Mathematical
Inequalities
& Applications
Volume 26, Number 3 (2023), 703–716 doi:10.7153/mia-2023-26-42

BOUNDEDNESS OF INTEGRAL OPERATORS OF DOUBLE PHASE

YOSHIHIRO MIZUTA AND TETSU SHIMOMURA*

(*Communicated by I. Perić*)

Abstract. Our aim in this note is to establish a Sobolev-type inequality and Trudinger-type inequality for fractional maximal and Riesz potential operators in the framework of general double phase functionals given by

$$\varphi(x,t) = \varphi_1(t) + \varphi_2(b(x)t), \;\; x \in \mathbb{R}^n, \;\; t \geqslant 0,$$

where φ_1, φ_2 are positive convex functions on $(0,\infty)$ and b is a nonnegative function on $[0,\infty)$ which is Hölder continuous of order $\theta \in (0,1]$.

1. Introduction

The classical Sobolev's inequality for Riesz potentials of L^p-functions (see, e.g. [1, Theorem 3.1.4 (b)]) has been extended to various function spaces. For Orlicz spaces, Sobolev's inequality was studied in e.g. [6, 22]. On the other hand, the classical Trudinger's inequality for Riesz potentials of L^p-functions (see, e.g. [1, Theorem 3.1.4 (c)]) has also been extended to function spaces as above. In [2, 21, 22], Trudinger type exponential integrability was studied on Orlicz spaces, as extensions of [9, 10, 12]. See also [11].

The double phase functional introduced by Zhikov ([28]) has been studied by many mathematicians. Regarding regularity theory of differential equations, Baroni, Colombo and Mingione [4, 7, 8] studied a double phase functional

$$\tilde{\varphi}(x,t) = t^p + a(x)t^q, \quad x \in \mathbb{R}^n, \;\; t \geqslant 0,$$

where $1 \leqslant p < q$, a is nonnegative, bounded and Hölder continuous in $\mathbb{R}^n$ of order $\theta \in (0,1]$. In [3], regularity for general functionals was studied under the condition $q \leqslant (1+\theta/n)p$. We refer the reader to [15, 19, 20, 23, 25] for Sobolev inequality and [16] for Trudinger's inequality in the double phase setting. For other recent works, see e.g. [5, 13, 14, 24, 26].

In the present note, we consider a general form of double phase functional given by

$$\varphi(x,t) = \varphi_1(t) + \varphi_2(b(x)t), \quad x \in \mathbb{R}^n, \;\; t \geqslant 0,$$

Mathematics subject classification (2020): 31B15, 46E35.
Keywords and phrases: Fractional maximal functions, Riesz potentials, Sobolev's inequality, Trudinger's inequality, Orlicz spaces, double phase functionals.
* Corresponding author.

where φ_1, φ_2 are positive convex functions on $(0,\infty)$ and b is a nonnegative function on $[0,\infty)$ which is Hölder continuous of order $\theta \in (0,1]$. For typical examples, see Section 2.

Our aim in this note is to establish a Sobolev-type inequality as well as a Trudinger-type inequality for fractional maximal and Riesz potential operators in the framework of general double phase functionals, as an extension of [15, 16, 19, 20]. By treating the general case, we can show new results (e.g. Corollaries 4.2, 6.5 and 6.13) which have not been found in the literature.

Throughout this paper, let C denote various constants independent of the variables in question and $C(a,b,\cdots)$ be a constant that depends on $a,b,\cdots$. Moreover, $f \sim g$ means that $C^{-1}g(r) \leqslant f(r) \leqslant Cg(r)$ for a constant $C > 0$.

2. Orlicz functions

Consider a positive convex function φ on $(0,\infty)$ satisfying

($\varphi0$) $\varphi(0) = \lim_{r\to 0} \varphi(r) = 0$;

($\varphi1$) $t \to t^{-p_1}\varphi(t)$ is almost increasing in $(0,\infty)$ for some $p_1 > 1$, that is, there exists a constant $A_1 \geqslant 1$ such that

$$s^{-p_1}\varphi(s) \leqslant A_1 t^{-p_1}\varphi(t) \quad \text{whenever } 0 < s < t.$$

The typical examples are

$$\varphi(r) = r^p(\log(c+r))^q, \exp(r^p) - 1, \text{etc.},$$

where $p > 1$ and c is chosen so that $c(p-1)+q \geqslant 0$. If $\varphi_1(r) = r^p(\log(e+r))^q$, then it may be replaced by

$$\varphi_2(r) = \int_0^r \{\sup_{0<s<t} s^p(\log(e+s))^q\} t^{-1}dt,$$

which is convex and $\varphi_1 \sim \varphi_2$.

