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THE POPOVICIU TYPE INEQUALITIES FOR s–CONVEX FUNCTIONS IN THE THIRD SENSE

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Abstract. In this article, firstly, some examples involving hypergeometric functions for the s-convex functions in the third sense are presented. Then, the Popoviciu type inequalities and some integral versions of these inequalities for the s-convex functions in the third sense are proved. In the last section, by using these obtained inequalities and given sample functions, inequality relations for special functions including beta, incomplete beta, hypergeometric, exponential integral, logarithmic integral functions and for special means including Stolarsky, power, arithmetic, Heronian, geometric means are derived.

1. Introduction

Convex functions are primarily defined by an inequality traced back to the studies of Jensen. The many generalizations and extensions of convex functions are obtained by modifying its definition inequality. s-convex, p-convex, B-convex and B^{-1}-convex functions are defined in this methodology [2, 10, 14, 15, 25].

s-convex functions are a generalization of convex functions via generic parameter s in $[0,1]$ and have various types. Their first emergence in literature is in the study by Orlicz [22], in which the s-convex function in the first sense was defined. The s-convex function in the second sense is introduced by Breckner [5] in linear topological spaces. Very recently, the s-convex functions in the third and fourth senses are introduced [10, 14]. While s-convex functions in the second and fourth senses are defined on convex sets, s-convex functions in the first and third senses are defined on special sets, namely, s-convex sets. One of the well-known applications of s-convex functions is on fractal sets such as the Mandelbrot and Julia sets [16, 17].

A great deal of properties of these generalizations of convex functions are studied in terms of some inequalities such as the Hermite-Hadamard, Fejer and Ostrowski inequalities. Moreover, the studies on the characterization of these new defined functions via mentioned inequalities have notable interest among enthusiasts [1, 8, 9, 12, 32, 33, 26, 31].

In literature, the characterization of the s-convex functions in the third sense are in terms of certain inequalities quite novel issue. The Hermite-Hadamard, Ostrowski inequalities for these functions are studied in [11, 27, 29].

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The Popoviciu inequality is an inequality that has been extensively studied on the characterization of convex functions. This inequality is introduced by a Romanian mathematician, Tiberiu Popoviciu in 1965 [24], which is connected to the Jensen inequality. The Popoviciu inequality is a powerful inequality and can be a powerful tool in some situations where the Jensen inequality is inappropriate. For a given convex function and a family of the several points in its domain, while the Jensen inequality sets a relation between weighted mean of images of the points and the image of weighted mean of the points; the Popoviciu inequality takes it a step forward by relating the weighted means of the functions values and the values taken at the barycenters of certain subfamilies of the given family of points. The refinements, extensions and generalizations of the Popoviciu inequality have been scrutinized in many studies [3, 6, 18, 19, 20, 21]. Some of applications have given in information theory [7]. Bougoffa introduced new Popoviciu type inequalities in [4] for convex functions and Pinheiro stated these inequalities for the s-convex functions in the first and second sense [23].

In this paper, we obtain two s-convex functions in the third sense involving hypergeometric functions via generator function given in [27]. Then we introduce the generalized Popoviciu type inequality of inequality given by Bougoffa in [4] and integral versions for the s-convex functions in the third sense. Using the generated functions and inequalities, we present some inequalities for hypergeometric, beta, incomplete beta, exponential integral, logarithmic integral functions and for Stolarsky, power, Heronian, arithmetic, geometric means.

2. Preliminaries

Let us setup basic notations and terminology. Throughout the paper, $\mathbb{R},\mathbb{R}_+,\mathbb{R}_{++}$ denote real numbers, $[0,\infty)$ and $(0,\infty)$, respectively.

In literature, the notion of the s-convex set is encountered in the study [14], in which the s-convex functions in the third sense are introduced. However, this notion corresponds p-convex set given in the study [25].

Let $A\subset\mathbb{R}^n$ and $s\in(0,1]$. The set A is called s-convex set if $\lambda_1x_1+\lambda_2x_2\in A$ for all $x_1,x_2\in A$ and $\lambda_1,\lambda_2\in[0,1]$ such that $\lambda_1^s+\lambda_2^s=1$. Not every convex set in $\mathbb{R}^n$ is s-convex. s-convex sets are intervals that accept 0 as the boundary point or interior point.

DEFINITION 1. [14] Let $A\subset\mathbb{R}^n$ be an s-convex set and $f:A\to\mathbb{R}$. The function f is said to be an s-convex function in the third sense if

$$f(\lambda_1x_1+\lambda_2x_2)\leqslant\lambda_1^{\frac{1}{s}}f(x_1)+\lambda_2^{\frac{1}{s}}f(x_2)$$

for all $x_1,x_2\in A$ and $\lambda_1,\lambda_2\in[0,1]$ such that $\lambda_1^s+\lambda_2^s=1$.

