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ESTIMATIONS OF COVERING FUNCTIONALS OF SIMPLICES

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Abstract. Let S_n be an n-dimensional simplex and $\Gamma_p(S_n)$ be the smallest positive number γ such that S_n can be covered by p translates of γS_n. We obtain an upper bound of the least positive number β such that $-S_n$ can be covered by two translates of βS_n, which is tight when $n=3$. In addition, we get the exact value of $\Gamma_{n+2}(S_n)$ and an upper bound of $\Gamma_{n+3}(S_n)$. We also provide the precise value of $\Gamma_6(S_3)$, new lower and upper bounds of $\Gamma_7(S_3)$, and an upper bound of $\Gamma_8(S_3)$.

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $e_1, e_2, \cdots, e_n$ be the standard orthogonal basis of $\mathbb{R}^n$. For $A \subseteq \mathbb{R}^n$, we denote by $\operatorname{aff} A$ the *affine hull* of A. A compact convex subset K of $\mathbb{R}^n$ having interior points is called a *convex body*, whose *relative interior*, *relative boundary*, *interior*, and *boundary* are denoted by $\operatorname{relint} K$, $\operatorname{relbd} K$, $\operatorname{int} K$, and $\operatorname{bd} K$, respectively. The set of *extreme points* of K is denoted by $\operatorname{ext} K$. We denote by $\mathscr{K}^n$ the collection of convex bodies in $\mathbb{R}^n$. For each $K \in \mathscr{K}^n$, we denote by $c(K)$ the least number of translates of $\operatorname{int} K$ needed to cover K. Concerning the least upper bound of $c(K)$ in $\mathscr{K}^n$, there is a long-standing conjecture:

CONJECTURE 1. (Hadwiger's covering conjecture) *For each $K \in \mathscr{K}^n$, we have*

$$c(K) \leqslant 2^n,$$

and the equality holds if and only if K is a parallelotope.

Although many in-depth studies have been carried out (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 21, 23, 25]), this conjecture is completely solved only in the two-dimensional case. Note that, for each $K \in \mathscr{K}^n$, $c(K)$ equals the least number of *smaller homothetic copies* of K (i.e., sets having the form $c + \gamma K$ with $\gamma \in (0,1)$

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and $c \in \mathbb{R}^n$) needed to cover K (cf., e.g., Theorem 34.3 in [8]). Therefore, $c(K) \leqslant p$ for some $p \in \mathbb{Z}^+$ if and only if

$$\Gamma_p(K) := \min\left\{\gamma > 0 \mid \exists \{c_j \mid j \in [p]\} \subseteq \mathbb{R}^n \text{ s.t. } K \subseteq \bigcup_{i\in[p]} (c_i + \gamma K)\right\} < 1,$$

where $[p] := \{i \in \mathbb{Z}^+ \mid 1 \leqslant i \leqslant p\}$. The map

$$\Gamma_p(\cdot):\ \mathscr{K}^n \to [0,1]$$
$$K \mapsto \Gamma_p(K)$$

is called the *covering functional with respect to* p. For each $p \in \mathbb{Z}^+$, $\Gamma_p(\cdot)$ is an affine invariant. More precisely, $\Gamma_p(K) = \Gamma_p(T(K))$ holds for each non-degenerate affine transformation T on $\mathbb{R}^n$.

The convex hull of $n+1$ affinely independent vectors in $\mathbb{R}^n$ is called an *n-simplex*, which is denoted by S_n. Any n-simplex is the image of *the standard n-simplex*

$$\Delta_n := \left\{(\alpha_1,\cdots,\alpha_n) \in \mathbb{R}^n \mid \sum_{i\in[n]} \alpha_i \leqslant 1 \quad \text{and} \quad \alpha_j \geqslant 0,\ \forall j \in [n]\right\} \tag{1}$$

under a non-degenerate affine transformation. Therefore, $\Gamma_m(\Delta_n) = \Gamma_m(S_n)$ holds for each pair of $m,n \in \mathbb{Z}^+$. In [13], M. Lassak provided exact values of $\Gamma_m(S_2)$ when $3 \leqslant m \leqslant 9$. Chuanming Zong [27] mentioned that $\Gamma_4(S_3) = 3/4$ and $\Gamma_5(S_3) = 9/13$. Fangyu Zhang et al. proved that $\Gamma_6(S_3) \leqslant 27/40$, $\Gamma_7(S_3) \leqslant 81/121$, and $\Gamma_8(S_3) \leqslant 5/8$ (cf. [26]). Exact values of $\Gamma_6(S_3)$, $\Gamma_7(S_3)$, and $\Gamma_8(S_3)$ were not known. In a recent work [15], Xia Li et al. obtained some estimations of $\Gamma_m(S_n)$ for large n. Moreover, they showed that, if $P \in \mathscr{K}^n$ is a convex polytope with $m+1$ vertices, then

$$\Gamma_p(P) \leqslant \Gamma_p(S_m), \tag{2}$$

which shows the importance of estimating $\Gamma_m(S_n)$. For this purpose, several lemmas are proved in Section 2. In Section 3, we provide the precise value of $\Gamma_{n+2}(\Delta_n)$. Meanwhile, we prove that $-\Delta_n$ can be covered by two translates of $(n-1)\Delta_n$ when $n \geqslant 3$ and that the coefficient is best possible if $n = 3$. Based on this result, we provide an upper bound of $\Gamma_{n+3}(\Delta_n)$ and the exact value of $\Gamma_6(\Delta_3)$. In Section 4, new lower and upper bounds of $\Gamma_7(\Delta_3)$ and an upper bound of $\Gamma_8(\Delta_3)$ are presented. Covering functionals of Δ_4 are also estimated by using results in [22]. By (2), results mentioned above yield also estimations of covering functionals of convex polytopes with few vertices.

2. Auxiliary Lemmas

For $c \in \mathbb{R}^n$ and $\gamma > 0$, set $\Delta_n^{c,\gamma} = c + \gamma\Delta_n$. For each $x \in \mathbb{R}^n$ and each $i \in [n]$, we denote by $p_i(x)$ the i-th coordinate of x. Clearly, we have

LEMMA 1. *Let $x = (\alpha_1,\ldots,\alpha_n)$ and $c = (\beta_1,\ldots,\beta_n)$ be two points in $\mathbb{R}^n$. Then $x \in \Delta_n^{c,\gamma}$ if and only if*

$$\sum_{i\in[n]} (\alpha_i - \beta_i) \leqslant \gamma \quad \textit{and} \quad \alpha_j - \beta_j \geqslant 0,\ \forall j \in [n]. \tag{3}$$

For a finite set $S \subseteq \mathbb{R}^n$, let $\gamma(S) = \min\{\gamma > 0 \mid \exists c \in \mathbb{R}^n \text{ s.t. } S \subseteq \Delta_n^{c,\gamma}\}$.

