.2 it will suffice to show that for any sparse collection $\mathscr{S}$, the associated $\bar{\varepsilon}$-sparse operator $S_{\bar{\varepsilon}}$ satisfies

$$\|S_{\bar{\varepsilon}} f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}.$$

By Lemma 2.15, there exists $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ with $\|g\|_{p'(\cdot)} \leqslant 1$ such that

$$\|S_{\bar{\varepsilon}} f\|_{p(\cdot)} \leqslant \|S_{\bar{\varepsilon}} f\|'_{p(\cdot)} \leqslant 2\int_{\mathbb{R}^n} S_{\bar{\varepsilon}} f(x) g(x)\,dx.$$

By Lemma 2.16, we have that

$$\begin{aligned}
\int_{\mathbb{R}^n} S_{\bar{\varepsilon}} f(x) g(x)\,dx &= \sum_{Q\in\mathscr{S}} \bar{\varepsilon}_Q ⨍_Q f(y)\,dy \int_Q g(x)\,dx \\
&\leqslant 2\sum_{Q\in\mathscr{S}} \bar{\varepsilon}_Q^{1/2} ⨍_Q f(y)\,dy\ \bar{\varepsilon}_Q^{1/2} ⨍_Q g(x)\,dx |E_Q| \\
&\leqslant 2\sum_{Q\in\mathscr{S}} \int_{E_Q} M_{\bar{\varepsilon}^{1/2}} f(t) M_{\bar{\varepsilon}^{1/2}} g(t) dt \\
&\leqslant 2\int_{\mathbb{R}^n} M_{\bar{\varepsilon}^{1/2}} f(t) M_{\bar{\varepsilon}^{1/2}} g(t) dt \\
&\leqslant 4\|M_{\bar{\varepsilon}^{1/2}} f\|_{p(\cdot)} \|M_{\bar{\varepsilon}^{1/2}} g\|_{p'(\cdot)}.
\end{aligned}$$

Since $p(\cdot) \in \bar{\varepsilon}_Q^{1/2} LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n)$, by Lemmas 2.14 and 2.13, we have that $p'(\cdot) \in \bar{\varepsilon}_Q^{1/2} LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n)$. Hence, by Theorem 1.2, we have

$$\|M_{\bar{\varepsilon}^{1/2}} f\|_{p(\cdot)} \|M_{\bar{\varepsilon}^{1/2}} g\|_{p'(\cdot)} \leqslant C\|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} \leqslant C\|f\|_{p(\cdot)}.$$

Therefore, $\|T_\varepsilon f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$ for $f \in L_c^\infty(\mathbb{R}^n)$.

Finally, since $L_c^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$ (see [2, Theorem 2.72]) and T_ε is linear, the desired inequality for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ follows by a standard approximation argument. □

Proof of Theorem 1.4. By Theorem 4.2, it suffices to show that for any sparse collection $\mathscr{S} \subset \mathscr{D}(Q_0)$ and non-negative sequence ε, the associated ε-sparse operator S_ε is compact on $L^{p(\cdot)}(Q_0)$. Fix $\mathscr{S} \subset \mathscr{D}(Q_0)$. For each $N \in \mathbb{N}$, define the set D_N by

$$D_N = \{Q \in \mathscr{D}(Q_0) : 2^{-N} \leqslant \ell(Q) \leqslant 2^N\},$$

and define the operator $S_{\varepsilon,N}$ by

$$S_{\varepsilon,N}f(x) = \sum_{Q\in D_N\cap\mathscr{S}} \varepsilon_Q \fint_Q f(y)\,dy\,\chi_Q(x).$$

Since Q_0 is bounded, D_N is a finite collection for all N. Hence $S_{\varepsilon,N}$ is a finite rank operator for all N. We claim that $S_{\varepsilon,N}$ converges to S_ε in operator norm: i.e., $S_{\varepsilon,N}f \to S_\varepsilon f$ uniformly for all f in the unit ball of $L^{p(\cdot)}(Q_0)$. Fix such an f; then

$$S_\varepsilon f - S_{\varepsilon,N}f = \sum_{Q\in D_N^c\cap\mathscr{S}} \varepsilon_Q \fint_Q f(y)\,dy\,\chi_Q.$$

By Lemma 2.15 there exists $g \in L^{p'(\cdot)}(Q_0)$ with $\|g\|_{p'(\cdot)} \leqslant 1$ such that

$$\Big\| \sum_{Q\in D_N^c\cap\mathscr{S}} \varepsilon_Q \fint_Q f(y)\,dy\,\chi_Q \Big\|_{p(\cdot)} \leqslant 2\int_{Q_0}\Big(\sum_{Q\in D_n^c\cap\mathscr{S}} \varepsilon_Q \fint_Q f(y)\,dy\,\chi_Q(x)\Big) g(x)\,dx.$$

