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CONTINUOUS SYMMETRIZATION AND CONTINUOUS INCREASING REFINEMENTS OF INEQUALITIES AND MONOTONICITY OF EIGENVALUES

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Abstract. *Continuous symmetrization* process and *continuous increasing* process are the tools used in this paper to refine Clausing inequality and Slater-Pečarić inequality. Also, we note on the monotonicity of the first eigenvalue of a Sturm-Liouville system.

1. Introduction

Continuous symmetrization process and *continuous increasing* process are the tools used in this paper to refine Clausing inequality and Slater-Pečarić inequality. Also, we note on the monotonicity of the first eigenvalue of a Sturm-Liouville system.

Clausing inequality says:

THEOREM 1. [5, Section 4.1(b)] *Let* ϕ *be continuous on* $[0,1]$ *and increasing on* $\left[0,\frac{1}{2}\right]$, *with* $\phi(x)=\phi(1-x)$. *Then, for a concave and positive function* f *on* $[0,1]$ *we have:*

$$\int_0^1 f(x)\,dx\int_0^1 \phi(x)\,dx \leqslant \int_0^1 f(x)\,\phi(x)\,dx \leqslant \int_0^1 f(x)\,dx\int_0^1 k(x)\,dx, \tag{1}$$

where $k(x)=4\min\{x,1-x\}\,\phi(x)$.

Lately this theorem has been proved in details by P. R. Mercer [8].

In Section 2 we refine this inequality.

Continuous symmetrization process, presented by Pólya and Szegö in their book [10, pages 200, 201, formula (1)], is applied in [1] to obtain a set of equimeasurable real functions $f(\alpha,x)$, where $\alpha\in[0,1]$. In [1, Introduction] and [3] the process is applied to functions that include convex functions. In Definition 1 and in Section 2 we make the needed adaptation for functions that include concave functions.

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DEFINITION 1. Let f be a continuous real function on $x \in [-1,1]$, non-decreasing on $[-1,l]$ and non-increasing on $[l,1]$. For $x \in [-1,l]$ we denote the function inverse to f by x_1 and for $x \in [l,1]$ by x_2.

In order to be able to use the process named continuous symmetrization (see [10, pages 200,201]) to build the set of functions $f(\alpha,x)$, we complete the graph of f as follows:

(A): when $f(-1) > f(1)$, we add the inverse function $x_1(y)$ defined on $y \in [f(-1), f(l)]$ an interval of definition $f(1) \leqslant y \leqslant f(-1)$, for which $x_1(y) = -1$ is a constant function,

and:

(B): when $f(-1) < f(1)$ we add the inverse function $x_2(y)$ defined on $[f(1), f(l)]$ an interval of definition $f(-1) \leqslant y \leqslant f(1)$ for which $x_2(y) = 1$ is a constant function.

We define the class of functions $f(\alpha,x)$, $\alpha \in [0,1]$, $x \in [-1,1]$, for which $f(0,x) = f(x)$ and $f(1,x)$ is the equimeasurable symmetrical rearrangement of f as follows:

For x in the interval $[-1, l(1-\alpha)]$ we denote the function inverse to $f(\alpha,x)$ by $x_{1,\alpha}$, and for $x \in [l(1-\alpha), 1]$ we denote the inverse function by $x_{2\alpha}$ where:

$$x_{1,\alpha}(y) = \left(1 - \frac{\alpha}{2}\right) x_1(y) - \frac{\alpha}{2} x_2(y), \quad \min(f(-1), f(1)) \leqslant y \leqslant f(l) \tag{2}$$

and

$$x_{2,\alpha}(y) = \left(1 - \frac{\alpha}{2}\right) x_2(y) - \frac{\alpha}{2} x_1(y), \quad \min(f(-1), f(1)) \leqslant y \leqslant f(l). \tag{3}$$

(By the addition of (A) and (B), the functions $x_{1,\alpha}$ and $x_{2,\alpha}$ are defined on $y \in [(\min(f(-1), f(1))), f(l)]$).

For more details on continuous symmetrization and its special cases, see [1] and [10].

As already mentioned, the functions $f(\alpha,x)$ are equimeasurable. On equimeasurable function see [7, Chapter. X], [10, Chapter VII], and the introduction in [1].

In Section 3 we use of the following Lemma 1 which is the same as [7, Theorem 399]:

LEMMA 1. *In order that an integrable function H should have the property*

$$\int_0^1 H(x)\, y(x)\, dx \leqslant 0$$

for all positive increasing and bounded $y(x)$, it is necessary and sufficient that

$$\int_x^1 H(t)\, dt \leqslant 0$$

holds for every $x \in [0,1]$.

DEFINITION 2. A function f on $[a,b]$ is called symmetrical decreasing if f is symmetrical on $[a,b]$ and increasing on $\left[a, \frac{a+b}{2}\right]$ (see [4, p. 509]).