In [14], it is shown that the function

$$f(x)=\begin{cases}a, & \text{if } x=0\\ bx^{\frac{1}{s}}+c, & \text{if } x>0\end{cases}\tag{1}$$

where $a,b,c \in \mathbb{R}$ with $b<0$ and $a,c \leqslant 0$ is an s-convex function in the third sense on $(0,\infty)$. Additionally, in case $a=c$, the function f is an s-convex function in the third sense on $[0,\infty)$.

The Popoviciu inequality given in [24] is one of the eminent tools in characterization of convex functions. A real-valued continuous function f defined on a convex subset $A \subset \mathbb{R}$ is convex if and only if it verifies the inequality

$$\begin{aligned}\frac{2}{3}\left[f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_1+x_3}{2}\right)\right]&\\ \leqslant \frac{f(x_1)+f(x_2)+f(x_3)}{3}+f\left(\frac{x_1+x_2+x_n}{3}\right)&\end{aligned} \tag{2}$$

for all $x_1,x_2,x_3 \in A$.

The Popoviciu inequality for a function defined on a convex subset $A \subset \mathbb{R}$ has been generalized in the same study as follows

$$\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right) \leqslant \frac{1}{k}\binom{n-2}{k-2}\left[\frac{n-k}{k-1}\sum_{i=1}^{n} f(x_i)+nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right] \tag{3}$$

for $k,n \in \mathbb{N}$ with $k \leqslant n$ and $x_1,x_2,\ldots,x_n \in \mathbb{R}$.

We need the following properties and definitions to express the findings.

THEOREM 1. [14] *Let A be an s-convex set and $f : A \to \mathbb{R}$ be an s-convex function in the third sense and $x_1,x_2,\ldots,x_m \in A$, $\lambda_1,\lambda_2,\ldots,\lambda_m \in \mathbb{R}_+$ with $\lambda_1^s+\lambda_2^s+\cdots+\lambda_m^s=1$. Then*

$$f(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_mx_m) \leqslant \lambda_1^{\frac{1}{s}}f(x_1)+\lambda_2^{\frac{1}{s}}f(x_2)+\cdots+\lambda_m^{\frac{1}{s}}f(x_m). \tag{4}$$

The inequality (4) can be referred to as the Jensen inequality for the s-convex functions in the third sense.

The following theorem states two ways of generation for the s-convex functions in the third sense.

THEOREM 2. [27] *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a nonincreasing s-convex function in the third sense and let $0<a<b$. Let $g,h : [0,1] \to \mathbb{R}$ be the functions defined as follows*

$$g(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx, \tag{5}$$

$$h(t)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy. \tag{6}$$

Then g and h are the s-convex functions in the third sense on $[0,1]$.

The hypergeometric functions are given below, which are encountered in the computation of integrals in applications. Although these functions on defined on complex numbers, for the sake of clarity, we give the definition on real numbers.

DEFINITION 2. [28] Let p,q be positive numbers, $a_1,\dots,a_p,b_1,\dots,b_q \in \mathbb{R}$ such that none of $b_1,\dots,b_q$ is a negative integer or zero and z be a complex number. The following serie

$$1+\frac{a_1a_2\cdots a_p}{b_1b_2\cdots b_q}z+\frac{a_1(a_1+1)a_2(a_2+1)\cdots a_p(a_p+1)}{b_1(b_1+1)b_2(b_2+1)\cdots b_q(b_q+1)}\frac{z^2}{2!}$$
$$+\cdots+\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!}+\cdots$$

where $(.)_n$ is the Pochhammer symbol, i.e.

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1)$$

is called generalized hypergeometric function and shown by ${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)$ if it converges.

In case $p \leqslant q$, ${}_pF_q$ converges on all complex plane. In case $p=q+1$, it is converges on unit disk $|z|<1$. Moreover, it converges for $z=1$ if $\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i > 0$. In case $p>q$, it converges only for $z=0$ or converges on complex plane provided that any of $a_1,\dots,a_p$ is a negative integer or zero. We will denote ${}_2F_1$ as F throughout the paper.

LEMMA 1. [13] (page 1006) *Let p,y,z be real numbers with $y,z\neq 0$. Then the following equalities hold*

$$F\left(\frac{1-p}{2},\frac{2-p}{2};\frac{3}{2};\frac{z^2}{y^2}\right)=\frac{(y+z)^p-(y-z)^p}{2pzy^{p-1}} \tag{7}$$

and

$$F\left(1-p,1;2;-\frac{z}{y}\right)=\frac{(y+z)^p-y^p}{pzy^{p-1}} \tag{8}$$

for $\left|\frac{z}{y}\right|<1$.

