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# THE PROOF OF A NOTABLE SYMMETRIC INEQUALITY

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*Abstract.* In this paper we give a proof of the inequality

$$\frac{1}{a_1^2+1}+\frac{1}{a_2^2+1}+\cdots+\frac{1}{a_n^2+1}\geqslant\frac{n}{2}$$

for nonnegative real numbers $a_1,a_2,\ldots,a_n$ satisfying

$$\sum_{1\leqslant i<j\leqslant n}a_ia_j=\frac{n(n-1)}{2}.$$

The inequality is an equality for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=a_2=\cdots=a_{n-1}=\sqrt{\frac{n}{n-2}}$ and $a_n=0$ (or any cyclic permutation).

## 1. Introduction

A proof of the inequality

$$\frac{1}{a_1^2+1}+\frac{1}{a_2^2+1}+\cdots+\frac{1}{a_n^2+1}\geqslant\frac{n}{2} \tag{1}$$

is given in [3] for $n\leqslant 8$ and nonnegative real numbers $a_1,a_2,\ldots,a_n$ under the constraint

$$\sum_{1\leqslant i<j\leqslant n}a_ia_j=\frac{n(n-1)}{2}. \tag{2}$$

Note that this inequality was proposed and proved for $n=3$ in 2005 (see [2]). Later, the inequality was given for $n=4$ at the Olympic Revenge Contest from Brazil-2013 (see [5]) and, in the same year, Henrique Vaz posted it on the website Art of Problem Solving [6], where the readers have presented three distinct proofs (for $n=4$).

In this paper, we give a proof for any integer $n\geqslant 3$. The proof is based on the following result in [3] (Theorem 4.1):

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THEOREM 1. *Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ and*

$$\sum_{1 \leqslant i < j \leqslant n} a_i a_j = \frac{n(n-1)}{2}.$$

*Let $k = \left\lfloor \frac{n}{2} \right\rfloor + 1$. If the inequality (1) holds for the particular cases*

*a)* $a_1 = a_2 = \cdots = a_k$ *and* $a_n = 0$,

*b)* $a_1 = a_2 = \cdots = a_k$ *and* $a_{n-1} = a_n > 0$,

*then it holds for all $a_1, a_2, \dots, a_n$.*

To prove the inequality (1) under the constraint (2), we will assume

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$$

and will use the induction method, Theorem 1 and the method of Lagrange multipliers (for fixed $\sum_{i=1}^n a_i$ and $\sum_{1 \leqslant i < j \leqslant n} a_i a_j$).

## 2. Method of Lagrange multipliers

Let $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0$ and

$$f_1(a_1, \dots, a_n) = \sum_{i=1}^n a_i,$$

$$f_2(a_1, \dots, a_n) = \sum_{1 \leqslant i < j \leqslant n} a_i a_j.$$

Under the constraints

$$\sum_{i=1}^n a_i = S_1,$$

$$\sum_{1 \leqslant i < j \leqslant n} a_i a_j = \frac{n(n-1)}{2} := S_2$$

(which define a smooth compact manifold), the minimum $m(S_1)$ of the expression

$$E = \frac{1}{a_1^2+1} + \frac{1}{a_2^2+1} + \cdots + \frac{1}{a_n^2+1}$$

exists. Thus, to prove the inequality (1) under the constraint (2), it suffices to show that $m(S_1) \geqslant \frac{n}{2}$, that means to prove the inequality (1) for $a_i$ chosen to minimize the expression $E$. The minimum of $E$ occurs at a point $(a_1, a_2, \dots, a_n)$ with $a_n = 0$, or at a point that satisfies the Lagrange multiplier equations

$$\frac{-a_i}{(a_i^2+1)^2} + \lambda - \mu a_i = 0, \qquad i = 1, 2, \dots, n \tag{3}$$

(where $\lambda$ and $\mu$ are real constants), or at a point where $\nabla f_1$ and $\nabla f_2$ are linearly dependent (that is when all $a_i$ are equal) – see [1, 4]. We claim that the equation $f(x)=0$, where

$$f(x)=\frac{-x}{(x^2+1)^2}+\lambda-\mu x\,,$$

has at most three distinct nonnegative roots. From

$$f'(x)=\frac{3x^2-1}{(x^2+1)^3}-\mu,\qquad f''(x)=\frac{12x(1-x^2)}{(x^2+1)^4},$$

it follows that $f'(x)$ is strictly increasing on $[0,1]$ and strictly decreasing on $[1,\infty)$. Since

$$f'(0)=-1-\mu<-\mu=f'(\infty),$$

there are four possible cases:

