Mathematical
Inequalities
& Applications
Volume 26, Number 4 (2023), 911–933 doi:10.7153/mia-2023-26-56

OLD AND NEW ON THE 3–CONVEX FUNCTIONS

DAN-ŞTEFAN MARINESCU AND CONSTANTIN P. NICULESCU*

(*Communicated by L. E. Persson*)

Abstract. This paper is aimed to review the current knowledge of 3-convex functions and to put some of the known results into a more general context. Several open questions are also raised.

1. Introduction

Higher order convexity was introduced by Hopf [13] and Popoviciu [26], [28], who defined it in terms of divided differences of a function. Assuming f a real-valued function defined on a real interval I, the divided differences of order $0,1,\ldots,n$ associated to a family $x_0,x_1,\ldots,x_n$ of $n+1$ distinct points are respectively defined by the formulas

$$\begin{aligned}
[x_0;f] &= f(x_0)\\
[x_0,x_1;f] &= \frac{f(x_1)-f(x_0)}{x_1-x_0}\\
&\cdots\\
[x_0,x_1,\ldots,x_n;f] &= \frac{[x_1,x_2,\ldots,x_n;f]-[x_0,x_1,\ldots,x_{n-1};f]}{x_n-x_0}\\
&= \sum\nolimits_{j=0}^{n} \frac{f(x_j)}{\prod_{k\neq j}(x_j-x_k)}.
\end{aligned}$$

Notice that all these divided differences are invariant under the permutation of points $x_0,x_1,\ldots,x_n$. As a consequence, we may always assume that $x_0<x_1<\cdots<x_n$.

A function f is called n-*convex* (respectively n-*concave*) if all divided differences $[x_0,x_1,\ldots,x_n;f]$ are nonnegative (respectively nonpositive). In particular,

- the convex functions of order 0 are precisely the nonnegative functions;
- the convex functions of order 1 are the nondecreasing functions;

Mathematics subject classification (2020): 26A51, 39B62, 26D15, 15A20.
Keywords and phrases: Higher order convexity, function with positive differences, spectral decomposition of real symmetric matrices.

* Corresponding author.

Paper MIA-26-56

- the convex functions of order 2 are nothing but the usual convex functions since in this case for all $x_0 < x_1 < x_2$ in I,

$$[x_0, x_1, x_2; f] = \frac{\frac{f(x_0)-f(x_1)}{x_0-x_1} - \frac{f(x_1)-f(x_2)}{x_1-x_2}}{x_0 - x_2} \geqslant 0,$$

  that is, $(x_2 - x_0) f(x_1) \leqslant (x_2 - x_1) f(x_0) + (x_1 - x_0) f(x_2)$.

While the properties of the above three classes of n-convex functions are well understood, only few relevant results are known in the case where $n \geqslant 3$. Besides the work of Popoviciu (see [26], [27], [28], [29] and [30]) we should mention here the contribution of Bennett [1], Bessenyei and Páles [2], Boas and Widder [4], Bojanić and Roulier [5], Brady [6], Bullen [7], [8], Kuczma [16], Marinescu and Monea [17], Pečarić and his collaborators [15], [14], [19], [25], Rajba [31], Szostok [36] and Wasowicz [40].

This paper is aimed to present an overview of the present state of art concerning the 3-convex functions, to add some new results, and to single out some open problems which seem of interest. The reason to restrict ourselves to this particular case is twofold: it offers a convenient framework to illustrate the richness of the class of 3-convex functions and also a context that keeps the technical aspects still simple and intuitive.

For the convenience of the reader, some very basic facts are recalled in Section 2.

Section 3 is mostly dedicated to the identity of 3-convex functions with the functions having positive differences up to order 3. See Theorem 2 below. This result, already known to Popoviciu, was rarely mentioned by the various books dedicated to convex functions, except that by Kuczma [16], that also includes a full proof. Our approach here combines results due to Hopf [13], Popoviciu [26], Boas and Widder [4] and Bennett [1] (to cite them according to their apparition). An important role is played by the fact that on an open interval the 3-convex functions are precisely the differentiable functions whose derivatives are convex functions. This offers the possibility to deduce results for the 3-convex functions from known results for the usual convex functions and vice-versa. See Theorem 3 and Theorem 4 and the comments that accompany them in Section 4.