LEMMA 2. [30] *Let $\alpha>0$ and ${}_pF_q$ be a generalized hypergeometric function given in Definition 2. Then*

$$\int z^{\alpha-1}\,{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)dz=\frac{z^\alpha}{\alpha}\,{}_{p+1}F_{q+1}(\alpha,a_1,\dots,a_p;\alpha+1,b_1,\dots,b_q;z).$$

3. Main results

In this section, firstly, two examples of the s-convex functions in the third sense, which are expressed with hypergeometric functions, are obtained. Secondly, Popoviciu type inequalities and their integral versions are given for the s-convex functions in the third sense.

THEOREM 3. *Let* $s \in (0,1]$. *Then*

$$g_1(t) = -2^{-\frac{1}{s}} F\left(-\frac{1}{2s}, \frac{1}{2} - \frac{1}{2s}; \frac{3}{2}; t^2\right) \tag{9}$$

is an s*-convex function in the third sense on* $[0,1)$.

Proof. Consider $f(x) = -x^{\frac{1}{s}}$ on $[0,1)$. Using (5) in Theorem 2, we have

$$g_1(t) = \begin{cases} \frac{s}{2^{1+\frac{1}{s}}(1+s)} \frac{(1-t)^{\frac{1}{s}+1} - (1+t)^{\frac{1}{s}+1}}{t}, & t \in (0,1] \\ -2^{-\frac{1}{s}}, & t = 0 \end{cases}.$$

From (7), the desired result is obtained. $\square$

THEOREM 4. *Let* $s \in (0,1]$. *Then,*

$$\begin{aligned} h_1(t) = \frac{s}{s+1}\Big(&F\Big(-\frac{1}{s}-1;1;1;t\Big) + F\Big(-\frac{1}{s}-1;1;1;1-t\Big) \\ &- F\Big(-\frac{1}{s}-1;1;2;1-t\Big) - F\Big(-\frac{1}{s}-1;1;2;t\Big)\Big) \end{aligned}$$

is an s*-convex function in the third sense on* $[0,1)$.

Proof. Consider $f(x) = -x^{\frac{1}{s}}$ on $[0,1)$. Using (6) in Theorem 2, we have

$$h_1(t) = \begin{cases} \frac{s^2}{(2s+1)(s+1)} \left(\frac{(1-t)^{\frac{1}{s}+2} + t^{\frac{1}{s}+2} - 1}{t(1-t)}\right), & t \in (0,1) \\ -\frac{s}{s+1}, & t = 0,1 \end{cases}. \tag{10}$$

Using

$$\frac{(1-t)^{\frac{1}{s}+2} + t^{\frac{1}{s}+2} - 1}{t(1-t)} = \frac{(1-t)^{\frac{1}{s}+2} - 1}{t} + \frac{t^{\frac{1}{s}+2} - 1}{1-t} + t^{\frac{1}{s}+1} + (1-t)^{\frac{1}{s}+1}$$

and (8), the desired result is obtained. $\square$

The next theorem states a Popoviciu type inequality for the s-convex functions in the third sense.

THEOREM 5. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be an* s*-convex function in the third sense and* $x_1,x_2,\ldots,x_n\in\mathbb{R}$*. Then*

$$\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} f\left(\frac{1}{k^{\frac{1}{s}}}\sum_{j=1}^{k}x_{i_j}\right)\leqslant g(k;s)\left[\sum_{i=1}^{n}f(x_i)-f\left(\frac{1}{n^{\frac{1}{s}}}\sum_{i=1}^{n}x_i\right)\right] \tag{11}$$

where

$$g(k;s)=\binom{n}{k}\left(\frac{n}{k}\right)^{\frac{1}{s^2}-1}\frac{1}{n^{\frac{1}{s^2}}-1}$$

for $k,n\in\mathbb{N}$ *with* $k\leqslant n$ *and* $x_1,x_2,\ldots x_n\in\mathbb{R}$.