In Section 2 we use Corollary 1 and Remark 1:

COROLLARY 1. [1, Corollary 1] *If ϕ is positive symmetrical decreasing and bounded on $[-1,1]$ and if H is an integrable function such that*

$$\int_{-s}^{s} H(x)\,dx \leqslant 0$$

holds for every s, $s \in [0,1]$, then

$$\int_{-1}^{1} H(x)\,\phi(x)\,dx \leqslant 0.$$

In Remark 1 we made the needed adaptation of the results of [1, Theorem 1d and Remark 2] for functions that include the concave functions.

REMARK 1. Let f be continuous real function on $x \in [-1,1]$, non-decreasing on $[-1,l]$ and non-increasing on $[l,1]$. Let $f(\alpha,x)$ be the function obtained from f by continuous symmetrization as in [10]. Then $\int_{-s}^{s} f(\alpha,x)\,dx$ is monotone non-decreasing in α, $\alpha \in [0,1]$ for $s \in [0,1]$.

In Section 2 we use Corollary 1 for *concave* functions.

In Section 3 we use another type of continuous process we name *continuous increasing* process. There we compare the upper bound obtained in Theorem 2 with the upper bound obtained by Slater-Pečarić in Theorem 3. Also, we note on the monotonicity of the first eigenvalue of Sturm-Liouville system using the same process.

THEOREM 2. [2, Theorem 1.2] *Let $f \in C^1$ and $f:[0,1] \to [0,1]$. Let f_- be the decreasing rearrangement of f satisfying $f_-(0)=1$ and $f_-(1)=0$. Let $u_-(x)$ be the inverse function of f_-.*

If for every $x \in [0,1]$

$$\int_{x}^{1} f_-(t)dt \leqslant \int_{x}^{1} u_-(t)dt, \tag{4}$$

holds. Then,

$$\varphi\left(\int_0^1 f(x)dx\right) \leqslant \int_0^1 \varphi'(x)f_-(x)\,dx \leqslant \int_0^1 \varphi(f(x))dx \leqslant \int_0^1 \varphi'(x)f_+(x)\,dx, \tag{5}$$

when $\varphi:[0,1] \to \mathbb{R}$ is a convex function and $\varphi(0)=0$.

If for every $x \in [0,1]$

$$\int_{x}^{1} f_-(t)dt \geqslant \int_{x}^{1} u_-(t)dt, \tag{6}$$

holds, then:

$$\varphi\left(\int_0^1 f(x)dx\right) \leqslant \int_0^1 \varphi(f(x))dx \leqslant \int_0^1 \varphi\prime(x)f_-(x)\,dx \leqslant \int_0^1 \varphi\prime(x)f_+(x)\,dx. \tag{7}$$

Jensen's inequality and Slater's companion inequality [11] (as generalized by Pečarić [9]) show that:

THEOREM 3. *If* φ *is a real convex function defined on* I *where* I *is the range of* f, *and if* $M \in I$, *then for all probability measures* μ *and all non-negative* μ*-integrable fuctions* f :

$$\varphi(m) \leqslant \int_\Omega \varphi(f(s))\,d\mu(s) \leqslant \varphi(M), \tag{8}$$

holds, where

$$m = \int_\Omega f(s)\,d\mu(s) \quad \text{and} \quad M = \frac{\int_\Omega f(s)\,C_{f(s)}\,d\mu(s)}{\int_\Omega C_{f(s)}\,d\mu(s)}, \tag{9}$$

and the function C should satisfy $\varphi'_-(x) \leqslant C_x \leqslant \varphi'_+(x)$ *where* φ'_- *and* φ'_+ *are the left and right derivatives of* φ.

We emphasize that in Section 2 we use the continuous symmetrization process as in [10, p. 201] to discuss the behavior of $\int_{-1}^{1} f(\alpha,x)\,\phi(x)\,dx$, $\alpha \in [0,1]$ when ϕ is a non-negative symmetrical decreasing function on $x \in [-1,1]$.

On the other hand, in Section 3, we use another type of continuous process we name *continuous increasing* process in order to generate a set of equimeasurable functions. Through this process we refine Slater-Pečarić inequality and inequalities related to the first eigenvalue of Sturm-Liouville system. We use the behavior of $\int_0^1 f(\alpha,x)\,T(x)\,dx$, $\alpha \in [0,1]$ when T is an increasing function on $x \in [0,1]$.

In this case $f \in C^1$, $f:[0,1] \to \mathbb{R}_+$ is increasing on $[0,l)$ and decreasing on $(l,1]$, x_1 is the inverse of $f \in [0,l)$ and x_2 is the inverse of $f \in (l,1]$.