Section 5 is devoted to an overview of the Hermite-Hadamard inequality in the context of continuous 3-convex functions. The central result is the remarkable extension obtained by Bessenyei and Páles [2], [3], that covers the general case of Borel probability measures on a compact interval $[a,b]$. The Hermite-Hadamard inequality for the usual convex functions is centered around the role played by the barycenter p and the extremal points. As $f(p) = \delta_p(f)$, in the 3-convexity variant, the role of δ_p is taken by a discrete probability measure supported at two points, one of them being inside the interval. Naturally, this raises the interesting problem how looks the analog of Choquet's theory in the framework of 3-convex functions. Several open problems in this connection are mentioned at the end of Section 5.

Section 6 presents a radiography of a recent result due to Ressel [32], concerning the connection between the functions which are continuous, nondecreasing, concave and 3-convex and the Hornich-Hlawka inequality. See Theorem 7. We show that this result is actually the juxtaposition of two distinct results covering complementary domains of the variables, one involving the properties of continuity and 3-convexity, while the other involving only the properties of monotonicity and concavity. See Lemma 4, and Lemma 5 below.

The paper ends with a section discussing the extension of the entire theory to the case of functions taking values in an ordered Banach space.

2. Preliminaries

A function f defined on an interval I is 3-*convex* if for every quadruple $x_0 < x_1 < x_2 < x_3$ of elements in I,

$$[x_0,x_1,x_2,x_3;f] = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} - \frac{f(x_1)}{(x_0-x_1)(x_1-x_2)(x_1-x_3)} + \frac{f(x_2)}{(x_0-x_2)(x_1-x_2)(x_2-x_3)} - \frac{f(x_3)}{(x_0-x_3)(x_1-x_3)(x_2-x_3)} \geqslant 0,$$

equivalently,

$$\begin{aligned}&(x_2-x_0)(x_3-x_0)(x_3-x_2)f(x_1)+(x_1-x_0)(x_2-x_0)(x_2-x_1)f(x_3)\\ &\geqslant (x_2-x_1)(x_3-x_1)(x_3-x_2)f(x_0)+(x_1-x_0)(x_3-x_0)(x_3-x_1)f(x_2).\end{aligned} \tag{1}$$

When the points x_0,x_1,x_2,x_3 are equidistant, that is, when $x_1 = x_0+h,\ x_2 = x_0+2h$, $x_3 = x_0+3h$ for some $h > 0$, the last inequality becomes

$$f(x_0+3h)-3f(x_0+2h)+3f(x_0+h)-f(x_0) \geqslant 0,$$

equivalently,

$$f(x_0)+3f\left(\frac{x_0+2x_3}{3}\right) \leqslant 3f\left(\frac{2x_0+x_3}{3}\right)+f(x_3). \tag{2}$$

If f is n-times differentiable, then a repeated application of Lagrange's mean value theorem yields the existence of a point $\xi \in (\min_k x_k, \max_k x_k)$ such that

$$[x_0,x_1,\ldots,x_n;f] = \frac{f^{(n)}(\xi)}{n!}.$$

As a consequence, one obtains the sufficiency part of the following practical criterion of 3-convexity.

LEMMA 1. *Suppose that f is a continuous function defined on an interval I which is 3-times differentiable on the interior of I. Then f is 3-convex if and only if its third derivative is nonnegative.*

The necessity part is also immediate by using the standard formulas for derivatives via iterated differences,

$$f'''(x_0) = \lim_{h\to 0+} \frac{f(x_0+3h)-3f(x_0+2h)+3f(x_0+h)-f(x_0)}{h^3}.$$

In connection with Lemma 1, it is worth mentioning a result due to Hopf [13], p. 24, and Popoviciu [26], p. 48, which asserts that every 3-convex function f defined on an open interval is differentiable and f' is convex. This can be easily turned into a characterization of 3-convexity in the framework of continuous functions. See Theorem 2 below.

Lemma 1 allows us to notice the existence of a large variety of 3-convex functions such as

$$x/(x+1),\ 1-e^{-\alpha x}\ (\text{for } \alpha>0),$$
$$\log(1+x),\ -x\log x,\ (x-1)/\log x,\ x^{\alpha}\ (\text{for } \alpha\in(0,1]\cup[2,\infty)),$$
$$-x^2+\sqrt{x},\ \sinh,\ \cosh,\ -\log(\Gamma(x)).$$

Also, the primitive of any continuous convex function is a 3-convex function.

The function $1-(x-3)+\frac{(x-3)^3}{6}$ is continuous and 3-convex on $\mathbb{R}_+$ but not n-convex for any $n\in\{0,1,2\}$.