We argue as in the proof of Theorem 1.3, but we split ε_Q into one factor of $\varepsilon_Q^{1-2\alpha}$ and two factors of ε_Q^{α} before using Lemma 2.16. This gives

$$\begin{aligned}
\int_{Q_0} \sum_{D_N^c\cap\mathscr{S}} \varepsilon_Q \fint_Q f(y)\,dy\,\chi_Q(x) g(x)\,dx &\leqslant 2 \sum_{Q\in D_n^c\cap\mathscr{S}} \fint_Q f(y)\,dy \fint_Q g(x)\,dx\,|E_Q| \\
&\leqslant 2 \sup_{Q\in D_N^c} \varepsilon_Q^{1-2\alpha} \sum_{Q\in D_N^c\cap\mathscr{S}} \int_{E_Q} M_{\varepsilon^\alpha} f(z) M_{\varepsilon^\alpha} g(z)dz \\
&\leqslant 2 \sup_{Q\in D_N^c} \varepsilon_Q^{1-2\alpha} \int_{Q_0} M_{\varepsilon^\alpha} f(z) M_{\varepsilon^\alpha} g(z)dz \\
&\leqslant 4 \sup_{Q\in D_N^c} \varepsilon_Q^{1-2\alpha} \|M_{\varepsilon^\alpha} f\|_{p(\cdot)} \|M_{\varepsilon^\alpha} g\|_{p'(\cdot)}.
\end{aligned}$$

Since $p(\cdot) \in \varepsilon^\alpha LH_0(Q_0)$, by Lemma 2.14 we have that $p'(\cdot) \in \varepsilon^\alpha LH_0(Q_0)$. Thus, by Lemma 3.1, we have that

$$\|M_{\varepsilon^\alpha} f\|_{p(\cdot)} \|M_{\varepsilon^\alpha} g\|_{p'(\cdot)} \leqslant C\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)} \leqslant C.$$

Therefore, to complete the proof we need to show that $\sup_{Q\in\mathscr{D}_N^c} \varepsilon_Q^{1-2\alpha} \to 0$ as $N\to\infty$. Choose N_0 such that $2^{N_0} = \ell(Q_0)$. Then for all $N \geqslant N_0$, there are no cubes $Q \in \mathscr{D}(Q_0)$ such that $\ell(Q) > 2^N$. Hence $D_N^c = \{Q \in \mathscr{D}(Q_0) : \ell(Q) < 2^{-N}\}$. Since we assume $\lim_{N\to\infty} \sup\{\varepsilon_Q : \ell(Q) < 2^{-N}\} = 0$, we have that $\sup_{Q\in D_N^c} \varepsilon_Q^{1-2\alpha} \to 0$ as $N\to\infty$. Thus, $S_{\varepsilon,N} \to S_\varepsilon$, and so S_ε is a limit of finite rank operators. Hence, S_ε is compact on $L^{p(\cdot)}(Q_0)$ (see [1, p. 174]). □

5. Examples

In this section, we give sufficient conditions on a sequence ε so that a specific exponent function $p(\cdot)$ is not locally log-Hölder continuous, but is in $\varepsilon LH_0(\mathbb{R})$ and satisfies the domination property. Let $0<a<1$ and define

$$p(x)=\begin{cases}2, & x\leqslant 0,\\ 2+(\log_2\frac{2}{x})^{-a}, & 0<x<1,\\ 3 & x\geqslant 1.\end{cases}$$

This exponent function is not in $LH_0(\mathbb{R})$: see [2, Example 4.44]. Our goal is to give sufficient conditions on ε so that $p(\cdot)\in\varepsilon LH_0(\mathbb{R})$. For each $n\in\mathbb{Z}$, $n\geqslant 0$, define $Q_n^j=[j2^{-n},(j+1)2^{-n})$ for $j=0,\dots,2^n-1$. Fix a constant $1\leqslant C\leqslant 2^{1/(2^a-1)}$ and define $\varepsilon_{Q_n^j}$ by

$$\varepsilon_{Q_n^j}=2^{-n}C^{(n+1)^a}.$$

For cubes of the form $Q=[0,2^k)$, $k\geqslant 1$, define $\varepsilon_Q=C|Q|$. For cubes Q such that $Q\cap[0,1)=\emptyset$, define $\varepsilon_Q=C$.

