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DISCRETE OPIAL TYPE INEQUALITIES FOR INTERVAL–VALUED FUNCTIONS

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Abstract. We introduce the forward (backward) gH-difference operator of interval sequences, and establish some new discrete Opial type inequalities for interval-valued functions. Further, we obtain generalizations of classical discrete Opial type inequalities. Some examples are presented to illustrate our results.

1. Introduction

The theory of inequalities has a long history but, from the applicative point of view, it fell into neglect for hundreds of years because of lack of applications to other branch of mathematics as well as other sciences, such as physics and engineering. Only in 1934 did Hardy, Littlewood and Pólya transformed the field of inequalities from a collection of isolated formulas into a systematic discipline [24]. After that, an enormous amount of effort has been devoted to the discovery of new types of inequalities and to applications of inequalities [1].

It is known that many physical problems in various applications are governed by finite difference equations. Moreover, discrete inequalities play an important role in the continuing development of the theory of difference equations. This importance seems to have increased considerably during the past decades. It has attracted the attention of a large number of researchers, stimulated new research directions, and influenced various aspects of difference equations and applications. Among the many types of inequalities, those associated with the names of Jensen [12, 17, 18], Hilbert [26, 45], Wirtinger [2, 4, 19], Chebyshev [36, 49], Gronwall–Bellman [20, 37] and Opial [5, 7, 23, 27, 34, 35] have deep roots and made a great impact on various branches of mathematics. The development of discrete inequalities resulted in a renewal of interest in the field and has attracted interest from more researchers [6, 10, 11, 16, 22, 28, 29, 30, 31, 39, 40, 41, 42].

More recently, some of classical inequalities have been extended to set-valued functions by Nikodem et al. [32], Štrboja et al. [44], and Zhang et al. [46, 47], especially to interval-valued functions by Chalco-Cano et al. [8, 9], Costa et al. [13, 14, 15], Flores-Franulič et al. [21], Román-Flores et al. [38], and Zhao et al. [48, 49, 50, 51].

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The present article is, in some sense, a continuation of the previous work [50]. Here, we establish some new discrete inequalities of Opial type involving sequences of intervals and their forward (backward) difference operator. Furthermore, our present results can be considered as tools for further research in interval difference equations and inequalities for interval-valued functions, among others.

The paper is organized as follows. Section 2 contains some necessary preliminaries. In Section 3, we present some new interval Opial type inequalities involving the backward gH-difference operator, and present some examples to illustrate our theorems. In Section 4, some new discrete Opial type inequalities, involving the forward gH-difference operator, are given. Finally, in the concluding Section 5, we summarize our results and outline some possible future work directions.

2. Preliminaries

We begin by recalling some basic notations, definitions, and results of interval analysis. We define an interval u by

$$u = [\underline{u}, \overline{u}] = \{t \in \mathbb{R} |\ \underline{u} \leqslant t \leqslant \overline{u}\}.$$

We write $len(u) = \overline{u} - \underline{u}$. If $len(u) = 0$, then u is called a degenerate interval. The set of all intervals of $\mathbb{R}$ is denoted by $\mathbb{R}_{\mathscr{I}}$. For $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}_{\mathscr{I}}$, λu is defined by

$$\lambda [\underline{u}, \overline{u}] = \begin{cases} [\lambda \underline{u}, \lambda \overline{u}] & \text{if } \lambda \geqslant 0, \\ [\lambda \overline{u}, \lambda \underline{u}] & \text{if } \lambda < 0. \end{cases}$$

For $u = [\underline{u}, \overline{u}]$ and $v = [\underline{v}, \overline{v}]$, the four arithmetic operators (+,-,$\cdot$,/) are defined by

$$u + v = [\underline{u} + \underline{v}, \overline{u} + \overline{v}],$$

$$u - v = [\underline{u} - \overline{v}, \overline{u} - \underline{v}],$$

$$u \cdot v = \left[\min\{\underline{u}\underline{v}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{u}\overline{v}\}, \max\{\underline{u}\underline{v}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{u}\overline{v}\}\right],$$

$$u/v = \left[\min\{\underline{u}/\underline{v}, \underline{u}/\overline{v}, \overline{u}/\underline{v}, \overline{u}/\overline{v}\}, \max\{\underline{u}/\underline{v}, \underline{u}/\overline{v}, \overline{u}/\underline{v}, \overline{u}/\overline{v}\}\right], \text{where } \ 0 \notin [\underline{v}, \overline{v}].$$

Note that $\mathbb{R}_{\mathscr{I}}$ with the above operations (i.e., the Minkowski addition and the scalar multiplication) is *not* a linear space since an interval does not have inverse element and, therefore, the subtraction does not have adequate properties. For example, when subtracting two intervals u and v, the width of the result is the sum of the widths of u and v, i.e.,

$$len(u - v) = len(u) + len(v).$$

To partially overcome this situation, Hukuhara [25] introduced the following H-difference:

$$u \ominus v = w \Leftrightarrow u = v + w.$$

Unfortunately, the H-difference does not always exist for any u and v.

In [43], Stefanini introduced the gH-difference as follows:

$$u \ominus_g v = w \Leftrightarrow \begin{cases} (a)\ u = v + w, \\ or\ (b)\ v = u + (-1)w. \end{cases}$$

The gH-difference always exists for any u and v. We also have

$$u \ominus_g v = \left[\min\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\}, \max\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\}\right].$$

The Hausdorff distance between u and v is defined by

$$d(u,v) = \max\left\{|\underline{u} - \underline{v}|, |\overline{u} - \overline{v}|\right\}.$$

Then, $(\mathbb{R}_{\mathscr{I}}, d)$ is a complete metric space. Note that $(\mathbb{R}_{\mathscr{I}}, +, \cdot)$ is a quasi-linear space (see [15]) equipped with the quasi-norm $\|\cdot\|$, which is given by

$$\|u\| = d(u,[0,0]) = d([\underline{u}, \overline{u}],[0,0]) = \max\{|\underline{u}|, |\overline{u}|\}$$

for all $u \in \mathbb{R}_{\mathscr{I}}$.