LEMMA 2. *Let $S \subseteq \mathbb{R}^n$ be a finite set. Then*

$$\gamma(S) = \max\left\{\sum_{i\in[n]}(p_i(x)-\beta_i) \mid x \in S\right\},$$

where $\beta_i = \min\{p_i(x) \mid x \in S\}$, $\forall i \in [n]$.

Proof. Let $\alpha = \max\left\{\sum_{i\in[n]}(p_i(x)-\beta_i) \mid x \in S\right\}$ and $c = (\beta_1,\cdots,\beta_n)$. For any $x \in S$, we have

$$\sum_{i\in[n]}(p_i(x)-\beta_i) \leqslant \alpha \quad \text{and} \quad p_j(x)-\beta_j \geqslant 0,\ \forall j \in [n].$$

Thus $S \subseteq \Delta_n^{c,\alpha}$, which implies that $\gamma(S) \leqslant \alpha$. Conversely, let $c' \in \mathbb{R}^n$ be a point satisfying $S \subseteq \Delta_n^{c',\gamma(S)}$. Then $p_i(c') \leqslant p_i(x)$ holds for each $x \in S$ and each $i \in [n]$. Hence $p_i(c') \leqslant \beta_i$, $\forall i \in [n]$, which implies that

$$\sum_{i\in[n]}(p_i(x)-\beta_i) \leqslant \sum_{i\in[n]}(p_i(x)-p_i(c')) \leqslant \gamma(S),\ \forall x \in S.$$

Therefore, $\alpha \leqslant \gamma(S)$. This completes the proof. $\square$

For $K \in \mathscr{K}^n$ and $p \in \mathbb{Z}^+$, a set C of p points satisfying

$$K \subseteq \Gamma_p(K)K + C = \bigcup_{c\in C}(\Gamma_p(K)K + c)$$

is called a *p-optimal configuration of K*.

LEMMA 3. *For $\gamma \in (0,1)$ and $c \in \mathbb{R}^n$, there exists $c' \in (1-\gamma)\Delta_n$ such that*

$$\Delta_n^{c,\gamma} \cap \Delta_n \subseteq \Delta_n^{c',\gamma}.$$

Proof. We only need to consider the case when $\Delta_n^{c,\gamma} \cap \Delta_n \neq \emptyset$ and $c = (\beta_1,\cdots,\beta_n) \notin (1-\gamma)\Delta_n$. Let $I = \{i \in [n] \mid \beta_i < 0\}$. We distinguish two cases.

Case 1. $I = \emptyset$. Then $\sum_{i\in[n]}\beta_i > 1-\gamma$. Put

$$\beta_i' = \frac{(1-\gamma)\beta_i}{\sum\limits_{j\in[n]}\beta_j},\ \forall i \in [n] \quad \text{and} \quad c' = (\beta_1',\cdots,\beta_n').$$

Then

$$0 \leqslant \beta_i' \leqslant \beta_i,\ \forall i \in [n] \quad \text{and} \quad c' \in (1-\gamma)\Delta_n.$$

For each point $x=(\alpha_1,\cdots,\alpha_n)\in\Delta_n^{c,\gamma}\cap\Delta_n$, we have

$$\alpha_i-\beta_i'\geqslant\alpha_i-\beta_i\geqslant 0,\ \forall i\in[n]\quad\text{and}\quad\sum_{j\in[n]}(\alpha_j-\beta_j')=\sum_{j\in[n]}\alpha_j-(1-\gamma)\leqslant\gamma.$$

Thus $\Delta_n^{c,\gamma}\cap\Delta_n\subseteq\Delta_n^{c',\gamma}$.

Case 2. $I\neq\emptyset$. Set $c'=(\beta_1',\cdots,\beta_n')$, where $\beta_i'=\beta_i$ if $i\in[n]\setminus I$ and $\beta_i'=0$ otherwise. Let $x=(\alpha_1,\cdots,\alpha_n)$ be an arbitrary point in $\Delta_n^{c,\gamma}\cap\Delta_n$. We have

$$\alpha_i-\beta_i'=\begin{cases}\alpha_i, & i\in I\\ \alpha_i-\beta_i, & i\in[n]\setminus I\end{cases}\geqslant 0,\quad\forall i\in[n]$$

and

$$\sum_{j\in[n]}(\alpha_j-\beta_j')\leqslant\sum_{j\in[n]}(\alpha_j-\beta_j)\leqslant\gamma.$$

Thus $\sum_{j\in[n]}\beta_j'\leqslant\sum_{j\in[n]}\alpha_j\leqslant 1$ and $x\in\Delta_n^{c',\gamma}\cap\Delta_n$. Hence

$$c'\in\Delta_n\quad\text{and}\quad\Delta_n^{c,\gamma}\cap\Delta_n\subseteq\Delta_n^{c',\gamma}\cap\Delta_n.$$

If $c'\in(1-\gamma)\Delta_n$, then the proof is complete. Otherwise, by Case 1, there exists $c''\in(1-\gamma)\Delta_n$ such that

$$\Delta_n^{c,\gamma}\cap\Delta_n\subseteq\Delta_n^{c',\gamma}\cap\Delta_n\subseteq\Delta_n^{c'',\gamma}.$$

I.e., c'' is a point with the desired property. $\square$

COROLLARY 4. *For each positive integer p, there exists a p-optimal configuration of Δ_n contained in $(1-\Gamma_p(\Delta_n))\Delta_n$.*

LEMMA 5. *For $\gamma>0$ and $c\in\mathbb{R}^n$, there exists $c'\in[-1,0]^n$ such that*

$$\Delta_n^{c,\gamma}\cap(-\Delta_n)\subseteq\Delta_n^{c',\gamma}.$$

Proof. It sufficies to consider the case when $\Delta_n^{c,\gamma}\cap(-\Delta_n)\neq\emptyset$ and $c=(\beta_1,\cdots,\beta_n)\notin[-1,0]^n$. Let

$$I=\{i\in[n]\mid\beta_i<-1\}.$$

Then $\beta_i\leqslant 0$ for each $i\in[n]$ and $I\neq\emptyset$. Set $c'=(\beta_1',\cdots,\beta_n')$, where $\beta_i'=\beta_i$ if $i\in[n]\setminus I$ and $\beta_i'=-1$ otherwise. Clearly, $c'\in[-1,0]^n$. For each point $x=(\alpha_1,\cdots,\alpha_n)$ in $\Delta_n^{c,\gamma}\cap(-\Delta_n)$, we have

$$\alpha_i-\beta_i'=\begin{cases}\alpha_i+1, & i\in I\\ \alpha_i-\beta_i, & i\in[n]\setminus I\end{cases}\geqslant 0,\quad\forall i\in[n]$$

and

$$\sum_{i\in[n]}(\alpha_i-\beta_i')\leqslant\sum_{i\in[n]}(\alpha_i-\beta_i)\leqslant\gamma.$$

Thus c' is a point with the desired property. $\square$

LEMMA 6. *Let* $\gamma \in (0,1)$, $c \in (1-\gamma)\Delta_n$, *and* $x \in \mathrm{bd}\,\Delta_n \cap \Delta_n^{c,\gamma}$. *We have*
(a) $c \in \mathrm{bd}\,((1-\gamma)\Delta_n)$,
(b) $c = o$ *if* $x = o$,
(c) $c = (1-\gamma)e_i$ *if* $x = e_i$ *for some* $i \in [n]$.