Given this sequence ε, we claim that $p(\cdot)\in\varepsilon LH_0(\mathbb{R})$. Fix $n\geqslant 0$. Since $(\log_2(2/x))^{-a}$ is an increasing function, it attains its infimum at the left endpoint and its supremum an the right endpoint of any cube. Consequently, for $j=0$, we have

$$p_-(Q_n^0)-p_+(Q_n^0)=-(n+1)^{-a}.$$

Thus, for the cube Q_n^0, we have

$$\left(\frac{|Q_n^0|}{\varepsilon_{Q_n^0}}\right)^{p_-(Q_n^0)-p_+(Q_n^0)}=\left(\frac{2^{-n}}{\varepsilon_{Q_n^0}}\right)^{-(n+1)^{-a}}=(C^{(n+1)^a})^{(n+1)^{-a}}=C.$$

For each $j\neq 0$, we have

$$p_-(Q_n^j)-p_+(Q_n^j)=[n+1-\log_2 j]^{-a}-[n+1-\log_2(j+1)]^{-a}.$$

Thus, for each j, we have that

$$\begin{aligned}\left(\frac{|Q_n^j|}{\varepsilon_{Q_n^j}}\right)^{p_-(Q_n^j)-p_+(Q_n^j)}&=(C^{(n+1)^a})^{p_+(Q_n^j)-p_-(Q_n^j)}\\&=C^{(n+1)^a(n+1-\log_2(j+1))^{-a}}C^{-(n+1)^a(n+1-\log_2 j)^{-a}}.\end{aligned}$$

This expression is bounded: since

$$0<\frac{n+1}{n+1-\log_2(j+1)}\leqslant n+1,$$

we have that $C^{(n+1)^a(n+1-\log_2(j+1))^{-a}} \leqslant C^{(n+1)^a}$. Moreover, since $j \geqslant 1$,

$$\frac{n+1}{n+1-\log_2 j} \geqslant 1,$$

and so we have that $C^{-(n+1)^a(n+1-\log_2 j)^{-a}} \leqslant C^{-1}$. Hence, for all $j = 1,\dots,2^n-1$,

$$\left(\frac{|Q_n^j|}{\varepsilon_{Q_n^j}}\right)^{p_-(Q_n^j)-p_+(Q_n^j)} \leqslant C^{(n+1)^a-1} = C(n,p(\cdot)).$$

Now consider cubes of the form $Q = [0,2^k)$, $k \geqslant 1$. For these cubes we have

$$\left(\frac{|Q|}{\varepsilon_Q}\right)^{p_-(Q)-p_+(Q)} = C^{-1}.$$

Finally, for cubes satisfying $Q \cap [0,1) = \emptyset$, we have that $p_-(Q)-p_+(Q) = 0$. Thus, $(|Q|/\varepsilon_Q)^{p_-(Q)-p_+(Q)} = 1$. Hence, $p(\cdot) \in \varepsilon LH_0(\mathbb{R})$.

We now show that the sequence ε has the domination property. First, if $Q \subset (-\infty,0)$ and $P \subset Q$, then $\varepsilon_P = \varepsilon_Q = C$. Also, if $Q \subset [1,\infty)$ and $P \subset Q$, then $\varepsilon_P = \varepsilon_Q = C$. If $Q = [0,2^k)$, $k \geqslant 1$, and $P \subset [1,\infty)$ with $P \subset Q$, then

$$\varepsilon_P = C \leqslant C|Q| = \varepsilon_Q.$$

If $Q = [0,2^k)$, $k \geqslant 1$, and $P = [j2^{-n},(j+1)2^{-n})$ for some $n \geqslant 0$ and $j = 0,\dots,2^n-1$, then $P \subset Q$. If $n = 0$, then

$$\varepsilon_P = C \leqslant C|Q| = \varepsilon_Q.$$

If $n \geqslant 1$, then $\varepsilon_P \leqslant \varepsilon_Q$ if and only if

$$\log_2 C \leqslant \frac{k+n}{(n+1)^a-1}.$$

Since $(1+n)/[(n+1)^a-1]$ increases as n increases, we have that

$$\frac{k+n}{(n+1)^a-1} \geqslant \frac{1+n}{(n+1)^a-1} \geqslant \frac{2}{2^a-1}.$$

But $C \leqslant 2^{1/(2^a-1)}$, so we have that $\log_2 C \leqslant \frac{k+n}{(n+1)^a-1}$ for all $k \geqslant 1$ and $n \geqslant 1$. Hence, $\varepsilon_P \leqslant \varepsilon_Q$.

Finally, we show that for any $n \geqslant 0$, if $P_{n+1}^m \subset Q_n^j$, then $\varepsilon_{P_{n+1}^m} \leqslant \varepsilon_{Q_n^j}$. For if this is the case, then the domination property holds for any $P,Q \subset [0,1)$ with $P \subset Q$. Let $n \geqslant 0$ and assume $P_{n+1}^m \subset Q_n^j$. Then $\varepsilon_{P_{n+1}^m} \leqslant \varepsilon_{Q_n^j}$ if and only if

$$\log_2 C \leqslant \frac{1}{(n+2)^a-(n+1)^a}.$$

Since $0 < a < 1$, $1/[(n+2)^a-(n+1)^a]$ increases as n increases, so we have that

$$\frac{1}{(n+2)^a-(n+1)^a} \geqslant \frac{1}{2^a-1}.$$

Thus, by our choice of C, $\varepsilon_{P_{n+1}^m} \leqslant \varepsilon_{Q_n^j}$. Hence, the sequence ε has the domination property.

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