- $f'(x)\geqslant 0$ for $x\in[0,\infty)$;
- $f'(x)<0$ for $x\in[0,x_1)$, $f'(x_1)=0$ and $f'(x)>0$ for $x\in(x_1,\infty)$;
- $f'(x)<0$ for $x\in[0,x_1)$, $f'(x)>0$ for $x\in(x_1,x_2)$ with $0<x_1<x_2$, and $f'(x)<0$ for $x\in(x_2,\infty)$;
- $f'(x)\leqslant 0$ for $x\in[0,\infty)$.

The equation $f(x)=0$ can have at most three distinct nonnegative roots. It can have three distinct nonnegative roots only in the third case, when $f(x)$ is decreasing on $[0,x_1]$, increasing on $[x_1,x_2]$ and decreasing on $[x_2,\infty)$. Since the Lagrange multiplier equations (3) can be satisfied only when all $a_i$ take at most three distinct nonnegative values, it suffices to consider the following three cases:

- $a_n=0$;
- $a_i$ take two distinct positive values;
- $a_i$ take three distinct positive values.

We will use the induction method.

## 3. Case $a_n=0$

We need to show that

$$\sum_{1\leqslant i<j\leqslant n-1} a_ia_j=\frac{n(n-1)}{2} \tag{4}$$

involves

$$\frac{1}{a_1^2+1}+\frac{1}{a_2^2+1}+\cdots+\frac{1}{a_{n-1}^2+1}\geqslant\frac{n-2}{2}. \tag{5}$$

Using the substitution

$$a_i = \sqrt{k}x_i, \quad k = \frac{n}{n-2}, \qquad i = 1,2,\ldots,n-1,$$

we need to prove that

$$\sum_{1\leqslant i<j\leqslant n-1} x_i x_j = \frac{(n-1)(n-2)}{2}$$

involves

$$\frac{1}{kx_1^2+1} + \frac{1}{kx_2^2+1} + \cdots + \frac{1}{kx_{n-1}^2+1} \geqslant \frac{n-2}{2},$$

which is equivalent to

$$\frac{k+1}{kx_1^2+1} + \frac{k+1}{kx_2^2+1} + \cdots + \frac{k+1}{kx_{n-1}^2+1} \geqslant n-1.$$

We will show that

$$\frac{k+1}{kx_1^2+1} + \frac{k+1}{kx_2^2+1} + \cdots + \frac{k+1}{kx_{n-1}^2+1} \geqslant \frac{2}{x_1^2+1} + \frac{2}{x_2^2+1} + \cdots + \frac{2}{x_{n-1}^2+1} \geqslant n-1.$$

The right inequality follows by the induction hypothesis, while the left inequality is equivalent to

$$b_1c_1 + b_2c_2 + \cdots + b_{n-1}c_{n-1} \geqslant 0,$$

where

$$b_i = \frac{2}{x_i^2+1} - 1, \quad c_i = \frac{1}{kx_i^2+1}, \qquad i = 1,2,\ldots,n-1.$$

By the induction hypothesis, we have

$$b_1 + b_2 + \cdots + b_{n-1} \geqslant 0.$$

Assuming $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{n-1}$, the sequences $(b_1, b_2, \ldots, b_{n-1})$ and $(c_1, c_2, \ldots, c_{n-1})$ are increasing. By the rearrangement inequality and the induction hypothesis, we have:

$$(n-1)(b_1c_1 + b_2c_2 + \cdots + b_{n-1}c_{n-1}) \geqslant (b_1 + b_2 + \cdots + b_{n-1})(c_1 + c_2 + \cdots + c_{n-1}) \geqslant 0.$$