Note here that

($\varphi1'$) $s^{-1}\varphi(s) \leqslant t^{-1}\varphi(t)$ whenever $0 < s < t$;

($\varphi2$) $\int_0^t \varphi(s)/s^2 ds \leqslant A_2\varphi(t)/t$ for $t > 0$;

(φ^{-1}) φ^{-1} is doubling, more precisely,

$$\varphi^{-1}(2r) \leqslant 2\varphi^{-1}(r) \quad \text{for } r > 0.$$

We define

$$\|f\|_{L^\varphi(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \varphi(|f(x)|/\lambda)dx \leqslant 1\right\}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let $L^\varphi(\mathbb{R}^n)$ denote the set all functions f such that $\|f\|_{L^\varphi(\mathbb{R}^n)} < \infty$. Note that $L^\varphi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ when $\varphi(r) = r^p$ for $p > 1$. The Hardy-Littlewood maximal function is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Our fundamental tool is the boundedness of maximal operator.

By using weak L^1 estimate in Stein [27, Chapter 1] and [17, Theorem 1.10.2], we have the boundedness of maximal operator as in [18, Lemma 2.5].

LEMMA 2.1. *Let φ be a positive convex function on $(0,\infty)$ satisfying $(\varphi 0)$ and $(\varphi 1)$. Then there exists a constant $C > 1$ such that*

$$\|Mf\|_{L^\varphi(\mathbb{R}^n)} \leqslant C\|f\|_{L^\varphi(\mathbb{R}^n)}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

3. Integrability of the fractional maximal functions

For $\alpha \geqslant 0$ the fractional maximal function is defined by

$$M_\alpha f(x) = \sup_{r>0} \frac{r^\alpha}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. When $\alpha = 0$, we write Mf instead of $M_\alpha f$ which is the usual maximal function.

In this section, we give integrability of $M_\alpha f$ in Orlicz spaces.

LEMMA 3.1. (cf. [6, Theorem 1]) *Let φ and φ^* be positive convex functions on $(0,\infty)$ satisfying $(\varphi 0)$ and $(\varphi 1)$. Suppose that*

$(\varphi\alpha)$ *$k(t) = t^\alpha \varphi^{-1}(t^{-n})$ is almost decreasing on $(0,\infty)$, that is, there exists a constant $K_1 > 0$ such that*

$$k(s) \leqslant K_1 k(t) \quad \textit{when } 0 < s < t;$$

$(\varphi\varphi^*\alpha)$ *there exists a constant $K_2 > 0$ such that*

$$\varphi^*(t\varphi(t)^{-\alpha/N}) \leqslant K_2\varphi(t) \quad \textit{for } t > 0.$$

Then there exists a constant $C > 1$ such that

$$\|M_\alpha f\|_{L^{\varphi^*}(\mathbb{R}^n)} \leqslant C\|f\|_{L^\varphi(\mathbb{R}^n)}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let f be a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\|f\|_{L^{\varphi}(\mathbb{R}^n)} \leqslant 1$. Let $t > 0$. If $0 < r < t$, then

$$\frac{r^{\alpha}}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy \leqslant t^{\alpha}Mf(x).$$

If $t \leqslant r$, then we have by Jensen's inequality, (φ^{-1}) and $(\varphi\alpha)$

$$\begin{aligned}\frac{r^{\alpha}}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy &\leqslant r^{\alpha}\varphi^{-1}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}\varphi(|f(y)|)dy\right)\\ &\leqslant Cr^{\alpha}\varphi^{-1}\left(r^{-n}\right)\\ &\leqslant Ct^{\alpha}\varphi^{-1}\left(t^{-n}\right),\end{aligned}$$

so that

$$M_{\alpha}f(x) \leqslant t^{\alpha}Mf(x) + Ct^{\alpha}\varphi^{-1}\left(t^{-n}\right).$$

Letting $t = \{\varphi(Mf(x))\}^{-1/n}$, we find

$$M_{\alpha}f(x) \leqslant C_1\{\varphi(Mf(x))\}^{-\alpha/n}Mf(x). \tag{1}$$

By $(\varphi\varphi^*\alpha)$,

$$\varphi^*(M_{\alpha}f(x)/C_1) \leqslant K_2\varphi(Mf(x)). \tag{2}$$

Hence we obtain by Lemma 2.1

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi^*(M_{\alpha}f(x)/C_1)dx &\leqslant K_2\int_{\mathbb{R}^n}\varphi(Mf(x))dx\\ &\leqslant C_2\int_{\mathbb{R}^n}\varphi(|f(x)|)dx\\ &\leqslant C_2,\end{aligned}$$

so that

$$\int_{\mathbb{R}^n}\varphi^*(M_{\alpha}f(x)/(C_1C_2))dx \leqslant 1.$$

Thus this lemma is proved. □

We say that φ^* is the Sobolev conjugate of φ.

4. Integrability of the fractional maximal functions of double phase

In this section, we show integrability of $M_{\alpha}f$ of double phase. Let φ_1 and φ_2 be positive convex functions on $(0,\infty)$ satisfying $(\varphi0)$ and $(\varphi1)$. For $0 \leqslant \theta \leqslant 1$ let b be a nonnegative function satisfying

$$|b(x) - b(y)| \leqslant C|x-y|^{\theta} \quad \text{for } x,y \in \mathbb{R}^n.$$

Let us consider the double phase functional

$$\varphi(x,t) = \varphi_1(t) + \varphi_2(b(x)t)$$

for $x \in \mathbb{R}^n$ and $t \geqslant 0$. Set

$$\varphi^*(x,t) = \varphi_1^*(t) + \varphi_2^*(b(x)t),$$

which plays the Sobolev conjugate of φ. The norm $\|\cdot\|_{L^{\varphi}(\mathbb{R}^n)}$ is defined as before.