Proof. Assume that $c = (\beta_1, \cdots, \beta_n)$ and $x = (\alpha_1, \cdots, \alpha_n)$.
(a) Otherwise, we have

$$\beta_i > 0,\ \forall i \in [n] \quad \text{and} \quad \sum_{j\in[n]} \beta_j < 1-\gamma.$$

Since $x \in \mathrm{bd}\,\Delta_n$, either $\sum_{j\in[n]} \alpha_j = 1$ or there exists $i \in [n]$ such that $\alpha_i = 0$. In the former case, we have

$$\sum_{j\in[n]} (\alpha_j - \beta_j) > \gamma,$$

a contradiction to (3); in the later case, we have $\alpha_i - \beta_i < 0$, yields also a contradiction.
(b) If $x = o$, then $0 \leqslant \beta_i \leqslant \alpha_i = 0,\ \forall\, i \in [n]$. Thus $c = o$.
(c) Without loss of generality, we may assume that $x = e_1$. By (3) again, $\beta_i = 0$ when $i \neq 1$. Therefore,

$$\sum_{j\in[n]} (\alpha_j - \beta_j) = \alpha_1 - \beta_1 = 1 - \beta_1 \leqslant \gamma,$$

which implies that $\beta_1 \geqslant 1-\gamma$. Since $c \in (1-\gamma)\Delta_n$, we have $\beta_1 \leqslant 1-\gamma$. Hence $\beta_1 = 1-\gamma$. $\square$

For $\gamma \in [0, \frac{n}{n+1}]$, set

$$P(n,\gamma) = \left\{ (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n \mid \gamma \leqslant \sum_{i\in[n]} \alpha_i \leqslant 1 \quad \text{and} \quad 0 \leqslant \alpha_j \leqslant 1-\gamma,\ \forall j \in [n] \right\}.$$

Obviously, $P(n,\alpha) \subseteq P(n,\gamma)$ if $\alpha \geqslant \gamma$. By (1), we have

$$\Delta_n = P(n,\gamma) \cup (\bigcup_{i=0}^{n} \Delta_n^{c_i,\gamma}), \tag{4}$$

where $c_0 = o$ and $c_i = (1-\gamma)e_i$, $\forall i \in [n]$. Indeed, it is clear that

$$P(n,\gamma) \cup (\bigcup_{i=0}^{n} \Delta_n^{c_i,\gamma}) \subseteq \Delta_n.$$

If $x = (\alpha_1, \cdots, \alpha_n) \in \Delta_n \setminus P(n,\gamma)$, then either $0 \leqslant \sum_{i\in[n]} \alpha_i < \gamma$, which implies that $x \in \Delta_n^{o,\gamma}$, or there exists $k \in [n]$ such that $\alpha_k \in (1-\gamma, 1]$, which shows that $x \in \Delta_n^{c_k,\gamma}$.

LEMMA 7. *For* $\gamma \in (0,1)$ *and* $p \in \mathbb{Z}^+$, $\Gamma_{p+n+1}(\Delta_n) \leqslant \gamma$ *if and only if* $P(n,\gamma)$ *can be covered by* p *translates of* $\gamma\Delta_n$.

Proof. Let $c_0 = o$ and $c_i = (1-\gamma)e_i$, $\forall i \in [n]$. If $\Gamma_{p+n+1}(\Delta_n) \leqslant \gamma$, then there exists a set $C \subseteq \mathbb{R}^n$ with $|C| = p+n+1$ such that

$$\Delta_n \subseteq C + \gamma\Delta_n.$$

By Lemma 3, we may assume that $C \subseteq (1-\gamma)\Delta_n$. Since o and $\{e_i \mid i \in [n]\}$ are contained in $\operatorname{bd}\Delta_n$, Lemma 6 shows that $\{c_i \mid i \in [n] \cup \{0\}\} \subseteq C$. For each $i \in [n] \cup \{0\}$, we have $\operatorname{int} P(n,\gamma) \cap \Delta_n^{c_i,\gamma} = \emptyset$. Thus

$$\operatorname{int} P(n,\gamma) \subseteq (C \setminus \{c_i \mid i \in [n] \cup \{0\}\}) + \gamma\Delta_n.$$

Since $\gamma\Delta_n$ is closed,

$$P(n,\gamma) \subseteq (C \setminus \{c_i \mid i \in [n] \cup \{0\}\}) + \gamma\Delta_n.$$

Conversely, let $C' \subseteq \mathbb{R}^n$ be a p-element set satisfying $P(n,\gamma) \subseteq C' + \gamma\Delta_n$. By (4), we have

$$\Delta_n \subseteq (C' \cup \{c_i \mid i \in [n] \cup \{0\}\}) + \gamma\Delta_n,$$

which implies that $\Gamma_{p+n+1}(\Delta_n) \leqslant \gamma$. □

LEMMA 8. *For* $n \in \mathbb{Z}^+$ *and* $\gamma \in [\frac{n-1}{n}, \frac{n}{n+1}]$, *we have*

$$P(n,\gamma) = (\gamma - n + n\gamma)\Delta_n + (1-\gamma)\sum_{i\in[n]} e_i.$$

Proof. The case when $\gamma = \frac{n}{n+1}$ is clear. In the following, we assume that $\gamma \in \left[\frac{n-1}{n}, \frac{n}{n+1}\right)$. Let $S = P(n,\gamma) - (1-\gamma)\sum_{i\in[n]} e_i$. Then

$$\begin{aligned} S &= \left\{(\beta_1,\cdots,\beta_n) \in \mathbb{R}^n \mid \beta_j + (1-\gamma) \in [0, 1-\gamma],\ \forall j \in [n],\ \sum_{i\in[n]} \beta_i + n(1-\gamma) \in [\gamma, 1]\right\} \\ &= \left\{(\beta_1,\cdots,\beta_n) \in \mathbb{R}^n \mid \beta_j \in [\gamma-1, 0],\ \forall j \in [n],\ \sum_{i\in[n]} \beta_i \in [\gamma - n + n\gamma, 1 - n + n\gamma]\right\}. \end{aligned}$$