## 4. Case where $a_i$ take two distinct positive values

We need to prove the inequality

$$\frac{a}{x^2+1} + \frac{b}{y^2+1} \geqslant \frac{a+b}{2}, \tag{6}$$

where $a, b$ are positive integer numbers and $x > y > 0$ such that

$$g(x,y) = d, \tag{7}$$

where

$$g(x,y) = a(a-1)x^2 + b(b-1)y^2 + 2abxy$$

and

$$d = a(a-1) + b(b-1) + 2ab = (a+b)(a+b-1).$$

Write the inequality (6) in the homogeneous form

$$\frac{a}{dx^2+g} + \frac{b}{dy^2+g} \geqslant \frac{a+b}{2g},$$

which is equivalent to

$$\frac{(a+b-1)(bx^2+ay^2)+g}{(dx^2+g)(dy^2+g)} \geqslant \frac{1}{2g},$$

or

$$g^2 - [a(a-1)-b(b-1)](x^2-y^2)g - d^2x^2y^2 \geqslant 0,$$

or

$$(a-1)(b-1)(x^4+y^4) + 2(a^2+b^2-a-b)xy(x^2+y^2)$$
$$-2(2a^2+2b^2+ab-3a-3b+1)x^2y^2 \geqslant 0,$$

or

$$(x-y)^2[(a-1)(b-1)(x^2+y^2)+2Axy] \geqslant 0\ ,$$

where

$$A = (a-1)^2 + (b-1)^2 + ab - 1.$$

Since $a,b \geqslant 1$, the last inequality is clearly true.

## 5. Case where $a_i$ take three distinct positive values

We need to prove the inequality

$$\frac{a}{x^2+1} + \frac{b}{y^2+1} + \frac{c}{z^2+1} \geqslant \frac{a+b+c}{2}, \tag{8}$$

where $a,b,c$ are positive integer numbers and $x > y > z > 0$ such that

$$a(a-1)x^2 + b(b-1)y^2 + c(c-1)z^2 + 2abxy + 2bcyz + 2cazx \tag{9}$$
$$= (a+b+c)(a+b+c-1).$$

According to Theorem 1, it suffices to consider

$$a = \left\lfloor \frac{a+b+c}{2} \right\rfloor + 1$$

and $c \geqslant 2$ (the case a) in Theorem 1 being proved at section 3). From

$$a \geqslant \frac{a+b+c-1}{2}+1,$$

we get

$$a \geqslant b+c+1 \geqslant b+3 \geqslant 4.$$

On the other hand, since $x > y > z$, from (9) we get

$$a(a-1)x^2+b(b-1)x^2+c(c-1)x^2+2abx^2+2bcx^2+2cax^2 \\ > (a+b+c)(a+b+c-1),$$

hence $x > 1$. Since

$$\frac{2}{x^2+1} = 1-\frac{x^2-1}{x^2+1} > 1-\frac{x^2-1}{2x} = 1-\frac{x}{2}+\frac{1}{2x},$$

$$\frac{2}{y^2+1} = 2-\frac{2y^2}{y^2+1} \geqslant 2-y$$

and

$$\frac{2}{z^2+1} \geqslant 2-z,$$

it suffices to show that

$$a\left(1-\frac{x}{2}+\frac{1}{2x}\right)+b(2-y)+c(2-z) \geqslant a+b+c,$$

which is equivalent to $F \geqslant 0$, where

$$F = \frac{a}{x}-ax-2by-2cz+2b+2c. \tag{10}$$

To prove the inequality $F \geqslant 0$, it is more convenient to consider

$$x \geqslant y \geqslant z \geqslant 0$$

instead of $x > y > z > 0$. For fixed $x$, taking into account the constraint (9), we may consider $y$ as a function of $z$. Clearly, $y(z)$ is decreasing on its domain $[m,M]$. Note that $m = 0$ when $y(0) \leqslant x$, and $m > 0$ when $y(0) > x$. In addition, we have $z = m > 0$ when $y = x$. By deriving (9) and (10), we get

$$y' = \frac{-c}{b} \cdot \frac{ax+by+(c-1)z}{ax+(b-1)y+cz} < 0,$$

hence

$$F'(z) = -2by'-2c = \frac{2c(y-z)}{ax+(b-1)y+cz} \geqslant 0.$$

Since $F(z)$ is increasing, the inequality $F(z) \geqslant 0$ holds if $F(m) \geqslant 0$. Thus, it suffices to prove the inequality $F \geqslant 0$ for $z = 0$ and for $y = x$.