Proof. Accepting $\lambda_{i_j}=\frac{1}{k^{\frac{1}{s}}}$ for $j\in\{1,2,\ldots,k\}$ with $i_j\in\{1,2,\ldots,n\}$ in (4), we have

$$f\left(\frac{1}{k^{\frac{1}{s}}}\sum_{j=1}^{k}x_{i_j}\right)\leqslant\frac{1}{k^{\frac{1}{s^2}}}\left(\sum_{j=1}^{k}f(x_{i_j})\right). \tag{12}$$

Summing all possible inequalities of (12) side by side, we have

$$\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} f\left(\frac{1}{k^{\frac{1}{s}}}\sum_{j=1}^{k}x_{i_j}\right)\leqslant\frac{\binom{n-1}{k-1}}{k^{\frac{1}{s^2}}}\left(\sum_{j=1}^{n}f(x_i)\right).$$

Since

$$\sum_{i=1}^{n}f(x_i)=\frac{n^{\frac{1}{s^2}}}{n^{\frac{1}{s^2}}-1}\left[\sum_{i=1}^{n}f(x_i)-\sum_{i=1}^{n}\frac{1}{n^{\frac{1}{s^2}}}f(x_i)\right],$$

we can write

$$\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} f\left(\frac{1}{k^{\frac{1}{s}}}\sum_{j=1}^{k}x_{i_j}\right)\leqslant\frac{\binom{n-1}{k-1}}{k^{\frac{1}{s^2}}}\frac{n^{\frac{1}{s^2}}}{n^{\frac{1}{s^2}}-1}\left[\sum_{i=1}^{n}f(x_i)-\sum_{i=1}^{n}\frac{1}{n^{\frac{1}{s^2}}}f(x_i)\right]. \tag{13}$$

From the s-convexity of f in the third sense, we can write

$$-\sum_{i=1}^{n}\frac{1}{n^{\frac{1}{s^2}}}f(x_i)\leqslant -f\left(\sum_{i=1}^{n}\frac{1}{n^{\frac{1}{s}}}x_i\right).$$

Using this fact in (13), one can get the required inequality. $\square$

In case $n=3$ and $k=2$ in Theorem 5, the following inequality similar to (2) for the s-convex functions in the third sense is obtained:

COROLLARY 1. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be an* s*-convex function of third sense. Then*

$$\begin{aligned}f\left(\frac{x_1+x_2}{2^{\frac{1}{s}}}\right)&+f\left(\frac{x_2+x_3}{2^{\frac{1}{s}}}\right)+f\left(\frac{x_1+x_3}{2^{\frac{1}{s}}}\right)\\&\leqslant\left(\frac{3}{2}\right)^{\frac{1}{s^2}-1}\frac{3}{3^{\frac{1}{s^2}}-1}\left[f(x_1)+f(x_2)+f(x_3)-f\left(\frac{x_1+x_2+x_3}{3^{\frac{1}{s}}}\right)\right]\end{aligned}$$

for all $x_1,x_2,x_3\in\mathbb{R}$.

Using Theorem 5, we deduce a necessary condition for a function to be s-convex functions in the third sense on real numbers.

COROLLARY 2. *Let $f:\mathbb{R}\to\mathbb{R}$ be an s-convex function of third sense. Then*

$$k^{\frac{1}{s^2}-1} f\left(k^{1-\frac{1}{s}}x\right) \leqslant \frac{n^{\frac{1}{s^2}-1}}{n^{\frac{1}{s^2}}-1}\left[nf(x) - f\left(n^{1-\frac{1}{s}}x\right)\right]$$

for $k,n\in\mathbb{N}$ with $k\leqslant n$ and $x\in\mathbb{R}$.

Proof. Letting $x_1 = x_2 = \cdots = x_n = x$ in Theorem 5 gives the result. $\square$

The inequality (3) can be restated as an integral inequality:

THEOREM 6. *Let $f:\mathbb{R}_{++}\to\mathbb{R}$ be an s-convex function in the third sense and $x_1,x_2,\ldots,x_n\in R$. Then*

$$\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\left(\frac{k^{\frac{1}{s}}}{\sum\limits_{j=1}^{k} x_{i_j}} \int\limits_0^{k^{-\frac{1}{s}}\sum_{j=1}^{k} x_{i_j}} f(t)dt\right)$$

$$\leqslant g(k;s)\left[\sum_{i=1}^{n}\left(\frac{1}{x_i}\int\limits_0^{x_i} f(t)dt\right) - \frac{n^{\frac{1}{s}}}{\sum\limits_{i=1}^{n} x_i}\int\limits_0^{n^{-\frac{1}{s}}\sum_{i=1}^{n} x_i} f(t)dt\right]$$

where

$$g(k;s) = \binom{n}{k}\left(\frac{n}{k}\right)^{\frac{1}{s^2}-1}\frac{1}{n^{\frac{1}{s^2}}-1}$$

for $k,n\in\mathbb{N}$ with $k\leqslant n$ and $x_1,x_2,\ldots,x_n\in\mathbb{R}$.

