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EXACT CONSTANTS IN ESTIMATES OF APPROXIMATION OF LIPSCHITZ CLASSES OF PERIODIC FUNCTIONS BY CESÀRO MEANS

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Abstract. We study the problem of computing the exact constant of approximation of classes of continuous functions by linear methods. Specifically, we describe the best constants of estimations for the rate of approximation of Lipschitz classes of periodic functions of several variables by Cesàro means of second and third order.

1. Introduction

Let $\mathbb{R}^d$ be the Euclidean space of vectors $\bar{x} = (x_1, x_2, \ldots, x_d)$, $\mathbb{T}^d = [-\pi;\pi]^d$. Let $C(\mathbb{T}^d)$ be the space of continuous 2π-periodic on each variables functions $f(\bar{x})$ with the norm

$$\|f\| := \|f\|_C = \max_{\bar{x}\in\mathbb{T}^d} |f(\bar{x})|.$$

Denote by $H^{\bar{\alpha}}(\mathbb{T}^d)$ the set of functions $f \in C(\mathbb{T}^d)$, such that

$$|f(\bar{x}) - f(\bar{x}')| \leqslant \sum_{i=1}^{d} |x_i - x_i'|^{\alpha_i}, \quad \bar{x},\ \bar{x}' \in \mathbb{T}^d,$$

where $0 < \alpha_i \leqslant 1$, $i = \overline{1,d}$ are fixed constants. If $\alpha_i = \alpha$, $i = \overline{1,d}$ then $H^{\bar{\alpha}}(\mathbb{T}^d) := H^{\alpha}(\mathbb{T}^d)$.

Let $f \in C(\mathbb{T}^d)$, $\bar{\delta} \in \mathbb{N}^d$. The rectangular Cesàro means $\sigma_{\bar{n}}^{(\bar{\delta})}[f]$ for function f are defined by relationship [25]

$$\sigma_{\bar{n}}^{(\bar{\delta})}[f](\bar{x}) = \pi^{-d} \int\limits_{\mathbb{T}^d} f\left(\bar{x} + \bar{t}\right) \prod_{i=1}^{d} K_{n_i}^{(\delta_i)}\left(t_i\right) dt_i,$$

where

$$K_{n_i}^{(\delta_i)}(t) = \frac{1}{2} + \sum_{k=1}^{n_i} \left(1 - \frac{k}{n_i+1}\right) \ldots \left(1 - \frac{k}{n_i+\delta_i}\right) \cos kt.$$

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If $\delta_i = 1$, $i = \overline{1,d}$, then rectangular Cesàro means $\sigma_{\bar{n}}^{(\bar{\delta})}[f]$ are rectangular Fejer means $\sigma_{\bar{n}}[f]$ of function f

$$\sigma_{\bar{n}}^{(\bar{1})}[f](\bar{x}) = \sigma_{\bar{n}}[f](\bar{x}) = \prod_{i=1}^{d}(n_i+1)^{-1}\sum_{i=1}^{d}\sum_{\nu_i=0}^{n_i} S_{\bar{\nu}}[f](\bar{x}),$$

where

$$S_{\bar{\nu}}[f](\bar{x}) = \pi^{-d}\int\limits_{\mathbb{T}^d} f(\bar{x}+\bar{t})\prod_{i=1}^{d} D_{\nu_i}(t_i)\,dt_i$$

is rectangular partial sum of Fourier series of function f, and $D_{\nu_i}(t_i)$ is Dirichlet kernel of power ν_i.

Let $U_n[f]$, $n \in \mathbb{N}$ be a sequence of linear polynomial operators defined on the set $C(\mathbb{T})$ and $\omega(f;\mu_n)$ be a modulus of continuity of function f for a given sequence $\mu_n > 0$. In the one-dimensional case, the problems of computing the exact constant in inequality

$$\|f - U_n[f]\| \leqslant A\omega(f;\mu_n)$$

were considered by Wang Xing-hua [24], Stechkin [20], Schurer and Steutel [17, 18], and others [2, 3, 7, 11]. Holland's survey [10] should also be mentioned. Exact constants in Jackson-type inequalities, as well as in their generalizations, are intensively studied in nowadays [6, 8, 13, 19, 21, 23]. Here it is difficult to mention all the relevant published research papers in this area. Recently, we have seen the appearance of several important works [1, 4, 22].

In the multidimensional case, finding the exact constant is not trivial. On class of continuous functions of two variables exact constant were obtained in [2] for Jackson polynomials, in [9] for Rogozinski polynomials, in [14] for Cesàro means of second order.