On $[a,b]$, u_i is called increasing if and only if $\underline{u_i}$ and $\overline{u_i}$ are increasing; u_i and v_i are synchronous (asynchronous) monotone if they have the same (opposite) monotonicity; u_i is μ-increasing if $len(u_i)$ is increasing. One defines u^λ by

$$u^\lambda = \{t^\lambda \,|\, t \in [\underline{u}, \overline{u}]\}.$$

For convenience, we now recall the classical Opial's inequality:

THEOREM 1. (continuous Opial inequality [33]) *Let $F \in C^1[0,h]$, $F(0) = F(h) = 0$ and $F(t) > 0$ for $t \in (0,h)$. Then,*

$$\int_0^h |F(t)F'(t)|dt \leqslant \frac{h}{4}\int_0^h \left(F'(t)\right)^2 dt, \tag{1}$$

where $\frac{h}{4}$ is the best possible.

A discrete analogue of Theorem 1 is the following:

THEOREM 2. (discrete Opial inequality [3]) *Let $\{u_i\}_{i=0}^n$ be a sequence of numbers with $u_0 = 0$ and $u_n = 0$. Then,*

$$\sum_{i=1}^{n-1} |u_i \Delta u_i| \leqslant \frac{1}{2}\left[\frac{n+1}{2}\right]\sum_{i=0}^{n-1} |\Delta u_i|^2, \tag{2}$$

where Δ is the forward difference operator and $[\cdot]$ is the greatest integer function.

Many generalizations of Theorem 2 are available in the literature: see, e.g., [5, 7, 23]. In Sections 3 and 4, we give several extensions of Theorem 2 for sequences of intervals.

3. Opial type inequalities involving the backward/nabla gH-difference operator

DEFINITION 1. Let $\{u_i\}$ be a sequence of intervals. We define the forward (delta) gH-difference operator Δu by

$$\Delta u_i = u_{i+1} \ominus_g u_i.$$

Similarly, we define the backward (nabla) gH-difference operator ∇u by

$$\nabla u_i = u_i \ominus_g u_{i-1}.$$

REMARK 1. Note that if $\{u_i\}$ is a sequence of degenerate intervals, then the forward (backward) gH-difference operator reduces to the classical forward (backward) difference operator.

Lemma 1 has been obtained by Lee in [27]. Here we give a new and more direct proof.

LEMMA 1. (cf. [27]) *Let $\{u_i\}_{i=1}^n$ be a non-decreasing sequence of non-negative real numbers, $u_0 = 0$, and $\lambda_1,\ \lambda_2 \geqslant 1$. Then,*

$$\sum_{i=1}^{n} u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} \leqslant \frac{\lambda_2 (n+1)^{\lambda_1}}{\lambda_1+\lambda_2} \sum_{i=1}^{n} (\nabla u_i)^{\lambda_1+\lambda_2}. \tag{3}$$

Proof. Since $\nabla u_i = u_i - u_{i-1}$, we have $u_i = \sum_{j=1}^{i} \nabla u_j$. We may rewrite (3) as

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{i} \nabla u_j \right)^{\lambda_1} (\nabla u_i)^{\lambda_2} \leqslant \frac{\lambda_2 (n+1)^{\lambda_1}}{\lambda_1+\lambda_2} \sum_{i=1}^{n} (\nabla u_i)^{\lambda_1+\lambda_2}. \tag{4}$$

We shall prove (4) by induction on n. Clearly, (4) holds with $n = 1$. Assume that it holds for n, so that

$$\begin{aligned} &\sum_{i=1}^{n+1} \left(\sum_{j=1}^{i} \nabla u_j \right)^{\lambda_1} (\nabla u_i)^{\lambda_2} \\ &\quad \leqslant \frac{\lambda_2 (n+1)^{\lambda_1}}{\lambda_1+\lambda_2} \left(\sum_{i=1}^{n} (\nabla u_i)^{\lambda_1+\lambda_2} + \frac{\lambda_1+\lambda_2}{\lambda_2} \left(\frac{1}{n+1} \sum_{j=1}^{n+1} \nabla u_j \right)^{\lambda_1} (\nabla u_{n+1})^{\lambda_2} \right) \qquad (5) \\ &\quad \leqslant \frac{\lambda_2 (n+1)^{\lambda_1}}{\lambda_1+\lambda_2} \left(\sum_{i=1}^{n} (\nabla u_i)^{\lambda_1+\lambda_2} + \frac{\lambda_1+\lambda_2}{\lambda_2} A_{n+1}^{\lambda_1} (\nabla u_{n+1})^{\lambda_2} \right), \end{aligned}$$

where $A_{n+1} = \frac{1}{n+1} \sum_{j=1}^{n+1} \nabla u_j$. Using Young's inequality, we have

$$A_{n+1}^{\lambda_1} (\nabla u_{n+1})^{\lambda_2} \leqslant \frac{\lambda_1}{\lambda_1+\lambda_2} A_{n+1}^{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2} (\nabla u_{n+1})^{\lambda_1+\lambda_2}.$$

Then,

$$\frac{\lambda_1+\lambda_2}{\lambda_2}A_{n+1}^{\lambda_1}\left(\nabla u_{n+1}\right)^{\lambda_2} \leqslant \frac{\lambda_1}{\lambda_2}A_{n+1}^{\lambda_1+\lambda_2}+\left(\nabla u_{n+1}\right)^{\lambda_1+\lambda_2}. \tag{6}$$