Proof. Let a be nonzero real number. Since

$$\int_0^1 f(at)dt = \frac{1}{a}\int_0^a f(x)dx$$

we have

$$\int_0^1 f(x_i t)dt = \frac{1}{x_i}\int\limits_0^{x_i} f(t)dt$$

for $i\in\{1,\ldots,n\}$ and

$$\int_0^1 f\left(\frac{t}{k^{\frac{1}{s}}}\sum_{j=1}^{k} x_{i_j}\right)dt = \frac{1}{k^{-\frac{1}{s}}\sum\limits_{j=1}^{k} x_{i_j}}\int\limits_0^{k^{-\frac{1}{s}}\sum_{j=1}^{k} x_{i_j}} f(t)dt \tag{14}$$

for fixed k and any $1 \leqslant i_j \leqslant n$. Writing (11) for tx_i with $t \in (0,1]$, integrating them on t and using of (14), we obtain the desired inequality. □

In case $n = 3$ and $k = 2$ in Theorem 6, the following integral inequality similar to integral version of (2) for the s-convex functions in the third sense is obtained:

COROLLARY 3. *Let $f : \mathbb{R}_{++} \to \mathbb{R}$ be an s-convex function in the third sense. Then*

$$\frac{1}{x_1+x_2}\int_0^{2^{-\frac{1}{s}}(x_1+x_2)} f(t)dt + \frac{1}{x_1+x_3}\int_0^{2^{-\frac{1}{s}}(x_1+x_3)} f(t)dt + \frac{1}{x_2+x_3}\int_0^{2^{-\frac{1}{s}}(x_2+x_3)} f(t)dt$$
$$\leqslant r(s)\left[\sum_{i=1}^{3}\left(\frac{1}{x_i}\int_0^{x_i} f(t)dt\right) - \frac{3^{\frac{1}{s}}}{x_1+x_2+x_3}\int_0^{3^{-\frac{1}{s}}(x_1+x_2+x_3)} f(t)dt\right]$$

where

$$r(s) = \frac{1}{2^{\frac{1}{s^2}-1}}\frac{3^{\frac{1}{s^2}}}{3^{\frac{1}{s^2}}-1}$$

for $x_1, x_2, x_3 \in \mathbb{R}_{++}$.

Using Theorem 6, we deduce a necessary condition expressed via integrals for a function to be s-convex functions in the third sense.

COROLLARY 4. *Let $f : \mathbb{R}_{++} \to \mathbb{R}$ be an s-convex function in the third sense. Then*

$$k^{\frac{1}{s}-1}\int_0^{k^{1-\frac{1}{s}}x} f(t)dt \leqslant \left(\frac{n}{k}\right)^{\frac{1}{s^2}-1}\frac{1}{n^{\frac{1}{s^2}}-1}\left[n\int_0^{x} f(t)dt - n^{\frac{1}{s}-1}\int_0^{n^{1-\frac{1}{s}}x} f(t)dt\right]$$

for $k, n \in \mathbb{N}$ with $k \leqslant n$ and $x \in \mathbb{R}_{++}$.

Proof. Letting $x_1 = x_2 = \cdots = x_n = x$ in Theorem 6 gives the result. □

4. Applications

In this section, essentially, we set forth some applications based on the obtained results. In the following applications, some inequality relations between special functions including beta, incomplete beta, hypergeometric, exponential integral and logarithmic integral functions are derived.

PROPOSITION 1. *Let $x > 2$. Then*

$$\left(2\cdot 3^{x^2} - 2^{x-1}(3^{x^2}-1)\right)B(x+1,x+1) \leqslant \frac{x(3^{x^2}-2)-1}{(x+1)(2x+1)} + 2^{2x+1}B_{\frac{1}{2}}(x+1,x+1)$$

where B and B_α are beta and incomplete beta functions, respectively, i.e.

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt \quad \text{and} \quad B_\alpha(a,b) = \int_0^\alpha t^{a-1}(1-t)^{b-1}dt$$

for $a,b > 0$.

Proof. Let $0 < t < 1$. Suppose $f(x) = -x^{\frac{1}{s}}$ on $(0,\infty)$ and $x_1 = 1-t,\ x_2 = 1-t,\ x_3 = t$ in Corollary 1. Then we have

$$\frac{3^{\frac{1}{s^2}}}{3^{\frac{1}{s^2}}-1}\left(2(1-t)^{\frac{1}{s}} + t^{\frac{1}{s}} - \left(\frac{2-t}{3^{\frac{1}{s}}}\right)^{\frac{1}{s}}\right) \leqslant 1 + 2^{\frac{1}{s}-1}(1-t)^{\frac{1}{s}}.$$