The polynomials with positive coefficients and the exponential are n-convex (on the positive axis) for every $n\geqslant 0$.

REMARK 1. All polynomials of degree less than or equal to 2 are both 3-convex and 3-concave. These functions together with the finite sums $\sum_{i=1}^{m} c_i\left((x-a_i)_+\right)^2$ with positive coefficients represent the building blocks of any 3-convex function. See Popoviciu [28], pp. 29–30 (and also [5] and [38]).

The continuous 3-convex functions on interval I constitute a convex cone in the vector space $C(I)$, of all continuous functions on I. This cone is closed under convolution, but not under usual product.

The extremal properties of 3-convex functions differ from those of convex functions.

The maximum (or minimum) of two 3-convex function is not necessarily a 3-convex function; consider the case of the functions $-x$ and x. Also, an interior critical point of a 3-convex function is not necessarily a point of minimum. It can be a point of inflection (the case of the cubic function x^3) or a point of maximum (the case of the function $-x^2+\sqrt{x}$).

The following approximation theorem due to Popoviciu [27] (see also [12], Theorem 1.3.1 (i), p. 20) allows us to reduce the reasoning with n-convex functions to the case where they are also differentiable.

THEOREM 1. (Popoviciu's approximation theorem) *If a continuous function f : $[0,1]\to\mathbb{R}$ is k-convex, then so are the Bernstein polynomials associated to it,*

$$B_n(f)(x)=\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}f\left(\frac{i}{n}\right).$$

Moreover, by the well-known property of simultaneous uniform approximation of a function and its derivatives by the Bernstein polynomials and their derivatives, it follows that $B_n(f)$ and any derivative (of any order) of it, converge uniformly to f and to its derivatives, correspondingly.

Using a change of variable, one can easily see that the approximation theorem extends to functions defined on compact intervals $[a,b]$ with $a<b$.

COROLLARY 1. *If $f:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous 3-convex function which is also nondecreasing and concave, then the same properties hold for f^α if $\alpha\in(0,1]$.*

Proof. According to Theorem 1, we may reduce the proof to the case where the involved function is of class C^3, in which case the conclusion follows from Lemma 1. □

REMARK 2. An important class of functions $f:\mathbb{R}_+\to\mathbb{R}_+$ which are continuous, nondecreasing concave and 3-convex is that of Bernstein functions. Recall that a function $\mathbb{R}_+\to\mathbb{R}_+$ is called a Bernstein function if it is continuous on $\mathbb{R}_+$, indefinitely differentiable on $(0,\infty)$ and

$$(-1)^{n+1}f^{(n)}\geqslant 0\quad\text{for all } n\geqslant 1.$$

Their theory is exposed in the monograph of Schilling, Song, and Vondraček [34]. According to Corollary 3.8, p. 28, in this monograph, the composition of two Bernstein functions is also a Bernstein function (which is not the case for the class of 3-convex functions).

3. Some characterizations of 3-convexity

The *difference operator* Δ_h (of step size $h\geqslant 0$) associates to each function f defined on an interval I the function $\Delta_h f$ defined by

$$(\Delta_h f)(x)=f(x+h)-f(x),$$

for all x such that the right-hand side formula makes sense. Notice that no restrictions are necessary if $I=\mathbb{R}^+$ or $I=\mathbb{R}$.

The difference operators are linear and commute to each other,

$$\Delta_{h_1}\Delta_{h_2}=\Delta_{h_2}\Delta_{h_1}.$$

They also verify the following property of invariance under translation:

$$\Delta_h(f\circ T_a)=(\Delta_h f)\circ T_a,$$

where T_a is the translation defined by the formula $T_a(x)=x+a$.

LEMMA 2. *If n is a positive integer, then the following formula holds:*

$$\Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_n}f(x)=\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{0,1\}}(-1)^{n-(\varepsilon_1+\cdots+\varepsilon_n)}f(x+\varepsilon_1h_1+\cdots+\varepsilon_nh_n).$$

The proof is immediate, by mathematical induction.

A function f has *positive differences of order* $n \geqslant 1$ if

$$\Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_n}f(x) \geqslant 0,$$

whenever the left-hand side is well defined. To outline a parallel to n-convexity, we say that a function f has positive differences of order 0 if $f \geqslant 0$.