The main aim of the paper is to present exact constant of the estimation of the approximation of classes $H^1(\mathbb{T}^d)$ by rectangular Cesàro means of third order. Also, we generalize the results of [14, 15] to the case of functions $f \in H^1(\mathbb{T}^d)$.

2. Results

Main result is the following.

THEOREM 1. *Let $f \in H^1(\mathbb{T}^d)$. For any $\bar{n} \in \mathbb{N}^d$ the inequalities hold*

$$\left\| f - \sigma_{\bar{n}}^{(\bar{2})}[f] \right\| \leqslant \frac{3\pi^2 - 8}{3\pi \ln 2}\sum_{i=1}^{d}\frac{\ln(n_i+1)}{n_i+1}, \tag{1}$$

$$\left\| f - \sigma_{\bar{n}}^{(\bar{3})}[f] \right\| \leqslant \frac{\pi^2 - 2}{\pi \ln 2}\sum_{i=1}^{d}\frac{\ln(n_i+1)}{n_i+1}. \tag{2}$$

The constants in (1), (2) are exact.

The proof of Theorem 1 is based on the following lemma.

LEMMA 1. *Let* $\bar{n} \in \mathbb{N}^d$. *Then*

$$\sup_{f \in H^1(\mathbb{T}^d)} \left\| f - \sigma_{\bar{n}}^{(\bar{\delta})}[f] \right\| = \pi^{-1} \sum_{i=1}^{d} \int_{\mathbb{T}} |t_i| K_{n_i}^{(\delta_i)}(t_i)\, dt_i. \tag{3}$$

Proof. First we find the upper estimate for the quantity

$$\sup_{f \in H^1(\mathbb{T}^d)} \left\| f - \sigma_{\bar{n}}^{(\bar{\delta})}[f] \right\|. \tag{4}$$

Since $f \in H^1(\mathbb{T}^d)$ and the quantity $K_{n_i}^{(\delta_i)}(t_i)$ satisfies the conditions (see [7], [25, Ch. III, §3])

$$K_{n_i}^{(\delta_i)}(t_i) \geqslant 0, \quad \pi^{-1} \int_{\mathbb{T}} K_{n_i}^{(\delta_i)}(t_i)\, dt_i = 1, \quad n_i \in \mathbb{N}, \quad i = \overline{1,d}, \tag{5}$$

then

$$\left| f(\bar{x}) - \sigma_{\bar{n}}^{(\bar{\delta})}[f](\bar{x}) \right| \leqslant \pi^{-d} \int_{\mathbb{T}^d} |f(\bar{x}) - f(\bar{x} + \bar{t})| \prod_{i=1}^{d} K_{n_i}^{(\delta_i)}(t_i)\, dt_i$$
$$\leqslant \pi^{-d} \int_{\mathbb{T}^d} \sum_{i=1}^{d} |t_i| \prod_{i=1}^{d} K_{n_i}^{(\delta_i)}(t_i)\, dt_i = \pi^{-1} \sum_{i=1}^{d} \int_{\mathbb{T}} |t_i| K_{n_i}^{(\delta_i)}(t_i)\, dt_i.$$

Next, we find the lower estimate of (4). Consider the function $f^*(\bar{x})$, which is a 2π-periodic continuation in each variable on set $\mathbb{R}^d$ of the function $\sum_{i=1}^{d} |x_i|$. We have $f^* \in H^1(\mathbb{T}^d)$, and

$$\left| f^*(\bar{0}) - \sigma_{\bar{n}}^{(\bar{\delta})}[f^*](\bar{0}) \right| = \pi^{-d} \int_{\mathbb{T}^d} \sum_{i=1}^{d} |t_i| \prod_{i=1}^{d} K_{n_i}^{(\delta_i)}(t_i)\, dt_i.$$

Thus

$$\sup_{f \in H^1(\mathbb{T}^d)} \left\| f - \sigma_{\bar{n}}^{(\bar{\delta})}[f] \right\| = \pi^{-d} \int_{\mathbb{T}^d} \sum_{i=1}^{d} |t_i| \prod_{i=1}^{d} K_{n_i}^{(\delta_i)}(t_i)\, dt_i.$$