Multiplying both side with $t^{\frac{1}{s}}$ and integrating both sides with respect to t on $[0,1]$, we have

$$\frac{3^{\frac{1}{s^2}}}{3^{\frac{1}{s^2}}-1}\left(2B\left(\frac{1}{s}+1,\frac{1}{s}+1\right) + \frac{s}{s+2} - \frac{2^{\frac{s+2}{s}}}{3^{\frac{1}{s^2}}}B_{\frac{1}{2}}\left(\frac{1}{s}+1,\frac{1}{s}+1\right)\right)$$
$$\leqslant \frac{s}{s+1} + 2^{\frac{1}{s}-1}B\left(\frac{1}{s}+1,\frac{1}{s}+1\right).$$

Then making substitution $\frac{1}{s} = x$ gives the desired result. $\square$

PROPOSITION 2. *Let $x > 1$. Then*

$$\frac{2^x}{1+x}F\left(1+x,-x,2+x,-\frac{1}{2}\right) + F\left(\frac{1}{2},-x,\frac{3}{2},-1\right)$$
$$\geqslant \frac{2\cdot 3^{x^2}}{3^{x^2}-1}\left(1 + \frac{2^x}{1+x} - \frac{2^{1+2x}-1}{3^{x^2}(2x+1)} + \frac{1}{1+2x}\right) - \frac{3^{x+1}-1}{2(1+x)}.$$

Proof. Let $t > 0$. Suppose $f(x) = -x^{\frac{1}{s}}$ and $x_1 = 1,\ x_2 = 2t,\ x_3 = t^2$ in Corollary 1. Then we have

$$\left(\frac{1+2t}{2^{\frac{1}{s}}}\right)^{\frac{1}{s}} + \left(\frac{1+t^2}{2^{\frac{1}{s}}}\right)^{\frac{1}{s}} + \left(\frac{2t+t^2}{2^{\frac{1}{s}}}\right)^{\frac{1}{s}}$$
$$\geqslant \left(\frac{3}{2}\right)^{\frac{1}{s^2}-1}\frac{3}{3^{\frac{1}{s^2}}-1}\left[1 + (2t)^{\frac{1}{s}} + t^{\frac{2}{s}} - \frac{1}{3^{\frac{1}{s^2}}}(1+t)^{\frac{2}{s}}\right].$$

Integrating both sides with respect to t on $[0,1]$, we have

$$\begin{aligned}
&2^{-\frac{1}{s^2}}\frac{s}{2(s+1)}(3^{\frac{1}{s}+1}-1)+2^{-\frac{1}{s^2}}F\Big(\frac{1}{2},-\frac{1}{s},\frac{3}{2},-1\Big)\\
&\quad+2^{\frac{s-1}{s^2}}\frac{s}{s+1}F\Big(1+\frac{1}{s},-\frac{1}{s},2+\frac{1}{s},-\frac{1}{2}\Big)\\
&\quad\geqslant\Big(\frac{3}{2}\Big)^{\frac{1}{s^2}-1}\frac{3}{3^{\frac{1}{s^2}}-1}\left(1+2^{\frac{1}{s}}\frac{s}{s+1}+\frac{s}{s+2}-\frac{2^{1+\frac{2}{s}}-1}{3^{\frac{1}{s^2}}}\frac{s}{s+2}\right).
\end{aligned}$$

Then making the substitution $\frac{1}{s}=x$ gives the desired result. $\square$

PROPOSITION 3. *Let n,k be natural numbers with $k\leqslant n$ and $s,x\in(0,1)$. Then*

$$\begin{aligned}
&k^{\frac{1}{s^2}-1}F\Big(-\frac{1}{2s},\frac{1}{2}-\frac{1}{2s};\frac{3}{2};(k^{1-\frac{1}{s}}x)^2\Big)\\
&\quad\geqslant\frac{n^{\frac{1}{s^2}}}{n^{\frac{1}{s^2}}-1}\left[nF\Big(-\frac{1}{2s},\frac{1}{2}-\frac{1}{2s};\frac{3}{2};x^2\Big)-F\Big(-\frac{1}{2s},\frac{1}{2}-\frac{1}{2s};\frac{3}{2};(n^{1-\frac{1}{s}}x)^2\Big)\right].
\end{aligned}$$

Proof. Using (9) in Theorem 2 and simplifying the expression, we have desired inequality. $\square$