Since $\gamma - 1 \leqslant \gamma - n + n\gamma < 0$ and $1 - n + n\gamma \geqslant 0$, we have

$$\begin{aligned} S &= \left\{(\beta_1,\cdots,\beta_n) \in \mathbb{R}^n \mid \beta_j \in [\gamma - n + n\gamma, 0],\ \forall j \in [n],\ \sum_{i\in[n]} \beta_i \in [\gamma - n + n\gamma, 0]\right\} \\ &= \left\{(\gamma - n + n\gamma)(\alpha_1,\cdots,\alpha_n) \in \mathbb{R}^n \mid \alpha_j \in [0,1],\ \forall j \in [n],\ \sum_{i\in[n]} \alpha_i \in [0,1]\right\} \\ &= (\gamma - n + n\gamma)\Delta_n. \end{aligned}$$

This completes the proof. □

LEMMA 9. *Supposed that* K *is bounded and* $\operatorname{relbd} K \subseteq \bigcup_{i\in[m]} K_i$, *where* K_i *is convex,* $\forall i \in [m]$. *If there exists* $p \in K$ *such that* $p \in \bigcap_{i\in[m]} K_i$, *then* $K \subseteq \bigcup_{i\in[m]} K_i$.

Proof. Let $x \in K$. We claim that there exist a number $\alpha \in [0,1]$ and a point $y \in \operatorname{relbd} K$ such that $x = \alpha p + (1-\alpha)y$. When $x \in \operatorname{relbd} K$, take $\alpha = 0$ and $y = x$. The case when $x = p$ is also clear. Now suppose that $x \in \operatorname{relint} K \setminus \{p\}$. Since K is bounded, there exists $y \in ([p,x\rangle \setminus [p,x]) \cap \operatorname{relbd} K$. Then $x \in [p,y]$. The claim is proved.

Since $\operatorname{relbd} K \subseteq \bigcup_{i\in[m]} K_i$, there exists $j \in [m]$ such that $y \in K_j$. By the convexity of K_j, we have

$$x \in K_j \subseteq \bigcup_{i\in[m]} K_i. \quad \square$$

3. Covering a simplex by its negative homothetic copies

Let $K \in \mathscr{K}^n$. For each $x \in K$, put

$$r_K(x) = \max\{\gamma \geqslant 0 \mid (1+\gamma)x - \gamma K \subseteq K\}.$$

The number

$$r_K = \max\{r_K(x) \mid x \in K\}$$

is called the *critical ratio* of K (cf. [24]). A point $x \in \operatorname{int} K$ satisfying $r_K(x) = r_K$ is called the *critical point* of K. It is shown that $r_K \geqslant 1/n$ holds for each $K \in \mathscr{K}^n$ and the equality holds if K is an n-simplex (cf. [24] again). Thus n is the least positive number γ such that $-S_n$ is contained in a translate of γS_n. Indeed, we may assume, without loss of generality, that o is a critical point of S_n. Then $-S_n \subseteq nS_n$. Suppose that there exist $c \in \mathbb{R}^n$ and $\beta \in (0,n)$ such that $-S_n \subseteq c + \beta S_n$. Then

$$\left(1+\frac{1}{\beta}\right)\left(-\frac{c}{1+\beta}\right) - \frac{1}{\beta}S_n \subseteq S_n \quad \text{and} \quad -c \in (1+\beta)S_n,$$

which implies that $r_{S_n} \geqslant 1/\beta > 1/n$, a contradiction.

THEOREM 10. *For* $n \in \mathbb{Z}^+$, *we have* $\Gamma_{n+2}(\Delta_n) = \frac{n^2}{n^2+n+1}$.

Proof. In [20], it is proved that $\Gamma_{n+2}(\Delta_n) \leqslant \frac{n^2}{n^2+n+1}$. We only need to prove the reverse inequality. Clearly, $\frac{n-1}{n} < \frac{n^2}{n^2+n+1} < \frac{n}{n+1}$. By Lemma 8, we have

$$P\left(n, \frac{n^2}{n^2+n+1}\right) = -\frac{n}{n^2+n+1}\Delta_n + \frac{n+1}{n^2+n+1}\sum_{i\in[n]} e_i.$$

By Lemma 7, $\Gamma_{n+2}(\Delta_n) \geqslant \frac{n^2}{n^2+n+1}$. $\square$

Lemma 8 shows that it is important to study the problem of covering a simplex by its negative homothetic copies. Januszewski et al. proved that K can be covered by two translates of $(-4/3)K$ for each $K \in \mathscr{K}^2$, and if K is a 2-simplex, then $-4/3$ is the best negative ratio (cf. [10]). Similar results are still missing for higher dimensions.

Let C be a finite set in $\mathbb{R}^n$ satisfying $\Delta_n \subseteq C - \gamma\Delta_n$ and P be a permutation of coordinates on $\mathbb{R}^n$. Then $\Delta_n \subseteq P(C) - \gamma\Delta_n$. We shall use this simple observation in the proof of the next result.

THEOREM 11. *For an integer $n \geqslant 3$, $-\Delta_n$ can be covered by two translates of $(n-1)\Delta_n$. When $n = 3$, the coefficient $n-1$ is best possible.*

Proof. Let

$$c_1 = (\beta - 1)e_1 - \sum_{i \in [n] \setminus \{1\}} e_i \quad \text{and} \quad c_2 = -e_1 + (\beta - 1) \sum_{i \in [n] \setminus \{1\}} e_i,$$

where $\beta = \frac{1}{n - [\frac{n}{2}]}$. Obviously, $-\Delta_n = I_1 \cup I_2 \cup I_3$, where

$$I_1 = \left\{ (\alpha_1, \cdots, \alpha_n) \in -\Delta_n \mid \alpha_1 \in [\beta - 1, 0] \quad \text{and} \quad \sum_{i \in [n]} \alpha_i \in [-1, \beta - 1] \right\},$$

$$I_2 = \left\{ (\alpha_1, \cdots, \alpha_n) \in -\Delta_n \mid \alpha_1 \in [\beta - 1, 0] \quad \text{and} \quad \sum_{i \in [n]} \alpha_i \in [\beta - 1, 0] \right\},$$

$$I_3 = \left\{ (\alpha_1, \cdots, \alpha_n) \in -\Delta_n \mid \alpha_1 \in [-1, \beta - 1] \quad \text{and} \quad \sum_{i \in [n]} \alpha_i \in [-1, \beta - 1] \right\}.$$