Thanks to Hölder's inequality, it follows that

$$\begin{aligned}
A_{n+1} &= \frac{1}{n+1}\sum_{j=1}^{n+1}\nabla u_j \\
&\leqslant \left(\sum_{j=1}^{n+1}\left(\frac{1}{n+1}\right)^{\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-1}}\right)^{\frac{\lambda_1+\lambda_2-1}{\lambda_1+\lambda_2}}\left(\sum_{j=1}^{n+1}\left(\nabla u_i\right)^{\lambda_1+\lambda_2}\right)^{\frac{1}{\lambda_1+\lambda_2}} \\
&\leqslant \left(\frac{1}{n+1}\right)^{\frac{1}{\lambda_1+\lambda_2}}\left(\sum_{j=1}^{n+1}\left(\nabla u_i\right)^{\lambda_1+\lambda_2}\right)^{\frac{1}{\lambda_1+\lambda_2}}.
\end{aligned}$$

Consequently, we get

$$A_{n+1}^{\lambda_1+\lambda_2} \leqslant \frac{1}{n+1}\sum_{j=1}^{n+1}\left(\nabla u_i\right)^{\lambda_1+\lambda_2}. \tag{7}$$

Thus, combining (5), (6) and (7), we have

$$\begin{aligned}
&\sum_{i=1}^{n+1}\left(\sum_{j=1}^{i}\nabla u_j\right)^{\lambda_1}\left(\nabla u_i\right)^{\lambda_2} \\
&\leqslant \frac{\lambda_2(n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\left(\sum_{i=1}^{n}\left(\nabla u_i\right)^{\lambda_1+\lambda_2}+\frac{\lambda_1+\lambda_2}{\lambda_2}A_{n+1}^{\lambda_1}\left(\nabla u_{n+1}\right)^{\lambda_2}\right) \\
&\leqslant \frac{\lambda_2(n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\left(\sum_{i=1}^{n}\left(\nabla u_i\right)^{\lambda_1+\lambda_2}+\frac{\lambda_1}{\lambda_2}A_{n+1}^{\lambda_1+\lambda_2}+\left(\nabla u_{n+1}\right)^{\lambda_1+\lambda_2}\right) \\
&\leqslant \frac{\lambda_2(n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\left(\sum_{i=1}^{n+1}\left(\nabla u_i\right)^{\lambda_1+\lambda_2}+\frac{\lambda_1}{n+1}\sum_{j=1}^{n+1}\left(\nabla u_i\right)^{\lambda_1+\lambda_2}\right) \\
&= \frac{\lambda_2(n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\cdot\frac{n+1+\lambda_1}{n+1}\cdot\sum_{j=1}^{n+1}\left(\nabla u_i\right)^{\lambda_1+\lambda_2} \\
&= \frac{\lambda_2\left[(n+1)^{\lambda_1}+\lambda_1(n+1)^{\lambda_1-1}\right]}{\lambda_1+\lambda_2}\sum_{j=1}^{n+1}\left(\nabla u_i\right)^{\lambda_1+\lambda_2} \\
&\leqslant \frac{\lambda_2(n+2)^{\lambda_1}}{\lambda_1+\lambda_2}\sum_{j=1}^{n+1}\left(\nabla u_i\right)^{\lambda_1+\lambda_2}.
\end{aligned}$$

The proof is complete. □

Thanks to Lemma 1, we can easily obtain the following Lemma 2, LEMMA 2. *Let* $\{u_i\}_{i=1}^n$ *be a sequence of numbers,* $u_0 = 0$*, and* $\lambda_1,\ \lambda_2 \geqslant 1$*. Then,*

$$\sum_{i=1}^{n} |u_i|^{\lambda_1} \left|\nabla u_i\right|^{\lambda_2} \leqslant \frac{\lambda_2 (n+1)^{\lambda_1}}{\lambda_1+\lambda_2} \sum_{i=1}^{n} \left|\nabla u_i\right|^{\lambda_1+\lambda_2}. \tag{8}$$

Proof. Since $|\nabla u_i| = |u_i - u_{i-1}|$, we have $|u_i| \leqslant \sum_{j=1}^{i} |\nabla u_j|$. The rest of proof is similar to that of Lemma 1 and is omitted here. □

We are now ready to formulate and prove our first original result.

THEOREM 3. *Let* $\{u_i\}_{i=1}^n$ *be a sequence of intervals,* $u_0 = [0,0]$*, and* $\lambda_1, \lambda_2 \geqslant 1$*. If* u_i *is monotone and* μ*-increasing, then*

$$\sum_{i=1}^{n} \left\| u_i^{\lambda_1} \left(\nabla u_i\right)^{\lambda_2} \right\| \leqslant \frac{\lambda_2 (n+1)^{\lambda_1}}{\lambda_1+\lambda_2} \sum_{i=1}^{n} \left\|\nabla u_i\right\|^{\lambda_1+\lambda_2}. \tag{9}$$

Proof. Suppose that u_i is increasing and μ-increasing. Then,

$$u_i^{\lambda_1} = \left[\underline{u_i}^{\lambda_1}, \overline{u_i}^{\lambda_1}\right], \quad \left(\nabla u_i\right)^{\lambda_2} = \left[\left(\underline{\nabla u_i}\right)^{\lambda_2}, \left(\overline{\nabla u_i}\right)^{\lambda_2}\right].$$

Consequently, we obtain that

$$u_i^{\lambda_1} \left(\nabla u_i\right)^{\lambda_2} = \left[\underline{u_i}^{\lambda_1} \left(\underline{\nabla u_i}\right)^{\lambda_2}, \overline{u_i}^{\lambda_1} \left(\overline{\nabla u_i}\right)^{\lambda_2}\right].$$

If u_i is decreasing and μ-increasing, then

$$u_i^{\lambda_1} = \begin{cases} \left[\underline{u_i}^{\lambda_1}, \overline{u_i}^{\lambda_1}\right] & \text{if } \lambda_1 \text{ is odd,} \\ \left[\overline{u_i}^{\lambda_1}, \underline{u_i}^{\lambda_1}\right] & \text{if } \lambda_1 \text{ is even,} \end{cases}$$

$$\left(\nabla u_i\right)^{\lambda_2} = \begin{cases} \left[\left(\underline{\nabla u_i}\right)^{\lambda_2}, \left(\overline{\nabla u_i}\right)^{\lambda_2}\right] & \text{if } \lambda_2 \text{ is odd,} \\ \left[\left(\overline{\nabla u_i}\right)^{\lambda_2}, \left(\underline{\nabla u_i}\right)^{\lambda_2}\right] & \text{if } \lambda_2 \text{ is even.} \end{cases}$$