PROPOSITION 4. *Let $t\in[0,1]$. If $x_1+x_2+x_3\leqslant 2^{\frac{1}{s}}$, then*

$$\begin{aligned}
&r\Big(\frac{x_1+x_2}{2^{\frac{1}{s}}}\Big)+r\Big(\frac{x_1+x_3}{2^{\frac{1}{s}}}\Big)+r\Big(\frac{x_2+x_3}{2^{\frac{1}{s}}}\Big)\\
&\qquad\geqslant h(s)\left[r(x_1)+r(x_2)+r(x_3)-r\Big(\frac{x_1+x_2+x_3}{3^{\frac{1}{s}}}\Big)\right]
\end{aligned}\tag{15}$$

where

$$r(t)={}_3F_2\Big(\frac{1}{2},\frac{1}{2}-\frac{1}{2s},-\frac{1}{2s};\frac{3}{2},\frac{3}{2};t^2\Big)\quad\text{and}\quad h(s)=\frac{1}{2^{\frac{1}{s^2}-2}}\frac{3^{\frac{1}{s^2}}}{3^{\frac{1}{s^2}}-1}.$$

Proof. Suppose $f(t)=g_1(t)$ on $[0,1]$ in Corollary 3. From Lemma 2, we have

$$\int_0^x F\Big(-\frac{1}{2s},\frac{1}{2}-\frac{1}{2s};\frac{3}{2};t^2\Big)dt=x\,{}_3F_2\Big(\frac{1}{2},\frac{1}{2}-\frac{1}{2s},-\frac{1}{2s};\frac{3}{2},\frac{3}{2};x^2\Big)$$

for $x\in[0,1]$. Using this fact, we have desired inequality. $\square$

REMARK 1. In Proposition 4, the choice of the points x_1,x_2,x_3 depends on s. If the condition $x_1+x_2+x_3\leqslant 2^{\frac{1}{s}}$ is replaced with $x_1+x_2+x_3\leqslant 2$, then the inequality 15 is satisfied for all s.

PROPOSITION 5. *Let* $0 < x < \frac{1}{2}$. *Then*

$$\mathrm{Ei}\left(\log\left(x^2+\frac{x}{2}+\frac{1}{4}\right)\right) \leqslant \mathrm{Ei}(\log x^2)+\frac{1}{2}\frac{x}{\ln(x)}+\mathrm{Ei}\left(\log\left(\frac{1}{4}\right)\right) \tag{16}$$

where Ei *denotes exponential integral, i.e.*

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt$$

for $x \in \mathbb{R}$.

Proof. Considering $n = k = 2$ and $f(x) = -x^{\frac{1}{s}}$ on $(0, \frac{1}{2})$ in Theorem 5, we have

$$(x_1+x_2)^{\frac{1}{s}} \geqslant x_1^{\frac{1}{s}} + x_2^{\frac{1}{s}}.$$

Accepting $u = \frac{1}{s}$, then, using $x_1, x_2 \in (0, \frac{1}{2})$, integrating both side with respect to u on $[1, \infty)$ yields to

$$\frac{x_1+x_2}{\ln(x_1+x_2)} \leqslant \frac{x_1}{\ln x_1} + \frac{x_2}{\ln x_2}.$$

Then integrating both side with respect to x_1 on $(0, \frac{1}{2})$, using

$$\int_0^x \frac{t}{\ln t} dt = \mathrm{Ei}(2\log(x)) \quad \text{and} \quad \lim_{t\to 0} \mathrm{Ei}(2\log(t)) = 0$$

we have

$$\mathrm{Ei}\left(2\log\left(\frac{1}{2}+x_2\right)\right) - \mathrm{Ei}(2\log(x_2)) \leqslant \mathrm{Ei}\left(\log\left(\frac{1}{4}\right)\right) + \frac{1}{2}\frac{x_2}{\ln(x_2)}. \quad \square$$

Note that in proposition above, $\mathrm{Ei}(\mathrm{x})$ is defined with respect to Cauchy Principal Value.

REMARK 2. Using the relation between exponential integral and logarithmic integral $\mathrm{Ei}(\ln x) = li(x)$, we can restate (16) in terms of logarithmic integral as follows

$$li\left(x^2+\frac{x}{2}+\frac{1}{4}\right) \leqslant li(x^2)+\frac{1}{2}\frac{x}{\ln(x)}+li\left(\frac{1}{4}\right).$$

In the following applications, we get inequalities between some bivariate means including Stolarsky, power, Heronian, arithmetic and geometric means.

PROPOSITION 6. *Let $x \geqslant 2$ and $t \in (0,1)$. Then*

$$\left(\frac{2}{x+1}P_x^x(2^{1-x},1-2^{1-x})-L_{x+1}^x(1,1-2^{1-x})-L_{x+1}^x(1,2^{1-x})\right) \leqslant 2^{2x-x^2}\left(\left(\frac{2}{x+1}P_x^x(t,1-t)-L_{x+1}^x(1,1-t)-L_{x+1}^x(1,t)\right)\right)$$

where P_q and L_q defined below are power mean and Stolarsky mean (generalized logarithmic mean), respectively,

$$P_q(u,v)=\left(\frac{u^q+v^q}{2}\right)^{\frac{1}{q}} \quad and \quad L_q(u,v)=\begin{cases} u, & if\ u=v \\ \left(\frac{u^q-v^q}{q(u-v)}\right)^{1/(q-1)}, & else \end{cases}$$

for $q>1$ and $u,v \in \mathbb{R}_{++}$.