Notice that a function has positive differences of first order if it is nondecreasing and it has positive differences of second order if it is convex (a simple exercise left to the reader). As was noticed by Popoviciu [28] (at the beginning of Section 24, p. 49), this remark works in the general case of n-convex continuous functions. A detailed proof can be found in the book of Kuczma [16]; see Theorem 15.6.1, p. 440. More comments are available in [24]. The next result concerns the case 3-convex functions.

THEOREM 2. *If $f : [0,A] \to \mathbb{R}$ is a continuous function, then the following assertions are equivalent:*

(i) *f is 3-convex;*

(ii) $\Delta_h\Delta_h\Delta_h f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x) \geqslant 0$ *for all* $x \in [0,A)$ *and* $h > 0$ *such that* $x + 3h \leqslant A$.

(iii) *f has positive differences of order 3, that is, it verifies the inequality*

$$\begin{aligned}\Delta_x\Delta_y\Delta_z f(t) = f(x+t) + f(y+t) + f(z+t) + f(x+y+z+t)\\ - f(x+y+t) - f(y+z+t) - f(z+x+t) - f(t) \geqslant 0\end{aligned}$$

whenever $x,y,z,t \in [0,A]$ *and* $x+y+z+t \leqslant A$;

(iv) *f is differentiable on $(0,A)$ and its derivative f' is a convex function.*

Proof. The implication $(i) \Longrightarrow (ii)$ is straightforward. The fact that $(i) \Longleftrightarrow (ii)$ under the presence of continuity is stated by Popoviciu in his book [28] (at the beginning of Section 24, p. 49).

The implication $(i) \Longrightarrow (iv)$ was noticed both by Hopf [13] and Popoviciu [26] (in the general case of n-convex functions).

The implications $(ii) \Longrightarrow (iii) \Longrightarrow (iv)$ are covered by the paper of Boas and Widder [4] (see Lemma 1 and the Theorem, p. 497).

The implication $(iv) \Longrightarrow (i)$ was noticed by Bennett ([1], Proposition 1), who used the identity

$$\begin{aligned}[a,b,c,d;f] = &\frac{1}{(b-a)(c-a)(d-a)}\int_a^b f'(t)dt\\ &- \frac{c+d-a-b}{(c-a)(c-b)(d-a)(d-b)}\int_b^c f'(t)dt\\ &+ \frac{1}{(d-a)(d-b)(d-c)}\int_c^d f'(t)dt,\end{aligned}$$

for all $a < b < c < d$. $\square$

It is worth mentioning that the property of 3-convex functions of having positive differences of order 3 can be also deduced by adapting the argument used by Popoviciu [29] for a weaker variant of it.

We shall need the following special case of the Hardy-Littlewood-Pólya inequality of majorization (see [23], Theorem 4.1.3, p. 186):

LEMMA 3. *If $g:[a,b]\to\mathbb{R}$ is a convex function and c and d are two points in $[a,b]$ such that $a+b=c+d$, then*

$$g(c)+g(d)\leqslant g(a)+g(b).$$

Popoviciu's alternative argument for the implication $(i)\Longrightarrow(iii)$ in Theorem 2 works as follows: According to Popoviciu's approximation theorem, we may restrict to the case where f is a 3-convex function of class C^{∞}. Then fix arbitrarily y,z,t in $[0,A]$ such that $y+z+t<A$ and consider the function

$$\begin{aligned}F(x)=&\,f(x+t)+f(y+t)+f(z+t)+f(x+y+z+t)\\&-f(x+y+t)-f(y+z+t)-f(z+x+t)-f(t)\end{aligned}$$

defined on the interval $[0,A-y-z-t]$. This function is also of class C^{∞} and

$$F^{'}(x)=f^{'}(x+t)+f^{'}(x+y+z+t)-f^{'}(x+y+t)-f^{'}(z+x+t).$$

From Lemma 1 we infer that F' is a convex function, so that $F'\geqslant 0$ according to Lemma 3. Therefore

$$F(x)\geqslant F(0)=0$$

for all x, a fact which is equivalent to the assertion (iii). The proof is done.

The connection of Theorem 2 (iii) with the Hornich-Hlawka inequality will be discussed in Section 6. The next section is devoted to the applications of Theorem 2 (iv).

4. Applications of Theorem 2 (iv)

Theorem 2 (iv) easily allows us to deduce results for the usual convex functions from those for the 3-convex functions (and vice-versa). Two examples are exhibited below. The first one is a refinement of the Jensen inequality.