Consequently, we obtain

$$u_i^{\lambda_1} \left(\nabla u_i\right)^{\lambda_2} = \begin{cases} \left[\overline{u_i}^{\lambda_1} \left(\overline{\nabla u_i}\right)^{\lambda_2}, \underline{u_i}^{\lambda_1} \left(\underline{\nabla u_i}\right)^{\lambda_2}\right] & \text{if } \lambda_1 \text{ and } \lambda_2 \text{ are odd,} \\ \left[\overline{u_i}^{\lambda_1} \left(\overline{\nabla u_i}\right)^{\lambda_2}, \underline{u_i}^{\lambda_1} \left(\underline{\nabla u_i}\right)^{\lambda_2}\right] & \text{if } \lambda_1 \text{ and } \lambda_2 \text{ are even,} \\ \left[\underline{u_i}^{\lambda_1} \left(\underline{\nabla u_i}\right)^{\lambda_2}, \overline{u_i}^{\lambda_1} \left(\overline{\nabla u_i}\right)^{\lambda_2}\right] & \text{if } \lambda_1 \text{ is odd and } \lambda_2 \text{ is even,} \\ \left[\underline{u_i}^{\lambda_1} \left(\underline{\nabla u_i}\right)^{\lambda_2}, \overline{u_i}^{\lambda_1} \left(\overline{\nabla u_i}\right)^{\lambda_2}\right] & \text{if } \lambda_1 \text{ is even and } \lambda_2 \text{ is odd.} \end{cases}$$

By Lemma 2, it follows that

$$
\begin{aligned}
&\sum_{i=1}^{n}\left\|u_i^{\lambda_1}\left(\nabla u_i\right)^{\lambda_2}\right\| \\
&=\sum_{i=1}^{n}\left\|\left[\min\left\{\underline{u_i}^{\lambda_1}\left(\underline{\nabla u_i}\right)^{\lambda_2},\overline{u_i}^{\lambda_1}\left(\overline{\nabla u_i}\right)^{\lambda_2}\right\},\max\left\{\underline{u_i}^{\lambda_1}\left(\underline{\nabla u_i}\right)^{\lambda_2},\overline{u_i}^{\lambda_1}\left(\overline{\nabla u_i}\right)^{\lambda_2}\right\}\right]\right\| \\
&=\sum_{i=1}^{n}\max\left\{\left|\underline{u_i}^{\lambda_1}\left(\underline{\nabla u_i}\right)^{\lambda_2}\right|,\left|\overline{u_i}^{\lambda_1}\left(\overline{\nabla u_i}\right)^{\lambda_2}\right|\right\} \\
&=\max\left\{\sum_{i=1}^{n}\left|\underline{u_i}^{\lambda_1}\left(\underline{\nabla u_i}\right)^{\lambda_2}\right|,\sum_{i=1}^{n}\left|\overline{u_i}^{\lambda_1}\left(\overline{\nabla u_i}\right)^{\lambda_2}\right|\right\} \\
&\leqslant\frac{\lambda_2(n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\max\left\{\sum_{i=1}^{n}\left|\underline{\nabla u_i}^{\lambda_1+\lambda_2}\right|,\sum_{i=1}^{n}\left|\overline{\nabla u_i}^{\lambda_1+\lambda_2}\right|\right\} \\
&\leqslant\frac{\lambda_2(n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\sum_{i=1}^{n}\max\left\{\left|\underline{\nabla u_i}\right|^{\lambda_1+\lambda_2},\left|\overline{\nabla u_i}\right|^{\lambda_1+\lambda_2}\right\} \\
&\leqslant\frac{\lambda_2(n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\sum_{i=1}^{n}\|\nabla u_i\|^{\lambda_1+\lambda_2}.
\end{aligned}
$$

This concludes the proof. □

Follows an example of application of our Theorem 3.

EXAMPLE 1. Suppose that $\{u_i\}_{i=0}^{n}=\{[0,0],[1,2],[2,4],\ldots,[n,2n]\}$ and $\lambda_1,\ \lambda_2\geqslant 1$. By Theorem 3, we have

$$
\begin{aligned}
\sum_{i=1}^{n}\left\|u_i^{\lambda_1}\left(\nabla u_i\right)^{\lambda_2}\right\| &=\sum_{i=1}^{n}\left\|\left[i^{\lambda_1},(2i)^{\lambda_1}\right]\cdot[1,2]^{\lambda_2}\right\| \\
&=2^{\lambda_1+\lambda_2}\sum_{i=1}^{n}i^{\lambda_1} \\
&\leqslant\frac{\lambda_2 n(n+1)^{\lambda_1}}{\lambda_1+\lambda_2}2^{\lambda_1+\lambda_2} \\
&=\frac{\lambda_2(n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\sum_{i=1}^{n}2^{\lambda_1+\lambda_2} \\
&=\frac{\lambda_2(n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\sum_{i=1}^{n}\left\|\nabla u_i\right\|^{\lambda_1+\lambda_2}.
\end{aligned}
$$

LEMMA 3. *Let $\{u_i\}_{i=1}^{m}$ be a sequence of numbers, $u_m=0$, and $\lambda_1,\ \lambda_2\geqslant 1$. Then,*

$$
\sum_{i=n}^{m-1}|u_i|^{\lambda_1}\left|\nabla u_i\right|^{\lambda_2}\leqslant\frac{\lambda_2(m-n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\sum_{i=n}^{m}\left|\nabla u_i\right|^{\lambda_1+\lambda_2}. \tag{10}
$$

Proof. Since $|\nabla u_i| = |u_i - u_{i-1}|$, we have $|u_i| \leqslant \sum_{j=i+1}^{m} |\nabla u_j|$. The rest of proof is similar to that of Lemma 1 and is omitted here. □

Similarly to Theorem 3, we obtain an analogous result when u_i is monotone but μ-decreasing instead of μ-increasing.