Proof. The function $h_1(t)$ given by (10) can be written for $t \in (0,1)$ as follows

$$h_1(t)=\frac{s^2}{(2s+1)(s+1)}\left(\frac{(1-t)^{\frac{1}{s}+2}-1}{t}+\frac{t^{\frac{1}{s}+2}-1}{1-t}+t^{\frac{1}{s}+1}+(1-t)^{\frac{1}{s}+1}\right).$$

From a definition of the Stolarsky mean and power mean, $h(t)$ can be expressed via these as below

$$h(t)=\frac{s}{s+1}\left(\frac{2s}{2s+1}P_{\frac{1}{s}+1}^{\frac{1}{s}+1}(t,1-t)-L_{\frac{1}{s}+2}^{\frac{1}{s}+1}(1,1-t)-L_{\frac{1}{s}+2}^{\frac{1}{s}+1}(1,t)\right).$$

Then using Theorem 5 for $n=k=2$ and $x_1=t$ and $x_2=1-t$ with $t\in(0,1)$, then making the substitution $\frac{1}{s}+1=x$, we have the desired inequality. $\square$

PROPOSITION 7. *Let a,b be positive real numbers and $x \in [1,\infty)$. Then*

$$H^x(a,b) \geqslant 3^{-x}\left(2^xA^x(a,b)+G^x(a,b)\right) \tag{17}$$

where A, H and G are arithmetic, Heronian and geometric means, i.e.

$$A(a,b)=\frac{a+b}{2}\ ,\ H(a,b)=\frac{a+\sqrt{ab}+b}{3}\ \ and\ \ G(a,b)=\sqrt{ab}.$$

Proof. Let a,b be positive real numbers. Applying (11) for $f(x)=-x^{\frac{1}{s}}$ with $s\in(0,1]$, $k=2$, $x_1=\frac{a+b}{3}$ and $x_2=\frac{\sqrt{ab}}{3}$, we have

$$\left(\frac{1}{2^{\frac{1}{s}}}\left(\frac{2}{3}\frac{a+b}{2}+\frac{1}{3}\sqrt{ab}\right)\right)^{\frac{1}{s}} \geqslant \frac{1}{2^{\frac{1}{s^2}}-1}\left[\left(\frac{2}{3}\frac{a+b}{2}\right)^{\frac{1}{s}}+\left(\frac{\sqrt{ab}}{3}\right)^{\frac{1}{s}}-\left(\frac{1}{2^{\frac{1}{s}}}\left(\frac{2}{3}\frac{a+b}{2}+\frac{1}{3}\sqrt{ab}\right)\right)^{\frac{1}{s}}\right].$$

From the definitions of means,

$$H^{\frac{1}{s}}(a,b) \geqslant 3^{-\frac{1}{s}}\left(2^{\frac{1}{s}}A^{\frac{1}{s}}(a,b)+G^{\frac{1}{s}}(a,b)\right).$$

Then by using the substitution $x=\frac{1}{s}$, thereby, $x\in[1,\infty)$, the desired inequality is obtained. □

Note that in case of $x=1$ in Proposition above, inequality becomes equality. Moreover, using $H(a,b)=\frac{2}{3}A(a,b)+\frac{1}{3}G(a,b)$, we can express (17) in terms of arithmetic and geometric means.

REMARK 3. It is possible to obtain more bivariate mean relations for the positive numbers a,b. For example, $x_1=A(a,b)$, $x_2=G(a,b)$, $x_3=H(a,b)$ in the light of the proof of the proposition above gives

$$\begin{aligned}&(A(a,b)+G(a,b))^{\frac{1}{s}}+(A(a,b)+H(a,b))^{\frac{1}{s}}+(G(a,b)+H(a,b))^{\frac{1}{s}}\\ &\geqslant \frac{3^{\frac{1}{s^2}-1}\cdot 2}{3^{\frac{1}{s^2}}-1}\left[A^{\frac{1}{s}}(a,b)+G^{\frac{1}{s}}(a,b)+H^{\frac{1}{s}}(a,b)-3^{-\frac{1}{s^2}}(A(a,b)+G(a,b)+H(a,b))^{\frac{1}{s}}\right].\end{aligned}$$

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