THEOREM 4.1. [cf. [20, Theorem 3.1]] *Suppose* $(\varphi_1\alpha)$, $(\varphi_1\alpha+\theta)$, $(\varphi_2\alpha)$, $(\varphi_1\varphi_1^*\alpha)$, $(\varphi_1\varphi_2^*\alpha+\theta)$ *and* $(\varphi_2\varphi_2^*\alpha)$ *hold. Then there exists a constant* $C > 1$ *such that*

$$\|M_\alpha f\|_{L^{\varphi^*}(\mathbb{R}^n)} \leqslant C\{\|f\|_{L^{\varphi_1}(\mathbb{R}^n)} + \|bf\|_{L^{\varphi_2}(\mathbb{R}^n)}\}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let f be a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L^{\varphi_1}(\mathbb{R}^n)} + \|bf\|_{L^{\varphi_2}(\mathbb{R}^n)} \leqslant 1.$$

In view of Lemma 3.1, we have by $(\varphi_1\alpha)$ and $(\varphi_1\varphi_1^*\alpha)$,

$$\int_{\mathbb{R}^n} \varphi_1^*(M_\alpha f(x))dx \leqslant C\int_{\mathbb{R}^n} \varphi_1(|f(x)|)dx \leqslant C.$$

Next note that

$$\begin{aligned}
&b(x)\frac{r^\alpha}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy\\
&= \frac{r^\alpha}{|B(x,r)|}\int_{B(x,r)}\{b(x)-b(y)\}|f(y)|dy + \frac{r^\alpha}{|B(x,r)|}\int_{B(x,r)} b(y)|f(y)|dy\\
&\leqslant \frac{r^\alpha}{|B(x,r)|}\int_{B(x,r)} C|x-y|^\theta|f(y)|dy + \frac{r^\alpha}{|B(x,r)|}\int_{B(x,r)} b(y)|f(y)|dy\\
&\leqslant C\frac{r^{\alpha+\theta}}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy + \frac{r^\alpha}{|B(x,r)|}\int_{B(x,r)} b(y)|f(y)|dy.
\end{aligned}$$

Therefore

$$b(x)M_\alpha f(x) \leqslant CM_{\alpha+\theta}f(x) + M_\alpha[bf](x),$$

so that Lemma 3.1 gives

$$\begin{aligned}
\|bM_\alpha f\|_{L^{\varphi_2^*}(\mathbb{R}^n)} &\leqslant C\left\{\|M_{\alpha+\theta}f\|_{L^{\varphi_2^*}(\mathbb{R}^n)} + \|M_\alpha[bf]\|_{L^{\varphi_2^*}(\mathbb{R}^n)}\right\}\\
&\leqslant C\left\{\|f\|_{L^{\varphi_1}(\mathbb{R}^n)} + \|bf\|_{L^{\varphi_2}(\mathbb{R}^n)}\right\},
\end{aligned}$$

which proves the result. □

COROLLARY 4.2. *Let* $1 < p < q$, $0 < \theta < 1$ *and*

$$1/q^* = 1/q - \alpha/n = 1/p - (\alpha+\theta)/n = 1/p^* - \theta/n > 0.$$

Then there exists a constant $C > 1$ *such that*

$$\|M_\alpha f\|_{L^{p^*}(\mathbb{R}^n)} + \|bM_\alpha f\|_{L^{q^*}(\mathbb{R}^n)} \leqslant C\{\|f\|_{L^p(\mathbb{R}^n)} + \|bf\|_{L^q(\mathbb{R}^n)}\}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

5. Riesz potentials

For $0 < \alpha < n$ and $f \in L^1_{loc}(\mathbb{R}^n)$ we define

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)dy.$$

In this section, we show integrability of $I_\alpha f$ of double phase.

LEMMA 5.1. [cf. [6, Theorem 2]] *Let φ and φ^* be positive convex functions on $(0,\infty)$ satisfying $(\varphi 0)$ and $(\varphi 1)$. Suppose $(\varphi\varphi^*\alpha)$ and*

$(\varphi\alpha+\varepsilon)$ $t^{\alpha+\varepsilon}\varphi^{-1}(t^{-n})$ *is almost decreasing in* $(0,\infty)$

for some $\varepsilon > 0$. Then there exists a constant $C > 1$ such that

$$\|I_\alpha f\|_{L^{\varphi^*}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{\varphi}(\mathbb{R}^n)}$$

for $f \in L^1_{loc}(\mathbb{R}^n)$.