In the same way as in Definition 1, in order to be able to use this process to build the set of functions $f(\alpha,x)$ we complete the graph of f as follows:

(C): when $f(0) > f(1)$, we add the inverse function x_1 an interval of definition $f(1) \leqslant y \leqslant f(0)$, for which $x_1(y) = 0$ is a constant function,

and:

(D): when $f(0) < f(1)$ we add the inverse function x_2 an interval of definition $f(0) \leqslant y \leqslant f(1)$ for which $x_2(y) = 1$ is a constant function.

We define $f(\alpha,x)$ by its inverses $x_{1,\alpha}$ and $x_{2,\alpha}$ as:

$$x_{1,\alpha}(y) = x_1(y) + \alpha(1 - x_2(y)), \quad \begin{cases} \min(f(0), f(1)) \leqslant y \leqslant f(l) \\ 0 \leqslant x_{1,\alpha} \leqslant l + \alpha(1-l) \end{cases}, \tag{10}$$

and

$$x_{2,\alpha}(y) = x_2(y) + \alpha(1 - x_2(y)), \quad \begin{cases} \min(f(0), f(1)) \leqslant y \leqslant f(l) \\ l + \alpha(1-l) \leqslant x_{2,\alpha} \leqslant 1 \end{cases}. \tag{11}$$

(By the addition of (C) and (D), $x_{1,\alpha}(y)$ and $x_{2,\alpha}(y)$ are defined on $y \in [(\min(f(0), f(1))), f(l)]$).

We finish the paper with a note on inequalities related to the first eigenvalue of a Sturm-Liouville system through continuous symmertization discussed in [2].

2. Refining Clausing inequality

In this section we show that our results refine Clausing inequality by adding inequalities after the integral $\int f(x)\phi(x)\,dx$ in (1) by using continuous symmetrization process as in Definition 1.

To prove Theorem 4 we first state Remark 2 which is essential for the proof of Theorem 4.

REMARK 2. Using Corollary 1 and Remark 1, when f is concave on $x \in [-1,1]$, we get, when ϕ is continuous and symmetrically decreasing on $[-1,1]$, that

$$\int_{-1}^{1} f(\alpha,x)\,\phi(x)\,dx$$

is increasing in α, $\alpha \in [0,1]$, and $f(1,x)$ is the equimeasurable symmetrical decreasing rearrangement of f.

THEOREM 4. *Let f be a non-negative concave function on $[-1,1]$ and let $f(\alpha,x)$, $\alpha \in [0,1]$ be the function obtained by continuous symmetrization process as in Definition 1. Let ϕ be non-negative and symmetrical decreasing on $[-1,1]$.*

Then:

a) the functions $f(\alpha,x)$, $\alpha \in [0,1]$, $x \in [-1,1]$ are concave equimeasurable and $f(1,x)$ is symmetrical decreasing rearrangement of f.

b) for $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$

$$\begin{aligned} &\frac{1}{2}\int_{-1}^{1} f(x)\,dx \int_{-1}^{1} \phi(x)\,dx \leqslant \int_{-1}^{1} f(x)\,\phi(x)\,dx \qquad (12)\\ &\leqslant \int_{-1}^{1} f(\alpha_1,x)\,\phi(x)\,dx \leqslant \int_{-1}^{1} f(\alpha_2,x)\,\phi(x)\,dx \\ &\leqslant \int_{-1}^{1} f(1,x)\,\phi(x)\,dx \leqslant \int_{-1}^{1} f(x)\,dx \int_{-1}^{1} g(x)\,\phi(x)\,dx, \end{aligned}$$

where g is a the symmetrical decreasing function:

$$g(x) = \begin{cases} 1+x, & -1 \leqslant x \leqslant 0 \\ 1-x, & 0 \leqslant x \leqslant 1. \end{cases}$$

Proof. We prove first that when the function f is concave so are the functions $f(\alpha,x)$, for all $\alpha \in [0,1]$ obtained by the continuous symmetrization process.

Let the function x_1 be the inverse of the function f in its increasing segment, and x_2 be the inverse of the function f in its decreasing segment.

When a continuous concave function f has an interval $[c,d]$ on which $f(x) = K$, where K is constant, then K is necessarily the maximum of f on the interval $[-1,1]$. Therefore using the continuous symmetrization process, $f(\alpha,x)$ gets its maximum K on the interval of length $d-c$, and it moves with α from $x_{1,0}(K) = a$, $x_{2,0}(K) = b$

toward $x_{1,1}(K) = -\left(\frac{b-a}{2}\right)$, $x_{2,1}(K) = \frac{b-a}{2}$. Hence in order to show that $f(\alpha,x)$ is concave it is enough to show it when f is strictly increasing on $[-1,l]$ and strictly decreasing on $[l,1]$.