Since

$$\begin{aligned} I_1 - c_1 = \Big\{ (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n \mid \alpha_1 \in [0, 1 - \beta],\ \alpha_j \in [0, 1],\ \forall j \in [n] \setminus \{1\}, \\ \sum_{i \in [n]} \alpha_i \in [n - 1 - \beta, n - 1] \Big\} \\ \subseteq (n-1)\Delta_n, \end{aligned}$$

we have $I_1 \subseteq c_1 + (n-1)\Delta_n$. Let $x = (\alpha_1, \cdots, \alpha_n) \in I_2 \cup I_3$. When n is odd, we have $\beta = 2/(n+1) \leqslant 1/2$. It follows that

$$\sum_{i \in [n]} \alpha_i - [(n-1)\beta - n] \leqslant -(n-1)\beta + n = \frac{n^2 - n + 2}{n+1} \leqslant n - 1. \tag{5}$$

When n is even, we have $n \geqslant 4$ and $\beta = 2/n \leqslant 1/2$. Hence

$$\sum_{i \in [n]} \alpha_i - [(n-1)\beta - n] \leqslant -(n-1)\beta + n \leqslant \frac{n^2 - 2n + 2}{n} \leqslant n - 1. \tag{6}$$

Moreover,

$$\alpha_j \geqslant \beta - 1,\ \forall j \in [n] \setminus \{1\}. \tag{7}$$

Since, otherwise, we would have

$$\sum_{i\in[n]}\alpha_i < \begin{cases}\beta-1, & x\in I_2,\\ 2(\beta-1)\leqslant -1, & x\in I_3,\end{cases}$$

which yields a contradiction. By (5), (6), and (7), we have $I_2\cup I_3\subseteq c_2+(n-1)\Delta_n$. Consequently, $-\Delta_n\subseteq\bigcup_{i\in[2]}(c_i+(n-1)\Delta_n)$.

In the following, we consider the case when $n=3$ and show that 2 is the least positive number γ such that $-\Delta_3$ can be covered by two translates of $\gamma\Delta_3$. Otherwise, there exist $\gamma\in(0,2)$ and a set $C=\{c_1,c_2\}$ such that $-\Delta_3\subseteq C+\gamma\Delta_3$. If there exists $c\in C$ such that $c+\gamma\Delta_3$ contains at least three vertices of $-\Delta_3$, then, by Lemma 2, $\gamma\geqslant 2$, a contradiction. Therefore, for any $c\in C$, $c+\gamma\Delta_3$ contains precisely two vertices of $-\Delta_3$. By applying a permutation of coordinates if necessary, we may assume that

$$\{-e_1,-e_2\}\subseteq c_1+\gamma\Delta_3 \quad\text{and}\quad \{-e_3,o\}\subseteq c_2+\gamma\Delta_3. \tag{8}$$

Applying Lemma 5, we may assume that $p_i(c_1),p_i(c_2)\in[-1,0]$, $\forall i\in[3]$. By Lemma 1, there exist real numbers α, β, and η such that

$$c_1=(-1,-1,\alpha) \quad\text{and}\quad c_2=(\beta,\eta,-1).$$

Since $-e_1$ and o are covered by different translates of $\gamma\Delta_3$, by (8), there exists $\mu_1\in[-1,0]$ such that

$$\{(\alpha_1,0,0)\in[-e_1,o]\mid \alpha_1\in[-1,\mu_1]\}\subseteq c_1+\gamma\Delta_3,$$
$$\{(\alpha_1,0,0)\in[-e_1,o]\mid \alpha_1\in[\mu_1,0]\}\subseteq c_2+\gamma\Delta_3.$$

For any point $x=(\alpha_1,\alpha_2,\alpha_3)\in[-e_1,o]\cap(c_1+\gamma\Delta_3)$, we have

$$\sum_{i\in[3]}\alpha_i-\sum_{i\in[3]}p_i(c_1)\leqslant\mu_1+2-\alpha\leqslant\gamma<2.$$

For each point $x=(\alpha_1,\alpha_2,\alpha_3)\in[-e_1,o]\cap(c_2+\gamma\Delta_3)$, we have

$$\alpha_1\geqslant\mu_1\geqslant p_1(c_2)=\beta \quad\text{and}\quad \sum_{i\in[3]}\alpha_i-\sum_{i\in[3]}p_i(c_2)\leqslant-\beta-\eta+1\leqslant\gamma<2.$$

Therefore, we have

$$\beta\leqslant\mu_1<\alpha \tag{9}$$

and

$$\beta+\eta>-1. \tag{10}$$

Similarly, there exists $\mu_2\in[-1,0]$ such that

$$\{(0,\alpha_2,-1-\alpha_2)\in[-e_2,-e_3]\mid \alpha_2\in[-1,\mu_2]\}\subseteq c_1+\gamma\Delta_3,$$
$$\{(0,\alpha_2,-1-\alpha_2)\in[-e_2,-e_3]\mid \alpha_2\in[\mu_2,0]\}\subseteq c_2+\gamma\Delta_3.$$

If $x = (\alpha_1, \alpha_2, \alpha_3) \in [-e_2, -e_3] \cap (c_1 + \gamma\Delta_3)$, then

$$\alpha_3 \geqslant -1 - \mu_2 \geqslant p_3(c_1) = \alpha;$$

if $x = (\alpha_1, \alpha_2, \alpha_3) \in [-e_2, -e_3] \cap (c_2 + \gamma\Delta_3)$, then

$$\alpha_2 \geqslant \mu_2 \geqslant p_2(c_2) = \eta.$$

Thus

$$\alpha + \eta \leqslant \alpha + \mu_2 \leqslant -1. \tag{11}$$

By (10) and (11), we have $\beta > \alpha$, a contradiction to (9). □

From Lemma 8 and Theorem 11, it follows that

COROLLARY 12. *For an integer $n \geqslant 3$, we have $\Gamma_{n+3}(\Delta_n) \leqslant \frac{n-1}{n}$, the equality holds when $n = 3$.*

By Theorem 10, Corollary 12, and (2), we have

COROLLARY 13. *For a convex polytope with m vertices P_m in $\mathbb{R}^n$, we have*

$$\Gamma_{m+2}(P_{m+1}) \leqslant \frac{m^2}{m^2+m+1} \quad \textit{and} \quad \Gamma_{m+3}(P_{m+1}) \leqslant \frac{m-1}{m}.$$