Proof. Let f be a function in $L^1_{loc}(\mathbb{R}^n)$ such that $\|f\|_{L^{\varphi}(\mathbb{R}^n)} \leqslant 1$. For $x \in \mathbb{R}^n$ and $r > 0$ write

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,r)} |x-y|^{\alpha-n} f(y)dy + \int_{\mathbb{R}^n\setminus B(x,r)} |x-y|^{\alpha-n} f(y)dy \\ &= I_1(x) + I_2(x). \end{aligned}$$

Note that

$$|I_1(x)| \leqslant Cr^\alpha Mf(x).$$

Further we see from Jensen's inequality, (φ^{-1}) and $(\varphi\alpha+\varepsilon)$ that

$$\begin{aligned} |I_2(x)| &\leqslant C\int_r^\infty t^{\alpha-n}\left(\int_{B(x,t)} |f(y)|dy\right) t^{-1}dt \\ &\leqslant C\int_r^\infty t^\alpha \varphi^{-1}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)} \varphi(|f(y)|)dy\right) t^{-1}dt \\ &\leqslant C\int_r^\infty t^\alpha \varphi^{-1}\left(t^{-n}\right) t^{-1}dt \\ &= C\int_r^\infty t^{\alpha+\varepsilon}\varphi^{-1}\left(t^{-n}\right) t^{-\varepsilon-1}dt \\ &\leqslant Cr^{\alpha+\varepsilon}\varphi^{-1}\left(r^{-n}\right)\int_r^\infty t^{-\varepsilon-1}dt \\ &\leqslant Cr^\alpha\varphi^{-1}\left(r^{-n}\right). \end{aligned}$$

Thus we obtain

$$|I_\alpha f(x)| \leqslant Cr^\alpha Mf(x) + Cr^\alpha\varphi^{-1}\left(r^{-n}\right).$$

Here, taking $r = \{\varphi(Mf(x))\}^{-1/n}$, we find

$$|I_\alpha f(x)| \leqslant C_1 Mf(x)\{\varphi(Mf(x))\}^{-\alpha/n}.$$

In view of $(\varphi\varphi^*\alpha)$, we establish

$$\int_{\mathbb{R}^n} \varphi^*(|I_\alpha f(x)|/C_1)dx \leqslant K_2 \int_{\mathbb{R}^n} \varphi(Mf(x))dx \leqslant C\int_{\mathbb{R}^n} \varphi(|f(x)|)dx,$$

which gives the result. □

THEOREM 5.2. *Let* φ_1 *and* φ_2 *be positive convex functions on* $(0,\infty)$ *satisfying* $(\varphi 0)$ *and* $(\varphi 1)$. *Let* $\varepsilon > 0$. *Suppose* $(\varphi_1\alpha+\varepsilon)$, $(\varphi_1\alpha+\theta+\varepsilon)$, $(\varphi_2\alpha+\varepsilon)$, $(\varphi_1\varphi_1^*\alpha)$, $(\varphi_1\varphi_2^*\alpha+\theta)$ *and* $(\varphi_2\varphi_2^*\alpha)$ *hold. Then there exists a constant* $C > 1$ *such that*

$$\|I_\alpha f\|_{L^{\varphi^*}(\mathbb{R}^n)} \leqslant C\{\|f\|_{L^{\varphi_1}(\mathbb{R}^n)} + \|bf\|_{L^{\varphi_2}(\mathbb{R}^n)}\}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let f be a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L^{\varphi_1}(\mathbb{R}^n)} + \|bf\|_{L^{\varphi_2}(\mathbb{R}^n)} \leqslant 1.$$

In view of Lemma 5.1, we have by $(\varphi_1\alpha+\varepsilon)$ and $(\varphi_1\varphi_1^*\alpha)$,

$$\int_{\mathbb{R}^n} \varphi_1^*(|I_\alpha f(x)|)dx \leqslant C\int_{\mathbb{R}^n} \varphi_1(|f(x)|)dx \leqslant C.$$

For $x \in \mathbb{R}^n$ and $r > 0$ we have

$$b(x)|I_\alpha f(x)| \leqslant C\int_{\mathbb{R}^n} |x-y|^{\alpha-n}\{b(x)-b(y)\}|f(y)|dy + C\int_{\mathbb{R}^n} |x-y|^{\alpha-n} b(y)|f(y)|dy \leqslant CI_{\alpha+\theta}|f|(x) + CI_\alpha[b|f|](x).$$

Therefore Lemma 5.1 gives

$$\|bI_\alpha f\|_{L^{\varphi_2^*}(\mathbb{R}^n)} \leqslant C\left\{\|I_{\alpha+\theta}|f|\|_{L^{\varphi_2^*}(\mathbb{R}^n)} + \|I_\alpha[b|f|]\|_{L^{\varphi_2^*}(\mathbb{R}^n)}\right\} \leqslant C\left\{\|f\|_{L^{\varphi_1}(\mathbb{R}^n)} + \|bf\|_{L^{\varphi_2}(\mathbb{R}^n)}\right\},$$

which obtains the result. □

COROLLARY 5.3. [cf. [15, Theorem 5.8]] *Let* $1 < p < q$, $0 < \theta < 1$ *and*

$$1/q^* = 1/q - \alpha/n = 1/p - (\alpha+\theta)/n = 1/p^* - \theta/n > 0.$$

Then there exists a constant $C > 1$ *such that*

$$\|I_\alpha f\|_{L^{p^*}(\mathbb{R}^n)} + \|bI_\alpha f\|_{L^{q^*}(\mathbb{R}^n)} \leqslant C\{\|f\|_{L^p(\mathbb{R}^n)} + \|bf\|_{L^q(\mathbb{R}^n)}\}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

6. Exponential integrability

In this section, we give exponential integrability of $M_\alpha f$ and $I_\alpha f$ of douoble phase.