When f is strictly increasing and concave, its inverse function is increasing and from the concavity it follows that $f^{-1}(tf(u_1)+(1-t)f(u_2)) \leqslant tu_1+(1-t)u_2$. Replacing $f(u_1)=v_1$ and $f(u_2)=v_2$ we get that f^{-1} which is denoted as $x_1(y)$ convex increasing. Similarly x_2 the inverse of a strictly decreasing and concave function is decreasing and concave.

Hence, when $\alpha \in [0,1]$

$$x_{1,\alpha}(y) = \left(1-\frac{\alpha}{2}\right)x_1(y) - \frac{\alpha}{2}x_2(y), \quad \begin{cases} f(-1) \leqslant y \leqslant f(l), \\ -1 \leqslant x_{1,\alpha} \leqslant (1-\alpha)l \end{cases}$$

is increasing and convex and similarly

$$x_{2,\alpha}(y) = \left(1-\frac{\alpha}{2}\right)x_2(y) - \frac{\alpha}{2}x_1(y), \quad \begin{cases} f(-1) \leqslant y \leqslant f(l), \\ (1-\alpha)l \leqslant x_{2,\alpha} \leqslant 1 \end{cases}$$

is *decreasing and concave*. Therefore by the same reasoning, $f(\alpha,x)$ when it is the inverse of $x_{1,\alpha}$ is concave increasing and when $f(\alpha,x)$ is the inverse of $x_{2,\alpha}$, it is concave decreasing, so that the set of functions $f(\alpha,x)$ are concave when $\alpha \in [-1,1]$. Part a) of the theorem is proved.

From Remark 2 we see that $\int_{-1}^{1} f(\alpha,x)\phi(x)\,dx$ is increasing in α, $\alpha \in [0,1]$, that is:

$$\begin{aligned} &\int_{-1}^{1} f(x)\phi(x)\,dx \\ &= \int_{-1}^{1} f(0,x)\phi(x)\,dx \leqslant \int_{-1}^{1} f(\alpha_1,x)\phi(x)\,dx \leqslant \int_{-1}^{1} f(\alpha_2,x)\phi(x)\,dx \\ &\leqslant \int_{-1}^{1} f(1,x)\phi(x)\,dx, \end{aligned} \tag{13}$$

where $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$

To complete the proof of the theorem we need to show that

$$\begin{aligned} \int_{-1}^{1} f(1,x)\phi(x)\,dx &\leqslant \int_{-1}^{1} f(1,x)\,dx \int_{-1}^{1} g(x)\phi(x)\,dx \\ &= \int_{-1}^{1} f(x)\,dx \int_{-1}^{1} g(x)\phi(x)\,dx, \end{aligned} \tag{14}$$

and

$$\frac{1}{2}\int_{-1}^{1} f(x)\,dx \int_{-1}^{1} \phi(x)\,dx \leqslant \int_{-1}^{1} f(x)\phi(x)\,dx. \tag{15}$$

As $f(1,x)$ is concave and f and $f(1,x)$ are equimeasurable, inequalities (14) and (15) are actually the right hand-side and the left hand-side of (1) proved in [8].

The proof of the theorem is complete. □

In the following example, we build the symmetrical rearrangement $f(1,x)$, for a given function $f(x)$, see Figure 1.

EXAMPLE 1. Let f be:

$$f(x) = y(x) = \begin{cases} y_1 = \frac{7}{9}x + 1, & -1 \leqslant x \leqslant \frac{3}{7}, \quad \frac{2}{9} \leqslant y_1 \leqslant \frac{4}{3}, \\ y_2 = -\frac{7}{9}x + \frac{7}{3}, & \frac{3}{7} \leqslant x \leqslant 1, \quad 0 \leqslant y_2 \leqslant \frac{4}{3}. \end{cases}$$

As $f\left(\frac{3}{7}\right) = \frac{4}{3}$, $f(1) = 0 < f(-1) = \frac{2}{9}$ then, according to Definition 1, in order to implement the continuous symmetrization process we add in such cases to the graph of the function the value $x_1(y) = -1$ for all $0 \leqslant y \leqslant \frac{2}{9}$.

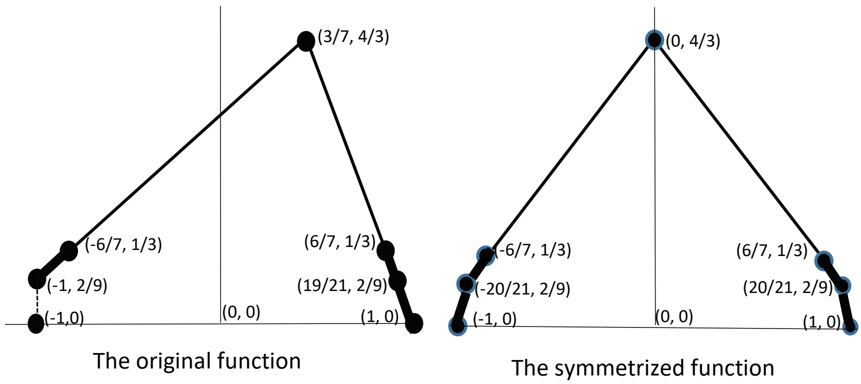

Figure 1.