4. New estimations for 3-simplies

When $\gamma \in (1/2, 2/3)$, $P(3,\gamma)$ is an octahedron with vertices:

$$v_1 = (1-\gamma, 0, 1-\gamma),\ v_2 = (2\gamma-1, 0, 1-\gamma),\ v_3 = (0, 2\gamma-1, 1-\gamma),$$
$$v_4 = (0, 1-\gamma, 1-\gamma),\ v_5 = (2\gamma-1, 1-\gamma, 1-\gamma),\ v_6 = (1-\gamma, 2\gamma-1, 1-\gamma),$$
$$v_7 = (0, 1-\gamma, 2\gamma-1),\ v_8 = (2\gamma-1, 1-\gamma, 0),\ v_9 = (1-\gamma, 2\gamma-1, 0),$$
$$v_{10} = (1-\gamma, 0, 2\gamma-1),\ v_{11} = (1-\gamma, 1-\gamma, 0),\ v_{12} = (1-\gamma, 1-\gamma, 2\gamma-1);$$

four triangular facets:

$$A_1 = \mathrm{conv}(\{v_3, v_4, v_7\}),\ A_2 = \mathrm{conv}(\{v_1, v_2, v_{10}\}),$$
$$A_3 = \mathrm{conv}(\{v_8, v_9, v_{11}\}),\ A_4 = \mathrm{conv}(\{v_5, v_6, v_{12}\});$$

and four hexagonal facets:

$$B_1 = \mathrm{conv}(\{v_1, v_2, v_3, v_4, v_5, v_6\}),\ B_2 = \mathrm{conv}(\{v_2, v_3, v_7, v_8, v_9, v_{10}\}),$$
$$B_3 = \mathrm{conv}(\{v_1, v_6, v_{12}, v_{11}, v_9, v_{10}\}),\ B_4 = \mathrm{conv}(\{v_4, v_5, v_{12}, v_{11}, v_8, v_7\}).$$

See Figure 1.

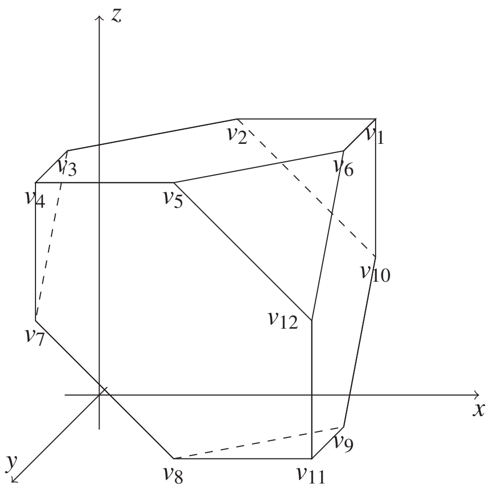

Figure 1: $P(3,\gamma)$ *when* $\gamma \in (1/2, 2/3)$

THEOREM 14. $\Gamma_7(\Delta_3) \leqslant 11/17$.

Proof. Let $C = \{c_1, c_2, c_3\}$, where $c_1 = (3/17, 3/17, 0)$, $c_2 = (0, 2/17, 2/17)$, and $c_3 = (3/34, 0, 3/17)$. Set

$$K_i = c_i + \frac{11}{17}\Delta_3, \quad \forall i \in [3].$$

Since

$$\frac{1}{4}\sum_{i\in[3]} e_i \in P\left(3, \frac{11}{17}\right) \cap \left(\bigcap_{j\in[3]} K_j\right),$$

by Lemma 9, it sufficies to show that $\operatorname{relbd} P(3, 11/17) \subseteq \bigcup_{j\in[3]} K_j$. By Lemma 1, we have

$$v_1 = \left(\frac{6}{17}, 0, \frac{6}{17}\right) \in K_3,\ v_2 = \left(\frac{5}{17}, 0, \frac{6}{17}\right) \in K_3,\ v_3 = \left(0, \frac{5}{17}, \frac{6}{17}\right) \in K_2,$$
$$v_4 = \left(0, \frac{6}{17}, \frac{6}{17}\right) \in K_2,\ v_5 = \left(\frac{5}{17}, \frac{6}{17}, \frac{6}{17}\right) \in K_1,\ v_6 = \left(\frac{6}{17}, \frac{5}{17}, \frac{6}{17}\right) \in K_1,$$
$$v_7 = \left(0, \frac{6}{17}, \frac{5}{17}\right) \in K_2,\ v_8 = \left(\frac{5}{17}, \frac{6}{17}, 0\right) \in K_1,\ v_9 = \left(\frac{6}{17}, \frac{5}{17}, 0\right) \in K_1,$$
$$v_{10} = \left(\frac{6}{17}, 0, \frac{5}{17}\right) \in K_3,\ v_{11} = \left(\frac{6}{17}, \frac{6}{17}, 0\right) \in K_1, \text{ and}$$
$$v_{12} = \left(\frac{6}{17}, \frac{6}{17}, \frac{5}{17}\right) \in K_1.$$

Hence, by the convexity of K_i, $\forall i \in [3]$, we have

$$A_1 \subseteq K_2, \quad A_2 \subseteq K_3, \quad A_3 \subseteq K_1, \quad \text{and} \quad A_4 \subseteq K_1.$$

By Lemma 1, $\lambda v_2 + (1-\lambda)v_3 \in K_2$ if $\lambda \in [0, 3/10]$; $\lambda v_2 + (1-\lambda)v_3 \in K_3$ if $\lambda \in [3/10, 1]$. Thus $[v_2, v_3] \subseteq \bigcup_{i \in [3]\setminus\{1\}} K_i$. Similarly,

$$[v_4, v_5] \subseteq \bigcup_{i\in[2]} K_i, \ [v_7, v_8] \subseteq \bigcup_{i\in[2]} K_i, \ [v_1, v_6] \subseteq \bigcup_{i\in[3]\setminus\{2\}} K_i, \quad \text{and}$$
$$[v_9, v_{10}] \subseteq \bigcup_{i\in[3]} K_i.$$

Let $b_1 = (3/17, 3/17, 6/17)$, $b_2 = (3/17, 4/17, 4/17)$, $b_3 = (6/17, 3/17, 3/17)$, and $b_4 = (3/17, 6/17, 3/17)$. Then

$$b_i \in B_i \cap \Big(\bigcap_{j\in[3]} K_j\Big) \quad \text{and} \quad \operatorname{relbd} B_i \subseteq \bigcup_{j\in[3]} K_j, \quad \forall i \in [4].$$