THEOREM 4. *Let $\{u_i\}_{i=0}^{m}$ be a sequence of intervals, $u_m = [0,0]$, and $\lambda_1, \lambda_2 \geqslant 1$. If u_i is monotone and μ-decreasing, then*

$$\sum_{i=n}^{m-1} \left\| u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} \right\| \leqslant \frac{\lambda_2 (m-n+1)^{\lambda_1}}{\lambda_1 + \lambda_2} \sum_{i=n}^{m} \|\nabla u_i\|^{\lambda_1+\lambda_2}. \tag{11}$$

Proof. Suppose that u_i is increasing and μ-decreasing. Then,

$$u_i^{\lambda_1} = \begin{cases} \left[\underline{u_i}^{\lambda_1}, \overline{u_i}^{\lambda_1}\right] & \text{if } \lambda_1 \text{ is odd}, \\ \left[\overline{u_i}^{\lambda_1}, \underline{u_i}^{\lambda_1}\right] & \text{if } \lambda_1 \text{ is even}, \end{cases}$$

and

$$(\nabla u_i)^{\lambda_2} = \left[\left(\overline{\nabla u_i}\right)^{\lambda_2}, \left(\underline{\nabla u_i}\right)^{\lambda_2} \right].$$

Consequently, we obtain

$$u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} = \begin{cases} \left[\underline{u_i}^{\lambda_1} \left(\underline{\nabla u_i}\right)^{\lambda_2}, \overline{u_i}^{\lambda_1} \left(\overline{\nabla u_i}\right)^{\lambda_2}\right] & \text{if } \lambda_1 \text{ is odd}, \\ \left[\overline{u_i}^{\lambda_1} \left(\overline{\nabla u_i}\right)^{\lambda_2}, \underline{u_i}^{\lambda_1} \left(\underline{\nabla u_i}\right)^{\lambda_2}\right] & \text{if } \lambda_1 \text{ is even}. \end{cases} \tag{12}$$

If u_i is decreasing and μ-decreasing, then

$$u_i^{\lambda_1} = \left[\underline{u_i}^{\lambda_1}, \overline{u_i}^{\lambda_1}\right],$$

$$(\nabla u_i)^{\lambda_2} = \begin{cases} \left[\left(\overline{\nabla u_i}\right)^{\lambda_2}, \left(\underline{\nabla u_i}\right)^{\lambda_2} \right] & \text{if } \lambda_2 \text{ is odd}, \\ \left[\left(\underline{\nabla u_i}\right)^{\lambda_2}, \left(\overline{\nabla u_i}\right)^{\lambda_2} \right] & \text{if } \lambda_2 \text{ is even}. \end{cases}$$

Consequently, we obtain

$$u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} = \begin{cases} \left[\overline{u_i}^{\lambda_1} \left(\overline{\nabla u_i}\right)^{\lambda_2}, \underline{u_i}^{\lambda_1} \left(\underline{\nabla u_i}\right)^{\lambda_2}\right] & \text{if } \lambda_2 \text{ is odd}, \\ \left[\underline{u_i}^{\lambda_1} \left(\underline{\nabla u_i}\right)^{\lambda_2}, \overline{u_i}^{\lambda_1} \left(\overline{\nabla u_i}\right)^{\lambda_2}\right] & \text{if } \lambda_2 \text{ is even}. \end{cases} \tag{13}$$

By (12), (13) and Lemma 3, it follows that

$$
\begin{aligned}
&\sum_{i=n}^{m-1}\left\|u_i^{\lambda_1}\left(\nabla u_i\right)^{\lambda_2}\right\| \\
&=\sum_{i=n}^{m-1}\left\|\left[\min\left\{\underline{u_i}^{\lambda_1}\left(\underline{\nabla u_i}\right)^{\lambda_2},\overline{u_i}^{\lambda_1}\left(\overline{\nabla u_i}\right)^{\lambda_2}\right\},\max\left\{\underline{u_i}^{\lambda_1}\left(\underline{\nabla u_i}\right)^{\lambda_2},\overline{u_i}^{\lambda_1}\left(\overline{\nabla u_i}\right)^{\lambda_2}\right\}\right]\right\| \\
&=\sum_{i=n}^{m-1}\max\left\{\left|\underline{u_i}^{\lambda_1}\left(\underline{\nabla u_i}\right)^{\lambda_2}\right|,\left|\overline{u_i}^{\lambda_1}\left(\overline{\nabla u_i}\right)^{\lambda_2}\right|\right\} \\
&=\max\left\{\sum_{i=n}^{m-1}\left|\underline{u_i}^{\lambda_1}\left(\underline{\nabla u_i}\right)^{\lambda_2}\right|,\sum_{i=n}^{m-1}\left|\overline{u_i}^{\lambda_1}\left(\overline{\nabla u_i}\right)^{\lambda_2}\right|\right\} \\
&\leqslant\frac{\lambda_2(m-n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\max\left\{\sum_{i=n}^{m}\left|\underline{\nabla u_i}\right|^{\lambda_1+\lambda_2},\sum_{i=n}^{m}\left|\overline{\nabla u_i}\right|^{\lambda_1+\lambda_2}\right\} \\
&\leqslant\frac{\lambda_2(m-n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\sum_{i=n}^{m}\max\left\{\left|\underline{\nabla u_i}\right|^{\lambda_1+\lambda_2},\left|\overline{\nabla u_i}\right|^{\lambda_1+\lambda_2}\right\} \\
&\leqslant\frac{\lambda_2(m-n+1)^{\lambda_1}}{\lambda_1+\lambda_2}\sum_{i=n}^{m}\|\nabla u_i\|^{\lambda_1+\lambda_2}.
\end{aligned}
$$