The symmetrized function $f(1,x)$ obtained by using (2) and (3) from $f(x)$ is:

$$f(1,x) = y^*(x) = \begin{cases} y_1 = \frac{14}{3}x + \frac{14}{3}, & -1 \leqslant x \leqslant -\frac{20}{21}, \quad 0 \leqslant y_1 \leqslant \frac{2}{9}, \\ y_2 = \frac{7}{6}x + \frac{8}{6}, & -\frac{20}{21} \leqslant x \leqslant 0, \quad \frac{2}{9} \leqslant y_2 \leqslant \frac{4}{3}, \\ y_3 = -\frac{7}{6}x + \frac{8}{6}, & 0 \leqslant x \leqslant \frac{20}{21}, \quad \frac{2}{9} \leqslant y_3 \leqslant \frac{4}{3}, \\ y_4 = -\frac{14}{3}x + \frac{14}{3}, & \frac{20}{21} \leqslant x \leqslant 1, \quad 0 \leqslant y_4 \leqslant \frac{2}{9}. \end{cases}$$

We see that $f(1,x)$ is continuous and that the given function f and the function $f(1,x)$ are equimeasurable concave on the interval $[-1,1]$.

We finish this section with a different extension of Theorem 1.

REMARK 3. If f is such that the positive function $\widehat{f}$ defined on $[0,1]$ as $\widehat{f}(x) = \frac{f(x)+f(1-x)}{2}$ is concave, then $\widehat{f}$ satisfies (1). As ϕ and $\widehat{f}$ are symmetric on $[0,1]$ it is obvious that also the function f satisfies (1) although f is not always concave. For example, such functions appear in [6] where it is shown that for the non-concave function $f(x) = x^3 + 64$ on the interval $[-4,2]$ its symetrized function $\widehat{f}$ is concave on the same interval.

3. Refinement of Slater-Pečarić inequalities and monotonicity of eigenvalues

We start this section with comparing Theorem 2 with Theorem 3. Both theorems produce upper bounds of $\int \varphi(f(s))\,d\mu(s)$.

In Theorem 5, sufficient condition for refining Slater-Pačarić inequality are proved by using the continuous increasing process defined by (10) and (11) when $f(0,x) = f_-(x)$ and $f_-(1,x) = f_+(x)$:

THEOREM 5. *Let $f \in C^1$, and $f : [0,1] \to [0,1]$. Let f_- be strictly decreasing rearrangement of f, satisfying $f_-(0) = 1$ and $f_-(1) = 0$. Let $f_-(\alpha_0, x)$ be an intermediate stage between $f_-(x) = f_-(0,x)$ and $f_+(x) = f_-(1,x)$ when using the continuous increasing process (10) and (11). Let u_- be the inverse function of f_-, $\varphi \in C^1$ and $\varphi : [0,1] \to \mathbb{R}$ be a convex function satisfying $\varphi(0) = 0$.*

If

$$\int_x^1 f_-(t)dt \leqslant \int_x^1 u_-(t)dt, \tag{16}$$

for every $x \in [0,1]$, then $f_-(\alpha,x)$ is continuous strictly increasing in $\alpha \in (0,1)$, and there always exists an $\alpha_0 \in [0,1]$ such that when $0 \leqslant M \leqslant 1$:

$$\int_0^1 \varphi(f(x))dx \leqslant \int_0^1 \varphi'(x) f_-(\alpha_0, x)\,dx \leqslant \varphi(M), \tag{17}$$

where $M = \frac{\int_0^1 f(x)\varphi'(f(x))dx}{\int_0^1 \varphi'(f(x))dx}$.

Proof. The function $f_-(\alpha,x)$ is an intermediate stage between the decreasing rearrangement $f_-(x) = f_-(0,x)$ of f and the increasing rearrangement $f_+(x) = f_-(1,x)$ of f. The proof of the continuity of $f(\alpha,x)$ follows step by step the proof of Theorem 1(c) in [A]. By using (10) and (11), in this case:

$$x_{1,\alpha}(y) = \alpha(1 - x_2(y)) \quad x \in [0,\alpha], \quad y \in [0,1], \tag{18}$$

$$x_{2,\alpha}(y) = x_2(y) + \alpha(1 - x_2(y)), \quad x \in [\alpha,1], \quad y \in [0,1], \tag{19}$$

because $x_1(y) = 0$ and $x_2(y) = u_-(y)$, where u_- is the inverse function of $f_-(0,x) = f_-(x)$.