6.1. Exponential integrability for fractional maximal functions

By Jensen's inequality we have

$$\frac{r^\alpha}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy \leqslant r^\alpha \varphi^{-1}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}\varphi(|f(y)|)dy\right).$$

If $r^\alpha\varphi^{-1}(r^{-n})$ is bounded, then $M_\alpha f$ is bounded when $\|f\|_{L^{\varphi_1}(\mathbb{R}^n)} < \infty$.

LEMMA 6.1. *Let φ and ψ be positive convex functions on $(0,\infty)$ satisfying $(\varphi 0)$ and $(\varphi 1)$. Suppose*

$(\varphi\psi\alpha*)$ *there exists a constant $K > 0$ such that*

$$\psi(r\varphi(r)^{-\alpha/n}) \leqslant K\{1+\varphi(r)\} \quad \textit{for } r > 0.$$

Let G be a bounded open set in $\mathbb{R}^n$. Then there exists a constant $C > 1$ such that

$$\|M_\alpha f\|_{L^\psi(G)} \leqslant C\|f\|_{L^\varphi(G)}$$

for $f \in L^1_{\mathrm{loc}}(G)$.

Proof. Let f be a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\|f\|_{L^\varphi(\mathbb{R}^n)} \leqslant 1$. For $x \in G$ and $t > 0$ we have by (1)

$$M_\alpha f(x) \leqslant C_1\{\varphi(Mf(x))\}^{-\alpha/n} Mf(x),$$

so that

$$\psi(M_\alpha f(x)/C_1) \leqslant K\{1+\varphi(Mf(x))\}$$

by $(\varphi\psi\alpha*)$, which gives the result. □

REMARK 6.2. Let $\varphi(r) = r^p(\log(c+r))^{-\varepsilon}$ for $p = n/\alpha > 1$ and $0 < \varepsilon < c(p-1)$. Set

$$\psi(r) = \exp(r^{p/\varepsilon}) - 1.$$

Then

$$\psi(r\varphi(r)^{-\alpha/n}) \leqslant C(1+\varphi(r)) \quad \text{for } r > 0.$$

COROLLARY 6.3. *Let $\varphi(r) = r^p(\log(c+r))^{-\varepsilon}$ for $p = n/\alpha > 1$ and $0 < \varepsilon < c(p-1)$. Set $\psi(r) = \exp(r^{p/\varepsilon}) - 1$ for $r > 0$. If G is a bounded open set in $\mathbb{R}^n$, then there exists a constant $C > 1$ such that*

$$\|M_\alpha f\|_{L^\psi(G)} \leqslant C\|f\|_{L^\varphi(G)}$$

for $f \in L^1_{\mathrm{loc}}(G)$.

THEOREM 6.4. *Let φ_1, φ_2 and ψ_2 be positive convex functions on $(0,\infty)$ satisfying $(\varphi 0)$ and $(\varphi 1)$. Suppose $(\varphi_1\alpha)$, $(\varphi_1\varphi_1^*\alpha)$, $(\varphi_1\psi_2\alpha+\theta*)$ and $(\varphi_2\psi_2\alpha*)$ hold. Set*

$$\psi(x,r) = \varphi_1^*(r) + \psi_2(b(x)r).$$

If G is a bounded open set in $\mathbb{R}^n$, then there exists a constant $C > 1$ such that

$$\|M_\alpha f\|_{L^\psi(G)} \leqslant C\{\|f\|_{L^{\varphi_1}(G)} + \|bf\|_{L^{\varphi_2}(G)}\}$$

for $f \in L^1_{\mathrm{loc}}(G)$.

Proof. As in the proof of Theorem 4.1, Theorem 6.4 is proved by Lemmas 3.1 and 6.1. □

COROLLARY 6.5. *Let $1 < p < q$, $0 < \theta < 1$, $0 < \varepsilon_1 < c(p-1)$, $0 < \varepsilon_2 < c(q-1)$ and*

$$1/q - \alpha/n = 1/p - (\alpha+\theta)/n = 0.$$

Set

$$\varphi_1(r) = r^p(\log(c+r))^{-\varepsilon_1}$$

and

$$\varphi_2(r) = r^q(\log(c+r))^{-\varepsilon_2}.$$

If $p/\varepsilon_1 = q/\varepsilon_2$ and

$$\psi_2(r) = \exp(r^{q/\varepsilon_2}) - 1.$$

Then there exists a constant $C > 1$ such that

$$\|bM_\alpha f\|_{L^{\psi_2}(G)} \leqslant C\{\|f\|_{L^{\varphi_1}(G)} + \|bf\|_{L^{\varphi_2}(G)}\}$$

for $f \in L^1_{\mathrm{loc}}(G)$.