This concludes the proof. □

EXAMPLE 2. Suppose that

$$
\{u_i\}_{i=1}^{n}=\left\{[1,2],\left[\frac{1}{2},1\right],\left[\frac{1}{i},\frac{2}{i}\right],\ldots,\left[\frac{1}{n-1},\frac{2}{n-1}\right],[0,0]\right\}
$$

and $\lambda_1=1$ and $\lambda_2=2$. By induction on n, we have

$$
\begin{aligned}
\sum_{i=2}^{n-1}\left\|u_i^{\lambda_1}\left(\nabla u_i\right)^{\lambda_2}\right\| &=\sum_{i=2}^{n-1}\left\|\left[\frac{1}{i},\frac{2}{i}\right]\cdot\left[\frac{-2}{i(i-1)},\frac{-1}{i(i-1)}\right]^2\right\| \\
&=\sum_{i=2}^{n-1}\left\|\left[\frac{1}{i},\frac{2}{i}\right]\cdot\left[\frac{1}{i^2(i-1)^2},\frac{4}{i^2(i-1)^2}\right]\right\| \\
&=\sum_{i=2}^{n-1}\frac{8}{i^3(i-1)^2} \\
&\leqslant\frac{2(n-1)}{3}\sum_{i=2}^{n}\frac{2^3}{i^3(i-1)^3} \\
&=\frac{\lambda_2(n-1)^{\lambda_1}}{\lambda_1+\lambda_2}\sum_{i=2}^{n}\|\nabla u_i\|^{\lambda_1+\lambda_2}.
\end{aligned}
$$

Theorem 3 is a special case of our next Theorem 5.

THEOREM 5. *Let* $\{u_i\}_{i=1}^n$ *be a sequence of intervals,* $u_0 = [0,0]$*, and* $\lambda_1, \lambda_2 \geqslant 1$*. If* $\{u_i\}_{i=1}^n$ *is piecewise alternate monotone, piecewise alternate* μ*-monotone, and there is no other point* i *such that* $u_i = [0,0]$*, then*

$$\sum_{i=1}^{n} \left\| u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} \right\| \leqslant \frac{\lambda_2 (n+1)^{\lambda_1}}{\lambda_1 + \lambda_2} \sum_{i=1}^{n} \|\nabla u_i\|^{\lambda_1+\lambda_2}. \tag{14}$$

Proof. First, suppose that there exists a finite number of points such that

$$1 = i_0 \leqslant i_1 < i_2 < \cdots < i_{k-1} < i_k = n$$

and u_i is piecewise alternate monotone and piecewise alternate μ-monotone. By Lemma 1, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \left\| u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} \right\| \\
&= \sum_{i=i_0}^{i_1} \left\| u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} \right\| + \sum_{i=i_1+1}^{i_2} \left\| u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} \right\| + \cdots + \sum_{i=i_{k-1}+1}^{i_k} \left\| u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} \right\| \\
&= \sum_{j=0}^{k-1} \sum_{i=i_j}^{i_{j+1}} \max\left\{ \left| \underline{u_i}^{\lambda_1} (\underline{\nabla u_i})^{\lambda_2} \right|, \left| \overline{u_i}^{\lambda_1} (\overline{\nabla u_i})^{\lambda_2} \right| \right\} \\
&= \sum_{i=1}^{n} \max\left\{ \left| \underline{u_i}^{\lambda_1} (\underline{\nabla u_i})^{\lambda_2} \right|, \left| \overline{u_i}^{\lambda_1} (\overline{\nabla u_i})^{\lambda_2} \right| \right\} \\
&\leqslant \frac{\lambda_2 (n+1)^{\lambda_1}}{\lambda_1 + \lambda_2} \sum_{i=1}^{n} \|\nabla u_i\|^{\lambda_1+\lambda_2}.
\end{aligned}$$

The proof is complete. □

Similarly, we can also generalize Theorem 4 as follows.

THEOREM 6. *Let* $\{u_i\}_{i=n}^m$ *be a sequence of intervals,* $u_m = [0,0]$*, and* $\lambda_1, \lambda_2 \geqslant 1$*. If* $\{u_i\}_{i=0}^m$ *is piecewise alternate monotone, piecewise alternate* μ*-monotone, and there is no other point* i *such that* $u_i = [0,0]$*, then*

$$\sum_{i=n}^{m-1} \left\| u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} \right\| \leqslant \frac{\lambda_2 (m-n+1)^{\lambda_1}}{\lambda_1 + \lambda_2} \sum_{i=n}^{m} \|\nabla u_i\|^{\lambda_1+\lambda_2}. \tag{15}$$

Proof. The proof is analogous to the one of Theorem 5. □

As an application of Theorems 5 and 6, we now obtain the following result.

THEOREM 7. *Let* $\{u_i\}_{i=0}^m$ *be a sequence of intervals,* $u_0 = u_m = [0,0]$*, and* $\lambda_1, \lambda_2 \geqslant 1$*. If* $\{u_i\}_{i=0}^m$ *is piecewise alternate monotone, piecewise alternate* μ*-monotone, and there is no other point* i *such that* $u_i = [0,0]$*, then*

$$\sum_{i=1}^{m-1} \left\| u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} \right\| \leqslant \frac{\lambda_2 \left(\left[\frac{m}{2}\right] + 1 \right)^{\lambda_1}}{\lambda_1 + \lambda_2} \sum_{i=1}^{m} \|\nabla u_i\|^{\lambda_1+\lambda_2}. \tag{16}$$

Proof. Let us take $n = \left[\frac{m}{2}\right]$. By Theorem 5, we have

$$\sum_{i=1}^{\left[\frac{m}{2}\right]} \left\| u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} \right\| \leqslant \frac{\lambda_2 \left(\left[\frac{m}{2}\right] + 1 \right)^{\lambda_1}}{\lambda_1 + \lambda_2} \sum_{i=1}^{\left[\frac{m}{2}\right]} \left\| \nabla u_i \right\|^{\lambda_1 + \lambda_2}. \tag{17}$$