From (18) and (19) when $0 < \alpha < \beta < 1$, $x_{1,\alpha}(y) < x_{1,\beta}(y)$ and $x_{2,\alpha}(y) < x_{2,\beta}(y)$. Hence when $x \in [0,\alpha]$ both $f(\alpha,x)$ and $f(\beta,x)$ are strictly increasing and $f(\alpha,x) > f(\beta,x)$, and when $x \in [\beta,1]$ both $f(\alpha,x)$ and $f(\beta,x)$ are strictly decreasing and $f(\alpha,x) < f(\beta,x)$. Therefore, $f(\alpha,x)$ cuts $f(\beta,x)$ exactly once when $x \in (\alpha,\beta)$, because in this interval $f(\alpha,x)$ is strictly decreasing in x and $f(\beta,x)$ is strictly increasing. Hence $\int_s^1 f(\alpha,x)\,dx$ is strictly increasing in $\alpha \in [0,1]$, and according to Lemma 1 we can see that also $\int_0^1 \varphi'(x) f(\alpha,x)\,dx$ is strictly increasing in α, $\alpha \in [0,1]$ when $\varphi \in C^1$ is convex. From Inequality (5) in Theorem 2 and because f and f_- are equimeasurable, $\int_0^1 \varphi(f(x))dx = \int_0^1 \varphi(f_-(x))dx$ we obtain that

$$\int_0^1 \varphi'(x) f_-(x)\,dx \leqslant \int_0^1 \varphi(f(x))dx \leqslant \int_0^1 \varphi'(x) f_+(x)\,dx. \tag{20}$$

Therefore, because of the strictly monotonicity in α of $\int_0^1 \varphi'(x) f(\alpha,x)\,dx$ on the values in $\left[\int_0^1 \varphi'(x) f_-(x)\,dx, \int_0^1 \varphi'(x) f_+(x)\,dx\right]$ there is $\alpha_0 \in (0,1]$ such that $\int_0^1 \varphi(f(x))dx \leqslant \int_0^1 \varphi'(x) f(\alpha_0,x)\,dx \leqslant \varphi(M) \leqslant \int_0^1 \varphi'(x) f_+(x)\,dx$ and Inequality (17) is proved. □

Theorem 5 shows that under the conditions stated there, there is an $\alpha_0 \in [0,1]$ such that the integral $\int_0^1 \varphi'(x) f(\alpha_0,x)\,dx$ is a better upper bound of $\int_0^1 \varphi(f(x))dx$ than the bound obtained by Slater-Pečarić inequality.

From Theorem 2 and Theorem 5 we obtain Corollary 2 which emphasizes that under our conditions and through the continuous increasing process (10) and (11), Jensen and Slater-Pečarić inequalities are refined:

COROLLARY 2. *Under the conditions of Theorem 5 on φ, f and M we can always refine Jensen and Slater-Pečarić inequalities and find $\alpha_0 \in [0,1]$ such that*

$$\varphi\left(\int_0^1 f(x)dx\right) \leqslant \int_0^1 \varphi'(x) f_-(x)\,dx \leqslant \int_0^1 \varphi(f(x))dx$$
$$\leqslant \int_0^1 \varphi'(x) f_-(\alpha_0,x)\,dx \leqslant \varphi(M),$$

when $M = \frac{\int_0^1 f(x)\varphi'(f(x))dx}{\int_0^1 \varphi'(f(x))dx}$, and $f_-(\alpha_0,x)$, $\alpha_0 \in (0,1]$ is an intermediate stage between $f_-(x) = f_-(0,x)$ the decreasing rearrangement of f and $f_+(x) = f_-(1,x)$ the increasing rearrangement of f obtained by the continuous increasing process (10) *and* (11).

Given the decreasing function $f(x) = 1 - x^2$, $x \in [0,1]$, we show in Example 2 cases that demonstrate refinements of Jensen and Slater Pečarić inequalities:

EXAMPLE 2. Let $f(x) = 1 - x^2$, $x \in [0,1]$. It is easy to compute that for every $x \in [0,1]$

$$\int_x^1 f_-(t)\,dt = \int_x^1 \left(1-t^2\right) dt \leqslant \int_x^1 \sqrt{1-t}dt = \int_x^1 u_-(t)\,dt.$$

Using (10) and (11) we see that for $\alpha \in [0,1]$

$$f(\alpha,x) = \begin{cases} \frac{x(2\alpha-x)}{\alpha^2}, & 0 \leqslant x \leqslant \alpha \\ 1 - \frac{(x-\alpha)^2}{(1-\alpha)^2}, & \alpha \leqslant x \leqslant 1 \end{cases}. \tag{21}$$