6.2. Exponential integrability for Riesz potentials

We say that a nonnegative function k on $(0,\infty)$ is of log type in $(0,\infty)$ if there exists a constant $K > 0$ such that

$$K^{-1}k(r) \leqslant k(r^2) \leqslant Kk(r) \quad \text{for } r > 0.$$

Finally we are interested in exponential integrability for Riesz potentials.

LEMMA 6.6. *Let φ and ψ be positive convex functions on $(0,\infty)$ satisfying $(\varphi 0)$ and $(\varphi 1)$ such that ψ^{-1} is of log type and there exists a positive continuous function k on $(0,\infty)$ satisfying*

(1) *$r^{\alpha-n}k(r^{-1})\varphi(k(r^{-1}))^{-1}$ is almost decreasing or bounded in $(0,\infty)$;*

(2) *there exists a constant $K_1>0$ such that*

$$r^{\alpha-n}k(r^{-1})\varphi(k(r^{-1}))^{-1}\leqslant K_1\psi^{-1}(1/r)\quad \textit{for } r>0;$$

(3) *there exists a constant $K_2>0$ such that*

$$\int_r^{d_G} t^{\alpha}k(t^{-1})t^{-1}dt\leqslant K_2\psi^{-1}(1/r)\quad \textit{for } r>0,$$

where d_G denotes the diameter of a bounded open set G in $\mathbb{R}^n$.

Then there exists a constant $C>1$ such that

$$\|I_\alpha f\|_{L^{\psi}(G)}\leqslant C\|f\|_{L^{\varphi}(G)}$$

for $f\in L^1_{\mathrm{loc}}(G)$

Proof. For $x\in G$ and $r>0$ write

$$\begin{aligned}I_\alpha f(x) &= \int_{G\cap B(x,r)}|x-y|^{\alpha-n}f(y)dy+\int_{G\setminus B(x,r)}|x-y|^{\alpha-n}f(y)dy\\ &= I_1(x)+I_2(x).\end{aligned}$$

Note that

$$|I_1(x)|\leqslant Cr^{\alpha}Mf(x).$$

Further we see from $(\varphi 1')$ and our assumptions (1)–(3) that

$$\begin{aligned}|I_2(x)| &= \int_{G\setminus B(x,r)}|x-y|^{\alpha-n}|f(y)|dy\\ &\leqslant \int_{G\setminus B(x,r)}|x-y|^{\alpha-n}k(|x-y|^{-1})dy\\ &\quad +\int_{G\setminus B(x,r)}|x-y|^{\alpha-n}|f(y)|\frac{|f(y)|^{-1}\varphi(|f(y)|)}{k(|x-y|^{-1})^{-1}\varphi(k(|x-y|^{-1}))}dy\\ &\leqslant C\int_r^{d_G}t^{\alpha}k(t^{-1})t^{-1}dt+C\{1+r^{\alpha-n}k(r^{-1})\varphi(k(r^{-1}))^{-1}\}\int_{G\setminus B(x,r)}\varphi|f(y)|)dy\\ &\leqslant C\psi^{-1}(1/r).\end{aligned}$$

Thus we obtain

$$|I_\alpha f(x)|\leqslant Cr^{\alpha}Mf(x)+C\psi^{-1}(1/r).$$

Here, taking $r = \{Mf(x)\}^{-1/\alpha}\{\psi^{-1}(Mf(x))\}^{1/\alpha}$, we find

$$|I_\alpha f(x)| \leqslant C\psi^{-1}(Mf(x))$$

since ψ^{-1} is of log type. In view of Jensen's inequality, Lemma 2.1 and (φ^{-1}), we establish

$$\begin{aligned}\int_G \psi(|I_\alpha f(x)|)dx &\leqslant C\int_G Mf(x)dx \\ &\leqslant C|G|\varphi^{-1}\left(\frac{1}{|G|}\int_G \varphi(Mf(x))dx\right) \\ &\leqslant C|G|\varphi^{-1}\left(\frac{1}{|G|}\int_G \varphi(|f(x)|)dx\right),\end{aligned}$$

which gives the result. □

REMARK 6.7. Let $\varphi(r) = r^p(\log(c+r))^a$ for $p = n/\alpha > 1$ and $c(p-1)+a \geqslant 0$, and $k(r) = r^\alpha(\log(e+r))^{-(1+a)/p}$. Then

(1) $r^{\alpha-n}k(r^{-1})\varphi(k(r^{-1}))^{-1} \sim (\log(e+r^{-1}))^{(p-1-a)/p}$;

(2) $t^\alpha k(t^{-1}) = (\log(e+t^{-1}))^{-(1+a)/p}$ and

$$\int_r^{d_G} t^\alpha k(t^{-1})t^{-1}dt \leqslant C(\log(e+r^{-1}))^{1-(1+a)/p}$$

when $1-(1+a)/p > 0$.

COROLLARY 6.8. *Let $\varphi(r) = r^p(\log(c+r))^a$ for $p = n/\alpha > 1$, $c \geqslant -a/(p-1)$ and $-1 < a < p-1$. Set $\psi(r) = \exp(r^{p/(p-1-a)}) - 1$ for $r > 0$. If G is a bounded open set in $\mathbb{R}^n$, then there exists a constant $C > 1$ such that*

$$\|I_\alpha f\|_{L^\psi(G)} \leqslant C\|f\|_{L^\varphi(G)}$$

for $f \in L^1_{\mathrm{loc}}(G)$.