Similarly, by Theorem 6, we have

$$\begin{aligned} \sum_{i=\left[\frac{m}{2}\right]+1}^{m-1} \left\| u_i^{\lambda_1} (\nabla u_i)^{\lambda_2} \right\| &\leqslant \frac{\lambda_2 \left(m - \left[\frac{m}{2}\right] \right)^{\lambda_1}}{\lambda_1 + \lambda_2} \sum_{i=\left[\frac{m}{2}\right]+1}^{m} \left\| \nabla u_i \right\|^{\lambda_1 + \lambda_2} \\ &\leqslant \frac{\lambda_2 \left(\left[\frac{m}{2}\right] + 1 \right)^{\lambda_1}}{\lambda_1 + \lambda_2} \sum_{i=\left[\frac{m}{2}\right]+1}^{m} \left\| \nabla u_i \right\|^{\lambda_1 + \lambda_2}. \end{aligned} \tag{18}$$

The intended relation (16) follows by adding the above two inequalities (17) and (18). □

EXAMPLE 3. Suppose that

$$\{u_i\}_{i=0}^{5} = \Big\{ [0,0], [1,2], [2,4], [3,6], [1,2], [0,0] \Big\}, \lambda_1 = 2, \lambda_2 = 3.$$

Then, we have

$$\sum_{i=1}^{4} \left\| u_i^2 (\nabla u_i)^3 \right\| = 704 < 6048 = \frac{3 \cdot 3^2}{5} \sum_{i=1}^{4} \left\| \nabla u_i \right\|^5.$$

Let $\lambda_1 = 1, \lambda_2 = 2$. Then, we have

$$\sum_{i=1}^{4} \left\| u_i (\nabla u_i)^2 \right\| = 80 < 184 = \frac{2 \cdot 3}{3} \sum_{i=1}^{4} \left\| \nabla u_i \right\|^3.$$

Now, we give new discrete Opial inequalities involving two interval sequences.

THEOREM 8. *Let $\{u_i\}_{i=0}^{n}$ and $\{v_i\}_{i=0}^{n}$ be two sequences of intervals, $u_0 = v_0 = [0,0]$. If u_i and v_i are synchronous monotone and μ-increasing, then*

$$\sum_{i=1}^{n} \left\| u_{i-1} \nabla v_i + v_i \nabla u_i \right\| \leqslant \frac{n}{2} \sum_{i=1}^{n} \left\| (\nabla u_i)^2 + (\nabla v_i)^2 \right\|. \tag{19}$$

Proof. Suppose that u_i and v_i are increasing and μ-increasing. Then, $u_i v_i$ is also increasing and μ-increasing. Consequently, we obtain that

$$\begin{aligned} u_{i-1} \nabla v_i + v_i \nabla u_i &= \left[\underline{u_{i-1}}, \overline{u_{i-1}} \right] \cdot \left[\underline{v_i} - \underline{v_{i-1}}, \overline{v_i} - \overline{v_{i-1}} \right] + \left[\underline{v_i}, \overline{v_i} \right] \cdot \left[\underline{u_i} - \underline{u_{i-1}}, \overline{u_i} - \overline{u_{i-1}} \right] \\ &= \left[\underline{u_{i-1}} (\underline{v_i} - \underline{v_{i-1}}) + \underline{v_i} (\underline{u_i} - \underline{u_{i-1}}),\ \overline{u_{i-1}} (\overline{v_i} - \overline{v_{i-1}}) + \overline{v_i} (\overline{u_i} - \overline{u_{i-1}}) \right] \\ &= \left[\underline{u_i} \cdot \underline{v_i} - \underline{u_{i-1}} \cdot \underline{v_{i-1}},\ \overline{u_i} \cdot \overline{v_i} - \overline{u_{i-1}} \cdot \overline{v_{i-1}} \right] \\ &= \nabla (u_i v_i). \end{aligned}$$

If u_i and v_i are decreasing and μ-increasing, then u_iv_i is increasing and μ-increasing. Consequently, we obtain that

$$\begin{aligned}
u_{i-1}\nabla v_i + v_i\nabla u_i &= \left[\underline{u_{i-1}}, \overline{u_{i-1}}\right]\cdot\left[\underline{v_i}-\underline{v_{i-1}}, \overline{v_i}-\overline{v_{i-1}}\right] + \left[\underline{v_i}, \overline{v_i}\right]\cdot\left[\underline{u_i}-\underline{u_{i-1}}, \overline{u_i}-\overline{u_{i-1}}\right]\\
&= \left[\overline{u_{i-1}}(\overline{v_i}-\overline{v_{i-1}}) + \overline{v_i}(\overline{u_i}-\overline{u_{i-1}}),\ \underline{u_{i-1}}(\underline{v_i}-\underline{v_{i-1}}) + \underline{v_i}(\underline{u_i}-\underline{u_{i-1}})\right]\\
&= \left[\overline{u_i}\cdot\overline{v_i} - \overline{u_{i-1}}\cdot\overline{v_{i-1}},\ \underline{u_i}\cdot\underline{v_i} - \underline{u_{i-1}}\cdot\underline{v_{i-1}}\right]\\
&= \nabla(u_iv_i).
\end{aligned}$$

Then, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\sum_{i=1}^{n}\left\|u_{i-1}\nabla v_i + v_i\nabla u_i\right\| &= \sum_{i=1}^{n}\max\left\{\underline{u_i}\cdot\underline{v_i} - \underline{u_{i-1}}\cdot\underline{v_{i-1}},\ \overline{u_i}\cdot\overline{v_i} - \overline{u_{i-1}}\cdot\overline{v_{i-1}}\right\}\\
&\leqslant \max\left\{\sum_{i=1}^{n}\left(\underline{u_i}\cdot\underline{v_i} - \underline{u_{i-1}}\cdot\underline{v_{i-1}}\right),\ \sum_{i=1}^{n}\left(\overline{u_i}\cdot\overline{v_i} - \overline{u_{i-1}}\cdot\overline{v_{i-1}}\right)\right\}\\
&\leqslant \max\left\{\underline{u_n}\cdot\underline{v_n},\ \overline{u_n}\cdot\overline{v_n}\right\}\\
&= \|u_nv_n\|\\
&= \left\|\sum_{i=1}^{n}\nabla u_i\right\|\cdot\left\|\sum_{i=1}^{n}\nabla v_i\right\|\\
&\leqslant \frac{n}{2}\sum_{i=1}^{n}\left(\|\nabla u_i\|^2 + \|\nabla v_i\|^2\right).
\end{aligned}$$

This concludes the proof. □

The following results are proved similarly to Theorem 8.