In the special cases where $\alpha = \frac{1}{2}$, $\alpha = \frac{1}{3}$ and $\alpha = \frac{2}{3}$:

$$f\left(\frac{1}{2},x\right) = 4x(1-x), \qquad 0 \leqslant x \leqslant 1,$$

$$f\left(\frac{1}{3},x\right)=\begin{cases}3x(2-3x), & 0\leqslant x\leqslant \frac{1}{3}\\ \frac{3}{4}(1-x)(3x+1), & \frac{1}{3}\leqslant x\leqslant 1\end{cases},$$

$$f\left(\frac{2}{3},x\right)=\begin{cases}\frac{3}{4}x(4-3x), & 0\leqslant x\leqslant \frac{2}{3}\\ (1-x)(9x-3), & \frac{2}{3}\leqslant x\leqslant 1\end{cases}.$$

and

$$f_{+}(x)=f(1,x)=x(2-x).$$

Computing $\int_0^1 \varphi'(x) f\left(\frac{1}{3},x\right) dx$ when $\varphi(x)=x^2$, in this case as well as in the case of $\int_0^1 \varphi'(x) f\left(\frac{1}{2},x\right) dx$ when $\varphi(x)=x^2$ we get that

$$\begin{aligned}\frac{1}{2}&=\varphi\left(\int_0^1 f(x)dx\right)\leqslant\int_0^1 \varphi(f(x))dx=\frac{8}{15}\\ &\leqslant\int_0^1 \varphi'(x)f\left(\frac{1}{3},x\right)dx\leqslant\int_0^1 \varphi'(x)f\left(\frac{1}{2},x\right)dx=\frac{2}{3}\\ &\leqslant\varphi\left(\frac{\int_0^1 f(x)\varphi'(f(x))dx}{\int_0^1 \varphi'(f(x))dx}\right)=\frac{4}{5},\end{aligned}$$

which are examples of refinement of Jensen and Slater-Pečarić inequalities.

REMARK 4. The inequality in Theorem 2 says that under the conditions stated there, in particular when $\varphi(0)=0$ and

$$\int_x^1 f_{-}(x)dx\geqslant\int_x^1 u_{-}(x)dx, \tag{22}$$

then

$$\varphi\left(\int_0^1 f(x)dx\right)\leqslant\int_0^1 \varphi(f(x))dx\leqslant\int_0^1 \varphi(u_{-}(x))dx. \tag{23}$$

This follows because

$$\int_0^1 \varphi(u_{-}(x))dx=\int_0^1 \varphi'(x)f_{-}(x)\,dx$$

and

$$\int_0^1 \varphi(f(x))dx=\int_0^1 \varphi(f_{-}(x))dx.$$

Hence when (22) is satisfied it is reasonable to compare also $\varphi(M)$ with the upper bounds of $\int_0^1 \varphi(f(x))dx$ obtained in Theorem 2, where

$$M=\frac{\int_0^1 f(x)\varphi'(f(x))dx}{\int_0^1 \varphi'(f(x))dx} \tag{24}$$

with

$$\varphi\left(\widetilde{M}\right)=\int_0^1 \varphi(u_-(x))dx. \tag{25}$$

We see that if a family of convex functions φ_p is such that:

a) (22) and therefore (23) are satisfied and

b) $\lim_{p\to\infty}\varphi_p\left(\frac{\int_0^1 f(x)\varphi_p'(f(x))dx}{\int_0^1 \varphi_p'(f(x))dx}\right) > \lim_{p\to\infty}\int_0^1 \varphi_p(u_-(x))dx$, ($u_-(x)$ is the inverse of $f_-(x)$, $x\in[0,1]$).

Then, there is always p_0 such that

$$\int_0^1 \varphi_p(u_-(x))dx < \varphi_p\left(\frac{\int_0^1 f(x)\varphi_p'(f(x))dx}{\int_0^1 \varphi_p'(f(x))dx}\right), \quad p\geqslant p_0.$$

This means that

$$\int_0^1 \varphi_p(f(x))dx \leqslant \int_0^1 \varphi_p(u_-(x))dx < \varphi_p\left(\frac{\int_0^1 f(x)\varphi_p'(f(x))dx}{\int_0^1 \varphi_p'(f(x))dx}\right), \quad p\geqslant p_0.$$

In other words, in addition to the proof in Theorem 5, there is a better bound of $\int_0^1 \varphi_p(f(x))dx$ than the bound obtained by Slater-Pečarić theorem also in other cases.

In the following example we demonstrate the results of Remark 4 for a specific f and a family of convex functions φ that although $\int_x^1 f_-(t)dt \geqslant \int_x^1 u_-(t)dt$, the upper bound $\varphi\left(\widetilde{M}\right)$ of $\int_0^1 \varphi(f(x))dx$ is better than that obtained from the Slater-Pečarić inequality.