REMARK 6.9. Let $\varphi(r) = r^p(\log(e+r))^{p-1}$ for $p = n/\alpha > 1$ and $k(r) = r^\alpha(\log(e+r))^{-1}(\log(e+(\log(e+r))))^{-1/p}$. Then

(1) $r^{\alpha-n}k(r^{-1})\varphi(k(r^{-1}))^{-1} \sim (\log(e+(\log(e+r^{-1}))))^{1-1/p}$;

(2) $t^\alpha k(t^{-1}) = (\log(e+t^{-1}))^{-1}(\log(e+(\log(e+t^{-1}))))^{-1/p}$ and

$$\int_r^{d_G} t^\alpha k(t^{-1})t^{-1}dt \leqslant C(\log(e+(\log(e+r^{-1}))))^{1-1/p}.$$

COROLLARY 6.10. *Let* $\varphi(r) = r^p(\log(e+r))^{p-1}$ *for* $p = n/\alpha > 1$. *Set* $\psi(r) = \exp(\exp(r^{p'})-1)-1$ *for* $r > 0$. *If* G *is a bounded open set in* $\mathbb{R}^n$, *then there exists a constant* $C > 1$ *such that*

$$\|I_\alpha f\|_{L^\psi(G)} \leqslant C\|f\|_{L^\varphi(G)}$$

for $f \in L^1_{\mathrm{loc}}(G)$.

Compare Corollaries 6.8 and 6.10 with [21, Theorems A and B].

THEOREM 6.11. *Let* $\{\alpha, \varphi_2, \psi_2\}$ *and* $\{\alpha+\theta, \varphi_1, \psi_1\}$ *be as in Lemma 6.6. Suppose there exists a constant* $K > 0$ *such that*

$$\psi_2(r) \leqslant K\{1+\psi_1(r)\} \quad \text{for } r > 0. \tag{3}$$

Then there exists a constant $C > 1$ *such that*

$$\|bI_\alpha f\|_{L^{\psi_2}(G)} \leqslant C\{\|f\|_{L^{\varphi_1}(G)} + \|bf\|_{L^{\varphi_2}(G)}\}$$

for $f \in L^1_{\mathrm{loc}}(G)$.

Proof. As in the proof of Theorem 5.2, Theorem 6.11 is proved by Lemma 6.6. We have only to note that in view of (3) and Lemma 6.6

$$\begin{aligned}\int_G \psi_2(I_{\alpha+\theta}|f|(x)|)dx &\leqslant K\int_G \{1+\psi_1(I_{\alpha+\theta}|f|(x)|)\}dx \\ &\leqslant C + C\int_G \varphi_1(|f(x)|)dx \leqslant C\end{aligned}$$

since $\{\alpha+\theta, \varphi_1, \psi_1\}$ is as in Lemma 6.6. □

COROLLARY 6.12. [cf. [16, Theorem 4.10]] *Let* $1 < p < q$, $0 < \theta < 1$ *and*

$$1/q - \alpha/n = 1/p - (\alpha+\theta)/n = 0.$$

Set

$$\psi_2(r) = \exp(r^{q'}) - 1.$$

Then there exists a constant $C > 1$ *such that*

$$\|I_\alpha f\|_{L^{p^*}(G)} + \|bI_\alpha f\|_{L^{\psi_2}(G)} \leqslant C\{\|f\|_{L^p(G)} + \|bf\|_{L^q(G)}\}$$

for $f \in L^1_{\mathrm{loc}}(G)$.

COROLLARY 6.13. *Let* $\varphi_1(r) = r^{p_1}(\log(c_1+r))^{a_1}$ *with* $p_1 = n/(\alpha+\theta)$ *and* $\varphi_2(r) = r^{p_2}(\log(c_2+r))^{a_2}$ *with* $p_2 = n/\alpha$. *Suppose* $c_1(p_1-1)+a_1 \geqslant 0$, $c_2(p_2-1)+a_2 \geqslant 0$, $-1 < a_1 < p_1-1$, $-1 < a_2 < p_2-1$,

$$\frac{p_1}{p_1-1-a_1} = \frac{p_2}{p_2-1-a_2} > 1$$

and set

$$\psi(r) = \exp(r^{p_1/(p_1-1-a_1)}) - 1.$$

Then there exists a constant $C > 1$ *such that*

$$\|bI_\alpha f\|_{L^\psi(G)} \leqslant C\{\|f\|_{L^{\varphi_1}(G)} + \|bf\|_{L^{\varphi_2}(G)}\}$$

for $f \in L^1_{\mathrm{loc}}(G)$.

REFERENCES

[1] D. R. ADAMS AND L. I. HEDBERG, *Function Spaces and Potential Theory*, Springer, 1996.

[2] A. ALBERICO AND A. CIANCHI, *Differentiability properties of Orlicz-Sobolev functions*, Ark. Mat. **43** (2005), 1–28.