The Lemma 1 is proved. □

Proof of Theorem 1. Taking into account the equality

$$\sup_{\varphi \in H^1(\mathbb{T})} \left\| \varphi - \sigma_n^{(\delta)}[\varphi] \right\| = \sigma_n^{(\delta)}[|t|](0), \tag{6}$$

and Lemma 1, we have

$$\sup_{f\in H^1(\mathbb{T}^d)} \left\| f-\sigma_{\bar{n}}^{(\bar{\delta})}[f]\right\| = \sum_{i=1}^{d} \sup_{\varphi\in H^1(\mathbb{T})} \left\| \varphi-\sigma_{n_i}^{(\delta_i)}[\varphi]\right\|. \tag{7}$$

Denote

$$\lambda_n^{(\delta)} := \sup_{\varphi\in H^1(\mathbb{T})} \frac{(n+1)\left\|\varphi-\sigma_n^{(\delta)}[\varphi]\right\|}{\ln(n+1)}. \tag{8}$$

In [15] it was shown that

$$\sup_{n\in\mathbb{N}} \lambda_n^{(2)} = \lambda_1^{(2)} = \frac{3\pi^2-8}{3\pi\ln 2},$$

and obtained the exact inequality

$$\left\|\varphi-\sigma_n^{(2)}[\varphi]\right\| \leqslant \frac{3\pi^2-8}{3\pi\ln 2}\frac{\ln(n+1)}{n+1}, \quad f\in H^1(\mathbb{T}), \quad n\in\mathbb{N}. \tag{9}$$

By (7), (9), we have

$$\sup_{f\in H^1(\mathbb{T}^d)} \left\| f-\sigma_{\bar{n}}^{(\bar{2})}[f]\right\| \leqslant \frac{3\pi^2-8}{3\pi\ln 2}\sum_{i=1}^{d}\frac{\ln(n_i+1)}{n_i+1}. \tag{10}$$

On the other hand, directly from (3) as $n_i=1$, $i=\overline{1,d}$, we find

$$\sup_{f\in H^1(\mathbb{T}^d)} \left\| f-\sigma_{\bar{n}}^{(\bar{2})}[f]\right\| = \lambda_1^{(2)} d\frac{\ln 2}{2}. \tag{11}$$

Combining (10), (11), we obtain the statement of the Theorem 1 in the part that concerns the approximation by means of $\sigma_{\bar{n}}^{(\bar{2})}[f]$.

Next, we find the exact estimate of the quantity

$$\sup_{\varphi\in H^1(\mathbb{T})} \left\|\varphi-\sigma_n^{(3)}[\varphi]\right\|.$$

For $n\in\mathbb{N}$, we have

$$\begin{aligned}
\sigma_n^{(3)}[f](x) &= \pi^{-1}\int_{\mathbb{T}} f(t)\left(\frac{1}{2}+\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{k}{n+2}\right)\right. \\
&\quad \left.\times\left(1-\frac{k}{n+3}\right)\cos k(t-x)\right)dt \\
&= \frac{1}{2\pi(n+1)(n+2)(n+3)}\int_{\mathbb{T}} f(t)\Bigg((n+1)(n+2)(n+3) \\
&\quad +2\sum_{k=1}^{n}(n-k+1)(n-k+2)(n-k+3)\cos k(t-x)\Bigg)dt.
\end{aligned}$$

Using the formulas

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2},$$

$$\sum_{k=1}^{n} \int_0^{\pi} t\cos(kt)\,dt = -2 \sum_{k=0}^{[\frac{n+1}{2}]-1} \frac{1}{(2k+1)^2},$$

we get

$$\begin{aligned}\lambda_n^{(3)} &= \frac{4}{\pi(n+2)(n+3)\ln(n+1)} \Bigg((n+1)(n+2)(n+3) \sum_{k=[\frac{n+1}{2}]}^{\infty} \frac{1}{(2k+1)^2} \\ &+((n+1)(n+2)+(n+2)(n+3)+(n+1)(n+3)) \sum_{k=0}^{[\frac{n+1}{2}]-1} \frac{1}{2k+1} \\ &-(3n+6) \sum_{k=0}^{[\frac{n+1}{2}]-1} 1 + \sum_{k=0}^{[\frac{n+1}{2}]-1} (2k+1) \Bigg). \end{aligned} \tag{12}$$