EXAMPLE 3. Let $f(x)=\sqrt{1-x}$, $u_-(x)=1-x^2$ and $\varphi(x)=x^p$, $x\in[0,1]$, $p\geqslant 1$. Then, as explained in Remark 4, from (22), (23), (24), (25) and $p\geqslant 5$ the inequalities

$$\begin{aligned}\varphi\left(\widetilde{M}(p)\right) &= \int_0^1 (u_-(x))^p dx = \int_0^1 \left(1-x^2\right)^p dx \leqslant \int_0^1 \left(1-x^2\right)^5 dx \\ &= \varphi\left(\widetilde{M}(5)\right) = 0.369408 \leqslant \frac{1}{e} = \lim_{p\to\infty}\left(\frac{p+1}{p+2}\right)^p \\ &\leqslant \varphi(M(p)) = \left(\frac{\int_0^1 f(x)\varphi'(f(x))dx}{\int_0^1 \varphi'(f(x))dx}\right)^p \\ &= \left(\frac{\int_0^1 (1-x)^{\frac{p}{2}}dx}{\int_0^1 (1-x)^{\frac{p}{2}-1}dx}\right)^p = \left(\frac{p+1}{p+2}\right)^p\end{aligned}$$

hold. The reason for this inequality is that $\left(\frac{p+1}{p+2}\right)^p$ is decreasing continuously in p towards $\frac{1}{e}$, and $\int_0^1 \left(1-x^2\right)^p dx$ is decreasing continuously in p for $p\geqslant 1$ and $\int_0^1 \left(1-x^2\right)^5 dx < \frac{1}{e}$.

We finish the paper by demonstrating how continuous symmetrization process defined by (10) and (11), bring about the monotonicity of the first eigenvalue of

$$y''(x)+\lambda(\alpha)f(\alpha,x)y(x)=0, \qquad y(0)=y'(1)=0, \qquad \alpha\in[0,1], \tag{26}$$

as a function of α.

In [2, Theorem 1.5] there is a condition that the function $f:[0,1]\to\mathbb{R}_+$ should be left balanced, that is $f(x)\geqslant f(1-x)$, $0\leqslant x\leqslant\frac{1}{2}$. In the following theorem this type of condition is redundant. For the convenience of the reader, a proof of the following theorem is presented.

THEOREM 6. *Let f be non-negative, continuous on $[0,1]$ increasing on $[0,l]$ and decreasing, on the interval $[l,1]$. Then, for $\alpha\in[0,1]$, $\lambda(\alpha)$, the first eigenvalue of (26) is decreasing in $\alpha\in[0,1]$, where $\lambda(0)$ is the first eigenvalue of (26) for $\alpha=0$, and $\lambda(1)$ is the first eigenvalue of (26) for $\alpha=1$, the increasing rearrangement of $f(x)$.*

Proof. Similarly to the proof of Theorem 5, it is easy to verify that

(a) $f(\alpha,x)$ is continuous on $[0,1]$, increasing in x on $[0,l(\alpha)]$ and decreasing in x on $[l(\alpha),1]$, where

$$l(\alpha)=l+\alpha(1-l), \quad l\in[0,1], \quad \alpha\in[0,1].$$

(b) $f(\alpha,x)$ is continuous in α, $\alpha\in[0,1]$,

(c) For $x\in(0,l(\alpha))$, $f(\alpha,x)\geqslant f(\beta,x)$, and for $x\in(l(\beta),1)$, $f(\alpha,x)\leqslant f(\beta,x)$, when $\alpha\leqslant\beta$.

Because $f(\alpha,x)$ are equimeasurable for $\alpha\in[0,1]$, and $f(\alpha,x)$ cuts $f(\beta,x)$ exactly once, and this occurs on $x\in(l(\alpha),l(\beta))$, where $f(\alpha,x)$ is decreasing in x and $f(\beta,x)$ is increasing in x, therefore $\int_x^1 f(\alpha,x)\,dx$ is increasing in α, $\alpha\in[0,1]$. As $y_{1,\alpha}(x)$, $\alpha\in[0,1]$, the first eigenfunctions of (26), are non-negative incresing in $x\in[0,1]$, hence $\int_0^1 f(\alpha,x)y_{1,\alpha}^2(x)\,dx$, are also increasing in α, $\alpha\in[0,1]$ and

$$\begin{aligned}\lambda(\alpha)&=\frac{\int_0^1 y_{1,\alpha}^{\prime 2}(x)\,dx}{\int_0^1 f(x,\alpha)y_{1,\alpha}^2(x)\,dx}\geqslant\frac{\int_0^1 y_{1,\alpha}^{\prime 2}(x)\,dx}{\int_0^1 f(x,\beta)y_{1,\alpha}^2(x)\,dx}\\&\geqslant\min\frac{\int_0^1 v^{\prime 2}(x)\,dx}{\int_0^1 f(x,\beta)v^2(x)\,dx}=\lambda(\beta), \quad 0\leqslant\alpha\leqslant\beta\leqslant1\end{aligned}$$

that is, $\lambda(\alpha)$ the first eigenvalue of (26) is non-increasing in α. □

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