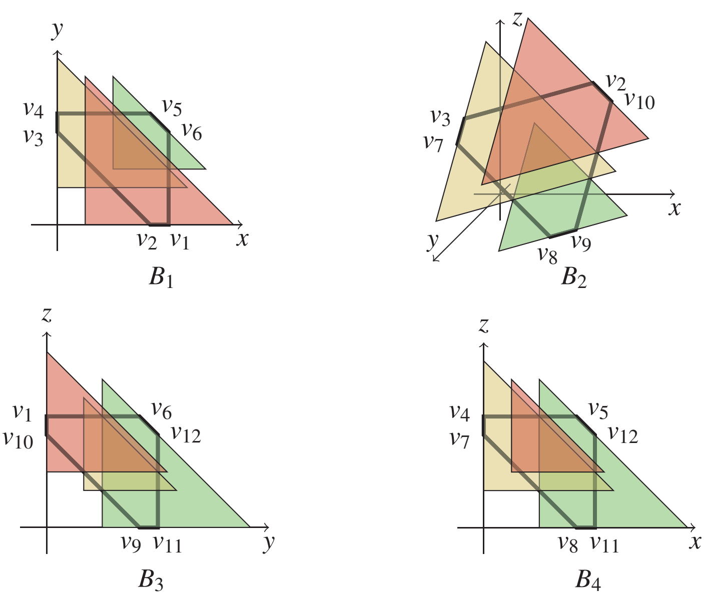

Figure 2: B_1, B_2, B_3, *and* B_4 *of* $P(3, 11/17)$. *For any* $i \in [4]$, *the green, yellow and red parts represent the intersection of* B_i *and* K_1, K_2, *and* K_3, *respectively.*

By Lemma 9, $B_i \subseteq \bigcup_{j\in[3]} K_j$, $\forall i \in [4]$, see Figure 2. Hence $\operatorname{relbd} P(3, 11/17) \subseteq \bigcup_{i\in[3]} K_i$. Applying Lemma 9 again, $P(3, 11/17) \subseteq \bigcup_{i\in[3]} K_i$. By Lemma 7, $\Gamma_7(\Delta_3) \leqslant 11/17$. $\square$

THEOREM 15. $\Gamma_7(\Delta_3) \geqslant 0.6$.

Proof. Suppose the contrary that there exist a number $\gamma \in (0,0.6)$ and a set $C \subseteq \mathbb{R}^3$ of 3 points satisfying $P(3,0.6) \subseteq C+\gamma\Delta_3$. Let c be an arbitrary point in C and

$$S=(c+\gamma\Delta_3)\cap \operatorname{ext}P(3,0.6).$$

Claim 1. For each $i \in [4]$ and each point $v \in \operatorname{ext}P(3,0.6)\setminus \operatorname{ext}A_i$, $\operatorname{ext}A_i \cup \{v\} \not\subseteq S$.

Otherwise, there exist $i_0 \in [4]$ and a point $v \in \operatorname{ext}P(3,0.6)\setminus \operatorname{ext}A_{i_0}$ such that $\operatorname{ext}A_{i_0} \cup \{v\} \subseteq S$. Then there exists $j_0 \in [4]\setminus\{i_0\}$ such that $v \in \operatorname{ext}A_{j_0}$. For the case when $i_0 \in [3]$, we may assume, without loss of generality, that $i_0 = 1$. Then

$$\min\{p_2(x) \mid x \in S\},\ \min\{p_3(x) \mid x \in S\} \leqslant 0.2 \quad \text{and} \quad \min\{p_1(x) \mid x \in S\} = 0.$$

If $j_0 \neq 4$, then $\min\left\{p_{j_0}(x) \mid x \in S\right\} = 0$. Therefore, by Lemma 2,

$$\gamma(S) \geqslant \begin{cases} \sum_{i\in[3]} p_i(v_4) - 0.2, & j_0 \neq 4 \\ \sum_{i\in[3]} p_i(v) - 0.4, & j_0 = 4 \end{cases} = 0.6,$$

a contradiction. Now suppose that $i_0 = 4$. We have

$$\min\{p_i(x) \mid x \in S\} \leqslant 0.2,\ \forall i \in [3] \quad \text{and} \quad \min\left\{p_{j_0}(x) \mid x \in S\right\} = 0.$$

By Lemma 2 again,

$$\gamma(S) \geqslant \sum_{i\in[3]} p_i(v_5) - 0.4 = 0.6,$$

which yields also a contradiction. This completes the proof of Claim 1.

Claim 2. S cannot contain points from three distinct triangular facets of $P(3,0.6)$.

Otherwise, there exist $i,\ j \in [3]$ with $i \neq j$ such that S contains a point $u \in \operatorname{ext}A_i$, a point $v \in \operatorname{ext}A_j$, and a point $w \in \operatorname{ext}P(3,0.6)\setminus(\operatorname{ext}A_i \cup \operatorname{ext}A_j)$. Then

$$\min\{p_i(x) \mid x \in S\} = \min\left\{p_j(x) \mid x \in S\right\} = 0 \quad \text{and} \quad p_i(w),\ p_j(w) > 0.$$

By Lemma 2, $\gamma(S) \geqslant p_i(w)+p_j(w) \geqslant 0.6$. This completes the proof of Claim 2.

Claim 3. S contains precisely four vertices of $P(3,0.6)$ and there are two triangular facets of $P(3,0.6)$, each one of which contains two points in S.

By Claim 1 and Claim 2, S contains at most four vertices of $P(3,0.6)$. Thus S contains precisely four vertices of $P(3,0.6)$. By Claim 2, S intersects at most two triangular facets of $P(3,0.6)$. By Claim 1, each of these two facets contains two points in S. This completes the proof of Claim 3.

Claim 3 shows that, for each $v \in \operatorname{ext}P(3,0.6)$, there exists a unique $c \in C$ such that $v \in c+\gamma\Delta_3$. Clearly, there exist a triangular facets F of $P(3,0.6)$ and two distinct points $c_1,\ c_2 \in C$ such that

$$(c_1+\gamma\Delta_3)\cap \operatorname{ext}F, \quad (c_2+\gamma\Delta_3)\cap \operatorname{ext}F \neq \emptyset.$$

Then we have $|(c_1+\gamma\Delta_3)\cap \operatorname{ext}F| = |(c_2+\gamma\Delta_3)\cap \operatorname{ext}F| = 2$, which is impossible. □

THEOREM 16. $\Gamma_8(\Delta_3) \leqslant 8/13$.