Applying the inequalities

$$\int_{[\frac{n+1}{2}]}^{\infty} \frac{dx}{(2x+1)^2} < \sum_{k=[\frac{n+1}{2}]}^{\infty} \frac{1}{(2k+1)^2} < \int_{[\frac{n+1}{2}]-1}^{\infty} \frac{dx}{(2x+1)^2},$$

$$\frac{1}{2}\frac{1}{n+2} < \sum_{k=[\frac{n+1}{2}]}^{\infty} \frac{1}{(2k+1)^2} < \frac{1}{2}\frac{1}{n-1}, \quad n = 2,3,\dots,$$

and [12, p. 208]

$$\frac{1}{2}\ln(2m+1) < \sum_{k=0}^{m-1} \frac{1}{2k+1} < 1 + \frac{1}{2}\ln(2m-1),$$

$$\frac{1}{2}\ln(n+1) < \sum_{k=0}^{[\frac{n+1}{2}]-1} \frac{1}{(2k+1)^2} < 1 + \frac{1}{2}\ln n, \quad n = 2,3,\dots,$$

we have

$$\begin{aligned}\lambda_n^{(3)} < \frac{4}{\pi\ln(n+1)} \bigg(\frac{n+1}{2(n-1)} &+ \frac{3n^2+12n+11}{(n+2)(n+3)} \left(1 + \frac{1}{2}\ln n\right) \\ &- \frac{3(n-1)}{2(n+3)} + \frac{n^2}{4(n+2)(n+3)} \bigg). \end{aligned} \tag{13}$$

We consider the function

$$\gamma(x) := \frac{4}{\pi \ln(x+1)} \left(\frac{x+1}{2(x-1)} + \frac{3x^2+12x+11}{(x+2)(x+3)} \left(1 + \frac{1}{2}\ln x\right) - \frac{3(x-1)}{2(x+3)} + \frac{x^2}{4(x+2)(x+3)} \right)$$

for $x \geqslant 5$.

Denote

$$\frac{4}{\pi}\left(\frac{x+1}{2(x-1)\ln(x+1)} + \frac{13x^2+48x+44}{4(x+2)(x+3)\ln(x+1)} + \frac{(3x^2+12x+11)\ln x}{2(x+2)(x+3)\ln(x+1)} + \frac{3(1-x)}{2(x+3)\ln(x+1)} \right) := \frac{4}{\pi}\sum_{i=1}^{4}\gamma_i.$$

Obviously, the functions $\gamma_1(x)$, $\gamma_2(x)$ are monotone decreasing for $x \geqslant 5$. Therefore

$$\max_{x\geqslant 5}\gamma_1 = \gamma_1(5), \quad \max_{x\geqslant 5}\gamma_2 = \gamma_2(5). \tag{14}$$

The functions $\gamma_3(x)$, $\gamma_4(x)$ are not monotone decreasing, but

$$\gamma_3(x) < \frac{3}{2}, \quad \gamma_4(x) < 0, \quad x \geqslant 5. \tag{15}$$

Combining (14), (15), we obtain $\gamma(x) < 3.42$, $x \geqslant 5$. So, $\lambda_n^{(3)} < 3.42$, $n \geqslant 5$. From (12), we find $\lambda_1^{(3)} > \lambda_2^{(3)} > \lambda_3^{(3)} > \lambda_4^{(3)} > 3.42$.
This implies the relation

$$\sup_{n\in\mathbb{N}}\lambda_n^{(3)} = \lambda_1^{(3)} = \frac{\pi^2-2}{\pi\ln 2},$$

and exact inequality

$$\left\| f - \sigma_n^{(3)}(f) \right\| \leqslant \frac{\pi^2-2}{\pi\ln 2}\frac{\ln(n+1)}{n+1}, \quad f \in H^1(\mathbb{T}), \quad n \in \mathbb{N}. \tag{16}$$

By (7), (16), we have

$$\sup_{f\in H^1(\mathbb{T}^d)} \left\| f - \sigma_{\bar{n}}^{(\bar{3})}[f] \right\| \leqslant \frac{\pi^2-2}{\pi\ln 2}\sum_{i=1}^{d}\frac{\ln(n_i+1)}{n_i+1}. \tag{17}$$

From (3) as $n_i = 1$, $i = \overline{1,d}$, we find

$$\sup_{f\in H^1(\mathbb{T}^d)} \left\| f - \sigma_{\bar{n}}^{(\bar{3})}[f] \right\| = \lambda_1^{(3)} d\frac{\ln 2}{2}. \tag{18}$$

Combining (17), (18), we obtain inequality (2). The proof of Theorem 1 is now complete. $\square$