*Case* 1: $z=0$. Taking into account (9) and (10), we need to show that

$$a(a-1)x^2+b(b-1)y^2+2abxy=(a+b+c)(a+b+c-1) \tag{11}$$

involves $F_1 \geqslant 0$, where

$$F_1=\frac{a}{x}-ax-2by+2b+2c. \tag{12}$$

Since $x \geqslant y$, from (11) we get

$$a(a-1)x^2+b(b-1)x^2+2abx^2 \geqslant (a+b+c)(a+b+c-1),$$

hence

$$\begin{aligned} x &\geqslant \sqrt{\frac{(a+b+c)(a+b+c-1)}{(a+b)(a+b-1)}} \\ &\geqslant \sqrt{\frac{(a+b+2)(a+b+1)}{(a+b)(a+b-1)}} > 1. \end{aligned}$$

Consider $x$ as function of $y$. From the constraint (11), it follows that $x(y)$ is a decreasing function on its domain $[0,M_1]$. Moreover, since $y \leqslant x$, $y$ has its maximum value $M_1$ when $y=x$. By deriving (11) and (12), we get

$$x'=\frac{-b}{a}\cdot\frac{ax+(b-1)y}{(a-1)x+by}<0$$

and

$$\begin{aligned} F_1'(y) &= -a\left(\frac{1}{x^2}+1\right)x'-2b \\ &= b\left(\frac{1}{x^2}+1\right)\frac{ax+(b-1)y}{(a-1)x+by}-2b. \end{aligned}$$

We will show that $F_1'(y) \leqslant 0$. This is equivalent to

$$(a-2)x^2+(b+1)xy \geqslant a+\frac{(b-1)y}{x}.$$

Since $x>1$, we have

$$(b+1)xy \geqslant (b-1)xy \geqslant \frac{(b-1)y}{x}.$$

Thus, we only need to show that

$$(a-2)x^2 \geqslant a.$$

It is true if

$$(a-2)(a+b+2)(a+b+1) \geqslant a(a+b)(a+b-1),$$

which is equivalent to
$$a(a-2) \geqslant (b+1)(b+2).$$
Since $a \geqslant b+3$, we get
$$\begin{aligned} a(a-2)-(b+1)(b+2) &\geqslant (b+3)(b+1)-(b+1)(b+2) \\ &= b+1 > 0. \end{aligned}$$

Because $F_1(y)$ is decreasing, the inequality $F_1(y) \geqslant 0$ holds if $F_1(M_1) \geqslant 0$. Thus, it suffices to show that $F_1 \geqslant 0$ for $y = x$. According to (11) and (12), we need to show that $b \geqslant 1$, $c \geqslant 2$, $a \geqslant b+c+1 \geqslant 4$ and
$$x = \sqrt{\frac{(a+b+c)(a+b+c-1)}{(a+b)(a+b-1)}}$$
involves
$$\frac{a}{x} - (a+2b)x + 2b + 2c \geqslant 0.$$
Write the inequality as
$$(a+b)\left(\frac{1}{x} - x\right) - b\left(x + \frac{1}{x} - 2\right) + 2c \geqslant 0.$$

For fixed $c$ and $a+b$, $x$ is also fixed. Since $b \leqslant a-c-1$ and the left side of the inequality has the minimum value when $b$ is maximum, it suffices to take $b = a-c-1$. So, we need to prove that
$$2(a-1)x \geqslant (3a-2c-2)x^2 - a$$
for
$$x = \sqrt{\frac{(2a-1)(2a-2)}{(2a-c-1)(2a-c-2)}}.$$
The inequality can be written as
$$2(a-1)\sqrt{\frac{(2a-1)(2a-2)}{(2a-c-1)(2a-c-2)}} \geqslant \frac{A}{(2a-c-1)(2a-c-2)},$$
where
$$A = -ac^2 - (4a^2 - 9a + 4)c + 4(a-1)^2(2a-1).$$
By squaring, we need to prove that
$$8(a-1)^3(2a-1)(2a-c-1)(2a-c-2) \geqslant A^2.$$

For fixed $a$ ($a \geqslant 4$), this inequality is equivalent to $cf(c) \geqslant 0$, where
$$f(c) = -a^2c^3 - 2a(4a^2 - 9a + 4)c^2 + Bc + C, \qquad c \in [2, a-2],$$

$$B = 16a^4 - 24a^3 - 9a^2 + 24a - 8,$$
$$C = 8(-4a^4 + 12a^3 - 13a^2 + 6a - 1).$$