THEOREM 9. *Let $\{u_i\}_{i=0}^m$ and $\{v_i\}_{i=0}^m$ be two sequences of intervals, $u_m = v_m = [0,0]$. If u_i and v_i are synchronous monotone and μ-decreasing, then*

$$\sum_{i=n+1}^{m}\left\|u_{i-1}\nabla v_i + v_i\nabla u_i\right\| \leqslant \frac{m-n}{2}\sum_{i=n+1}^{m}\left\|(\nabla u_i)^2 + (\nabla v_i)^2\right\|.$$

THEOREM 10. *Let $\{u_i\}_{i=0}^m$ and $\{v_i\}_{i=0}^m$ be two sequences of intervals, $u_0 = v_0 = [0,0]$. If $\{u_i\}_{i=0}^m$ is piecewise alternate monotone, piecewise alternate μ-monotone, and there is no other point i such that $u_i = [0,0]$ and $v_i = [0,0]$, then*

$$\sum_{i=1}^{n}\left\|u_{i-1}\nabla v_i + v_i\nabla u_i\right\| \leqslant \frac{n}{2}\sum_{i=1}^{n}\left\|(\nabla u_i)^2 + (\nabla v_i)^2\right\|.$$

THEOREM 11. *Let $\{u_i\}_{i=0}^m$ and $\{v_i\}_{i=0}^m$ be two sequences of intervals, $u_m = v_m = [0,0]$. If $\{u_i\}_{i=0}^m$ is piecewise alternate monotone, piecewise alternate μ-monotone, and there is no other point i such that $u_i = [0,0]$ and $v_i = [0,0]$, then*

$$\sum_{i=n+1}^{m}\left\|u_{i-1}\nabla v_i + v_i\nabla u_i\right\| \leqslant \frac{m-n}{2}\sum_{i=n+1}^{m}\left\|(\nabla u_i)^2 + (\nabla v_i)^2\right\|.$$

THEOREM 12. *Let $\{u_i\}_{i=0}^m$ and $\{v_i\}_{i=0}^m$ be two sequences of intervals, $u_1 = v_1 = [0,0]$, and $u_m = v_m = [0,0]$. If $\{u_i\}_{i=0}^m$ is piecewise alternate monotone, piecewise alternate μ-monotone, and there is no other point i such that $u_i = [0,0]$ and $v_i = [0,0]$, then*

$$\sum_{i=1}^{m} \left\| u_{i-1}\nabla v_i + v_i \nabla u_i \right\| \leqslant \frac{\left[\frac{m+1}{2}\right]}{2} \sum_{i=1}^{m} \left\| (\nabla u_i)^2 + (\nabla v_i)^2 \right\|.$$

4. Opial type inequalities involving the forward/delta gH-difference operator

In Section 3, we obtained several Opial type inequalities involving the backward gH-difference operator. Similar arguments can be used to establish discrete Opial type inequalities concerning the forward gH-difference operator. The proofs of the results formulated here are left to the interested reader.

THEOREM 13. (delta version of Theorem 3) *Let $\{u_i\}_{i=1}^n$ be a sequence of intervals, $u_0 = [0,0]$, and $\lambda_1, \lambda_2 \geqslant 1$. If u_i is monotone and μ-increasing, then*

$$\sum_{i=0}^{n-1} \left\| u_i^{\lambda_1} (\Delta u_i)^{\lambda_2} \right\| \leqslant \frac{\lambda_2 (n+1)^{\lambda_1}}{\lambda_1 + \lambda_2} \sum_{i=0}^{n-1} \left\| \Delta u_i \right\|^{\lambda_1 + \lambda_2}.$$

THEOREM 14. (delta version of Theorem 4) *Let $\{u_i\}_{i=0}^m$ be a sequence of intervals, $u_m = [0,0]$, and $\lambda_1, \lambda_2 \geqslant 1$. If u_i is monotone and μ-decreasing, then*

$$\sum_{i=n}^{m-1} \left\| u_i^{\lambda_1} (\Delta u_i)^{\lambda_2} \right\| \leqslant \frac{\lambda_2 (m-n+1)^{\lambda_1}}{\lambda_1 + \lambda_2} \sum_{i=n}^{m} \left\| \Delta u_i \right\|^{\lambda_1 + \lambda_2}.$$

THEOREM 15. (delta version of Theorem 7) *Let $\{u_i\}_{i=0}^m$ be a sequence of intervals, $u_0 = u_m = [0,0]$, and $\lambda_1,\ \lambda_2 \geqslant 1$. If $\{u_i\}_{i=0}^m$ is piecewise alternate monotone, piecewise alternate μ-monotone, and there is no other point i such that $u_i = [0,0]$, then*

$$\sum_{i=1}^{m-1} \left\| u_i^{\lambda_1} (\Delta u_i)^{\lambda_2} \right\| \leqslant \frac{\lambda_2 \left(\left[\frac{m}{2}\right] + 1 \right)^{\lambda_1}}{\lambda_1 + \lambda_2} \sum_{i=1}^{m} \left\| \Delta u_i \right\|^{\lambda_1 + \lambda_2}.$$

5. Conclusions

We investigated discrete Opial type inequalities for interval-valued functions, and obtained several new interval discrete Opial type inequalities. Our results generalize many known discrete Opial type inequalities, and will be useful in developing the theory of interval difference inequalities and interval difference equations. As future research directions, we intend to investigate interval discrete Opial type inequalities on time scales, and give some applications to interval difference equations.

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