3. Concluding Remarks

We considered the problem of exact constants in Jackson-type inequalities for approximation of continuous 2π-periodic functions of several variables by Cesàro means $\sigma_{\bar{n}}^{(\bar{\delta})}[f]$ of their Fourier series. We studied specific cases of this problem for the Lipshitz class $H^1(\mathbb{T}^d)$. For the values $\bar{\delta} = \bar{2}, \bar{3}$, our results are new in the literature. The solved problems open up a certain range of questions that seem interesting to us for further research. The first question that arises is it possible that Lemma 1 holds for general Lipschitz classes $H^{\bar{\alpha}}(\mathbb{T}^d)$? Or some of the results that follow? The answer to this question is positive. The analogue of Lemma 1 holds for the class $C(\mathbb{T})$. Also, we know the asymptotic equality [5]

$$\sup_{\substack{f\in C(\mathbb{T})\\ f\neq const}} \frac{\left\|f-\sigma_n^{(\delta)}[f]\right\|}{\omega\left(f;\frac{\ln(n+1)}{n}\right)} = 1+\frac{2\delta}{\pi}-\frac{2\delta\ln\ln(n+1)}{\pi\ln(n+1)}+O\left(\frac{1}{\ln(n+1)}\right), \quad \delta\geqslant 1,$$

and hence the exact inequality

$$\left\|f-\sigma_n^{\delta}[f]\right\| < \left(1+\frac{2\delta}{\pi}\right)\omega\left(f;\frac{\ln(n+1)}{n}\right), \quad \forall f\in C(\mathbb{T}), \quad n>N(\delta).$$

In our case, establishing estimates of the type of formula (13) for the class $H^{\alpha}(\mathbb{T})$ requires the use of the tool of special functions. Further work on this issue will be presented in the near future.

Next, in connection with the established results for $\lambda_n^{(2)}$, $\lambda_n^{(3)}$, the question arises, what about $\lambda_n^{(4)}$, and $\lambda_n^{(\delta)}$? Is there some conjecture on the general case? Or roundabout argument? Is $\lambda_1^{(\delta)}$ computable in general? Is there an counterexample that this is not the $\sup\limits_{n\in\mathbb{N}}\lambda_n^{(\delta)}$? In the general case, we know the following relations

$$\lambda_1^{(\delta)} = \frac{\pi^2(\delta+1)-8}{\pi(\delta+1)\ln 2}, \quad \delta\in\mathbb{N},$$

$$\lambda_n^{(\delta)} = \frac{4}{\pi\ln(n+1)}\left((n+1)\sum_{[\frac{n+1}{2}]}^{\infty}\frac{1}{(2k+1)^2}+(-1)^{\delta-1}\frac{(n+1)!}{(n+\delta)!}\sum_{k=0}^{[\frac{n+1}{2}]-1}(2k+1)^{\delta-2}\right.$$
$$\left.+\frac{(n+1)!}{(n+\delta)!}\sum_{j=1}^{\delta-1}(-1)^{\delta+1}\sum_{\zeta=0}^{j}\prod_{i=1}^{\delta-1}(n+i+\zeta)\sum_{k=0}^{[\frac{n+1}{2}]-1}(2k+1)^{j-2}\right), \quad (19)$$

and recursive formulas

$$\lambda_n^{(\delta+1)} = \frac{n+\delta}{n+\delta+1}\lambda_n^{(\delta)}+\frac{\pi}{2}\frac{n+1}{(n+\delta+1)\ln(n+1)},$$

$$\lambda_n^{(\delta)} = \frac{n+1}{n+\delta}\lambda_n^{(1)} + (\delta-1)\frac{\pi}{2}\frac{n+1}{(n+\delta)\ln(n+1)}.$$

From (19) follows the asymptotic formula

$$\lambda_n^{(\delta)} = \frac{2\delta}{\pi} + O\left(\frac{1}{\ln(n+1)}\right), \quad n \to \infty,$$

which was first proved in [16]. The method proposed in the work is universal and in combination with formula (19) allows to calculate $\sup\limits_{n\in\mathbb{N}} \lambda_n^{(\delta)}$ for arbitrary concrete $\delta \in \mathbb{N}$. In the general case, the calculation of $\sup\limits_{n\in\mathbb{N}} \lambda_n^{(\delta)}$ is practically impossible due to the great computational difficulties.

Finally, we comment that using the methodology of the work, it is possible to obtain exact inequalities for the case of approximation of classes $H^1(\mathbb{T}^d)$ by Cesàro means $\sigma_{\bar{n}}^{(\bar{\delta})}[f]$ with different values of components $\delta_i \in \{1,2,3\}$, $i = \overline{1,d}$ without any additional conditions.

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