Since

$$f''(c) = -6a^2c - 4a(4a^2 - 9a + 4) < 0,$$

$f(c)$ is concave. Therefore, to prove that $f(c) \geqslant 0$, it suffices to show that $f(2) \geqslant 0$ and $f(a-2) \geqslant 0$. We have

$$f(2) = 2(a-2)(8a^2 - 13a + 6) > 0$$

and

$$\begin{aligned} f(a-2) &= 7a^5 - 32a^4 + 11a^3 + 50a^2 - 40a + 8 \\ &> a(7a^4 - 32a^3 + 11a^2 + 50a - 120) \\ &= a(a-4)(7a^3 - 4a^2 - 5a + 30) \\ &\geqslant a^2(a-4)(7a^2 - 4a - 5) \geqslant 0. \end{aligned}$$

*Case* 2: $y = x$. Taking into account (9) and (10), we need to show that

$$(a+b)(a+b-1)x^2 + c(c-1)z^2 + 2(a+b)cxz = (a+b+c)(a+b+c-1) \tag{13}$$

involves $F_2 \geqslant 0$, where

$$F_2 = \frac{a}{x} - (a+2b)x - 2cz + 2b + 2c. \tag{14}$$

Consider $x$ as function of $z$. From the constraint (13), it follows that $x(z)$ is a decreasing function on its domain $[0, M_2]$. Moreover, since $z \leqslant x$, $z$ has its maximum value $M_2$ when $z = x$. By deriving (13) and (14), we get

$$x' = \frac{-c}{a+b} \cdot \frac{(a+b)x + (c-1)z}{(a+b-1)x + cz} < 0$$

and

$$\begin{aligned} F_2'(z) &= -\left(\frac{a}{x^2} + a + 2b\right)x' - 2c \\ &= \left(\frac{a}{x^2} + a + 2b\right) \cdot \frac{c}{a+b} \cdot \frac{(a+b)x + (c-1)z}{(a+b-1)x + cz} - 2c. \end{aligned}$$

We will show that $F_2'(z) \leqslant 0$. This is equivalent to

$$\frac{2(a+b)[(a+b-1)x + cz]}{(a+b)x + (c-1)z} \geqslant \frac{a}{x^2} + a + 2b,$$

$$\frac{(a-2)(a+b)x + (ac+a+2b)z}{(a+b)x + (c-1)z} \geqslant \frac{a}{x^2},$$

which can be written in the homogeneous form

$$\frac{(a-2)(a+b)x^3+(ac+a+2b)x^2z}{a[(a+b)x+(c-1)z]}$$
$$\geqslant \frac{(a+b)(a+b-1)x^2+c(c-1)z^2+2(a+b)cxz}{(a+b+c)(a+b+c-1)},$$

or

$$(x-z)(Ax^2+Bxz+Cz^2)\geqslant 0, \tag{15}$$

where

$$\begin{aligned} A &= (a+b)[(a+b+c)(a+b+c-1)(a-2)-a(a+b)(a+b-1)] \\ &\geqslant (a+b)[(a+b+2)(a+b+1)(a-2)-a(a+b)(a+b-1)] \\ &= 2(a+b)[a(a-2)-b^2-3b-2] \\ &\geqslant 2(a+b)[(b+3)(b+1)-b^2-3b-2] \\ &= 2(a+b)(b+1)>0, \\ B &= ac(c-1)(3a+3b+c-1)>0, \\ C &= ac(c-1)^2>0. \end{aligned}$$

Since $x\geqslant z$ and $A,B,C>0$, the inequality (15) is true. Finally, since $F_2(z)$ is decreasing, the inequality $F_2(z)\geqslant 0$ holds if $F_2(M_2)\geqslant 0$. Thus, it suffices to show that $F_2\geqslant 0$ for $z=x$. From the constraint (13), we get $z=x=1$, hence $F_2=0$.

The proof is completed. The equality occurs for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=a_2=\cdots=a_{n-1}=\sqrt{\dfrac{n}{n-2}}$ and $a_n=0$ (or any cyclic permutation).

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