Proof. Let $C = \{c_1, c_2, c_3, c_4\}$, where

$$c_1 = \left(\frac{5}{26}, 0, 0\right), \quad c_2 = \left(0, \frac{9}{52}, \frac{1}{26}\right),$$
$$c_3 = \left(\frac{17}{104}, \frac{17}{104}, \frac{3}{52}\right), \quad c_4 = \left(\frac{3}{52}, \frac{1}{26}, \frac{3}{13}\right).$$

Set $K_i = c_i + (8/13)\Delta_3$, $\forall i \in [4]$. Note that

$$\frac{1}{4}\sum_{i\in[3]} e_i \in P\left(3, \frac{8}{13}\right) \cap \left(\bigcap_{j\in[4]} K_j\right).$$

Therefore, we only need to show that $\operatorname{relbd} P(3, 8/13) \subseteq \bigcup_{j\in[4]} K_j$. By Lemma 1, we have

$$v_1 = \left(\frac{5}{13}, 0, \frac{5}{13}\right) \in K_1,\ v_2 = \left(\frac{3}{13}, 0, \frac{5}{13}\right) \in K_1,\ v_3 = \left(0, \frac{3}{13}, \frac{5}{13}\right) \in K_2,$$
$$v_4 = \left(0, \frac{5}{13}, \frac{5}{13}\right) \in K_2,\ v_5 = \left(\frac{3}{13}, \frac{5}{13}, \frac{5}{13}\right) \in K_3,\ v_6 = \left(\frac{5}{13}, \frac{3}{13}, \frac{5}{13}\right) \in K_3,$$
$$v_7 = \left(0, \frac{5}{13}, \frac{3}{13}\right) \in K_2,\ v_8 = \left(\frac{3}{13}, \frac{5}{13}, 0\right) \in K_1,\ v_9 = \left(\frac{5}{13}, \frac{3}{13}, 0\right) \in K_1,$$
$$v_{10} = \left(\frac{5}{13}, 0, \frac{3}{13}\right) \in K_1,\ v_{11} = \left(\frac{5}{13}, \frac{5}{13}, 0\right) \in K_1, \text{ and}$$
$$v_{12} = \left(\frac{5}{13}, \frac{5}{13}, \frac{3}{13}\right) \in K_3.$$

Hence

$$A_1 \subseteq K_2, \quad A_2 \subseteq K_1, \quad A_3 \subseteq K_1, \quad \text{and} \quad A_4 \subseteq K_3.$$

By Lemma 1, $\lambda v_2 + (1-\lambda)v_3 \in K_2$ if $\lambda \in [0, 1/4]$; $\lambda v_2 + (1-\lambda)v_3 \in K_4$ if $\lambda \in [1/4, 5/6]$; $\lambda v_2 + (1-\lambda)v_3 \in K_1$ if $\lambda \in [5/6, 1]$. Thus $[v_2, v_3] \subseteq \bigcup_{i\in[4]\setminus\{3\}} K_i$. Similarly,

$$[v_4, v_5] \subseteq \bigcup_{i\in[4]\setminus\{1\}} K_i, \quad [v_7, v_8] \subseteq \bigcup_{i\in[2]} K_i,$$
$$[v_1, v_6] \subseteq \bigcup_{i\in[4]\setminus\{2\}} K_i, \quad [v_{11}, v_{12}] \subseteq \bigcup_{i\in[3]} K_i.$$

Put

$$b_1 = \left(\frac{8}{39}, \frac{8}{39}, \frac{5}{13}\right), \quad b_2 = \left(\frac{5}{26}, \frac{5}{26}, \frac{3}{13}\right),$$
$$b_3 = \left(\frac{5}{13}, \frac{9}{52}, \frac{3}{13}\right), \quad b_4 = \left(\frac{5}{26}, \frac{5}{13}, \frac{3}{13}\right).$$

Hence

$$b_j \in B_j \cap \Big(\bigcap_{i\in[4]} K_i\Big) \quad \text{and} \quad \operatorname{relbd} B_j \subseteq \bigcup_{i\in[4]} K_i, \ \forall j \in [4].$$

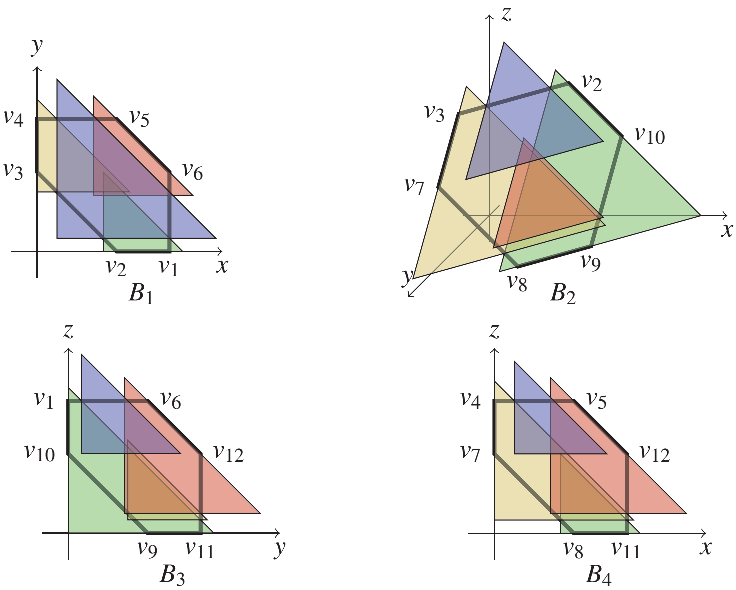

Figure 3: *B_1, B_2, B_3, and B_4 of $P(3,8/13)$. For any $i \in [4]$, the green, yellow, red, and blue parts represent the intersection of B_i and K_1, K_2, K_3, and K_4 respectively.*

By Lemma 9, $B_j \subseteq \bigcup_{i\in[4]} K_i$, $\forall j \in [4]$, see Figure 3. Therefore, $\operatorname{relbd} P(3,8/13) \subseteq \bigcup_{i\in[4]} K_i$. By Lemma 9 again, $P(3,8/13) \subseteq \bigcup_{i\in[4]} K_i$. This completes the proof. □

In [22], Senlin Wu and Ke Xu proved that, if $K \in \mathscr{K}^n$, then

$$\Gamma_{m+1}(C) \leqslant \frac{1}{2-\Gamma_m(K)}, \ \forall m \in \mathbb{Z}^+, \tag{12}$$

where $p \in \mathbb{R}^{n+1} \setminus \mathbb{R}^n \times \{0\}$ and $C = \operatorname{conv}((K \times \{0\}) \cup \{p\})$. Therefore, we obtain that $\Gamma_8(\Delta_4) \leqslant 17/23$ and $\Gamma_9(\Delta_4) \leqslant 13/18$ by Theorem 14 and Theorem 16, respectively.

As we have mentioned in the introduction, Fangyu Zhang et al. proved that $\Gamma_6(\Delta_3) \leqslant 27/40$, $\Gamma_7(\Delta_3) \leqslant 81/121$, and $\Gamma_8(\Delta_3) \leqslant 5/8$ (cf. [26]); Senlin Wu and Ke Xu [22] proved that $\Gamma_6(C) \leqslant 15/22$, $\Gamma_7(C) \leqslant 2/3$, and $\Gamma_8(C) \leqslant 11/17$, where C is a cone whose base is a triangle. Compared with these known estimations, we provide better estimations about $\Gamma_p(\Delta_3)$ when $p \in \{6,7,8\}$.

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