[3] P. BARONI, M. COLOMBO AND G. MINGIONE, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations **57** (2018), no. 2, Paper No. 62, 48 pp.

[4] P. BARONI, M. COLOMBO AND G. MINGIONE, *Non-autonomous functionals, borderline cases and related function classes*, St Petersburg Math. J. **27** (2016), 347–379.

[5] S. S. BYUN, N. CHO AND Y. YOUN, *Global gradient estimates for a borderline case of double phase problems with measure data*, J. Math. Anal. Appl. **501** (2021), 124072, 31 pp.

[6] A. CIANCHI, *Strong and weak type inequalities for some classical operators in Orlicz spaces*, J. London Math. Soc. (2) **60** (1999), no. 1, 187–202.

[7] M. COLOMBO AND G. MINGIONE, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215** (2015), no. 2, 443–496.

[8] M. COLOMBO AND G. MINGIONE, *Bounded minimizers of double phase variational integrals*, Arch. Rat. Mech. Anal. **218** (2015), 219–273.

[9] D. E. EDMUNDS, P. GURKA AND B. OPIC, *Double exponential integrability, Bessel potentials and embedding theorems*, Studia Math. **115** (1995), 151–181.

[10] D. E. EDMUNDS, P. GURKA AND B. OPIC, *Sharpness of embeddings in logarithmic Bessel-potential spaces*, Proc. Royal Soc. Edinburgh. **126** (1996), 995–1009.

[11] D. E. EDMUNDS AND R. HURRI-SYRJÄNEN, *Sobolev inequalities of exponential type*, Israel. J. Math. **123** (2001), 61–92.

[12] D. E. EDMUNDS AND M. KRBEC, *Two limiting cases of Sobolev imbeddings*, Houston J. Math. **21** (1995), 119–128.

[13] C. DE FILIPPIS AND G. PALATUCCI, *Hölder regularity for nonlocal double phase equations*, J. Differential Equations **267** (2019), no. 1, 547–586.

[14] P. HARJULEHTO, P. HÄSTÖ AND M. LEE, *Hölder continuity of quasiminimizers and* ω*-minimizers of functionals with generalized Orlicz growth*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) XXII (2021), 549–582.

[15] F.-Y. MAEDA, Y. MIZUTA, T. OHNO AND T. SHIMOMURA, *Sobolev's inequality for double phase functionals with variable exponents*, Forum Math. **31** (2019), 517–527.

[16] F.-Y. MAEDA, Y. MIZUTA, T. OHNO AND T. SHIMOMURA, *Trudinger's inequality for double phase functionals with variable exponents*, Czechoslovak Math. J. **71** (2021), 511–528.

[17] Y. MIZUTA, *Potential Theory in Euclidean Spaces*, Gakkōtosyo, Tokyo, 1996.

[18] Y. MIZUTA, T. OHNO AND T. SHIMOMURA, *Integrability of maximal functions for generalized Lebesgue spaces with variable exponent*, Math. Nachr. **281** (2008), 386–395.

[19] Y. MIZUTA, T. OHNO AND T. SHIMOMURA, *Sobolev's theorem for double phase functionals*, Math. Ineq. Appl. **23** (2020), 17–33.

[20] Y. MIZUTA, T. OHNO AND T. SHIMOMURA, *Boundedness of fractional maximal operators for double phase functionals with variable exponents*, J. Math. Anal. Appl. **501** (2021), no. 1, 124360.

[21] Y. MIZUTA AND T. SHIMOMURA, *Exponential integrability for Riesz potentials of functions in Orlicz classes*, Hiroshima Math. J. **28** (1998), 355–371.

[22] Y. MIZUTA AND T. SHIMOMURA, *Continuity properties of Riesz potentials of Orlicz functions*, Tohoku Math. J. **61** (2009), 225–240.

[23] Y. MIZUTA AND T. SHIMOMURA, *Hardy-Sobolev inequalities in the unit ball for double phase functionals*, J. Math. Anal. Appl. **501** (2021), 124133, 17 pp.

[24] Y. MIZUTA AND T. SHIMOMURA, *Boundary growth of Sobolev functions of monotone type for double phase functionals*, Ann. Fenn. Math. **47** (2022), 23–37.

[25] Y. MIZUTA AND T. SHIMOMURA, *Hardy-Sobolev inequalities and boundary growth of Sobolev functions for double phase functionals on the half space*, to appear in Ricerche di Matematica.

[26] M. A. RAGUSA AND A. TACHIKAWA, *Regularity for minimizers for functionals of double phase with variable exponents*, Adv. Nonlinear Anal. **9** (2020), no. 1, 710–728.

[27] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, 1970.

[28] V. V. ZHIKOV, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), 675–710.

(Received September 22, 2022)

Yoshihiro Mizuta
Graduate School of Advanced Science and Engineering
Hiroshima University
Higashi-Hiroshima 739-8521, Japan
e-mail: yomizuta@hiroshima-u.ac.jp

Tetsu Shimomura
Department of Mathematics
Graduate School of Humanities and Social Sciences
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
e-mail: tshimo@hiroshima-u.ac.jp