THEOREM 3. *A continuous function f defined on an interval I is convex if and only if*

$$\frac{1}{|J|}\int_J f(x)\mathrm{d}x\geqslant\frac{1}{|K|}\int_K f(x)\mathrm{d}x,$$

whenever $K\subset J$ are two compact subintervals of I with the same midpoint.

Proof. Suppose that f is convex and that the two intervals under attention are $J=[a,b]$ and $K=[a+\varepsilon,b-\varepsilon]$ (where $\varepsilon\in(0,(b-a)/2)$). Every primitive F of f is a 3-convex function of class C^1 so, according to the formula (1) (applied to the points $a<a+\varepsilon<b-\varepsilon<b$), it verifies the inequality

$$F(b)-F(a)\geqslant\frac{b-a}{b-a-2\varepsilon}\left[F(b-\varepsilon)-F(a+\varepsilon)\right].$$

Thus

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x&=\frac{1}{b-a}\int_a^b F'(x)\mathrm{d}x=\frac{F(b)-F(a)}{b-a}\\&\geqslant\frac{1}{b-a-2\varepsilon}\left[F(b-\varepsilon)-F(a+\varepsilon)\right]\\&=\frac{1}{b-a-2\varepsilon}\int_{a+\varepsilon}^{b-\varepsilon}f(x)\mathrm{d}x\end{aligned}$$

and the proof of the necessity part is done.

For the sufficiency part, notice that f verifies the condition

$$\frac{1}{|I_n|}\int_{I_n}f(x)dx\searrow f\left(\frac{a+b}{2}\right),$$

whenever $I_0=[a,b]\supset I_1\supset I_2\supset\cdots$ is a sequence of nested compact subintervals of I that shrink to $(a+b)/2$, supposed to be their common midpoint. Therefore f is a continuous function such that

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\geqslant f\left(\frac{a+b}{2}\right)$$

whenever $a<b$ in I. As it is well known, this property implies that f is a convex function. See [23], Exercise 1, p. 63. □

REMARK 3. In the same manner one can prove that a continuous function f defined on an interval I is convex if and only if

$$\frac{1}{2\varepsilon}\int_a^{a+\varepsilon}f(x)\mathrm{d}x+\frac{1}{2\varepsilon}\int_{b-\varepsilon}^b f(x)\mathrm{d}x\geqslant\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x,$$

whenever $a<b$ in I and $\varepsilon\in(0,(b-a)/2)$. This represents a refinement of the right-hand side of the Hermite-Hadamard inequality (see [23], Section 1.10, pp. 59–64).

Combining Theorem 3 and Remark 3 one obtains double inequalities such as

$$\begin{aligned}&\frac{4}{b-a}\int_a^{(3a+b)/4}f(x)\mathrm{d}x+\frac{4}{b-a}\int_{(a+3b)/4}^b f(x)\mathrm{d}x\\&\qquad\geqslant\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\geqslant\frac{3}{b-a}\int_{(2a+b)/3}^{(a+2b)/3}f(x)\mathrm{d}x,\end{aligned}$$

for every convex function $f:[a,b]\to\mathbb{R}$. Continuity of f is not necessary. See [23], Proposition 1.1.3, p. 3.

Similarly, one can use the theory of convex functions to characterize the 3-convex functions.

THEOREM 4. *Suppose that $f:I\to\mathbb{R}$ is a function continuous on I and of class C^1 on the interior of I. Then the following assertions are equivalent:*

(i) f is 3-convex;

(ii) f verifies the inequality

$$f'\left(\frac{a+b}{2}\right)\leqslant\frac{f(b)-f(a)}{b-a}, \tag{3}$$

for all points $a<b$ in $\operatorname{int}I$;

(ii) f verifies the inequality

$$\frac{f(b)-f(a)}{b-a}\leqslant\frac{1}{2}\left(\frac{f'(a)+f'(b)}{2}+f'\left(\frac{a+b}{2}\right)\right), \tag{4}$$

for all points $a<b$ in $\operatorname{int}I$.

Proof. $(i)\Longrightarrow(ii)$ According to Theorem 2, f is differentiable on (a,b) and f' is a convex function on this interval. Taking into account the Jensen inequality (see [23], Corollary 1.7.4, p. 43), for every $\varepsilon\in(0,(b-a)/2)$, we have

$$f'\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_{a+\varepsilon}^{b-\varepsilon}f'(x)\mathrm{d}x=\frac{f(b-\varepsilon)-f(a+\varepsilon)}{b-a}$$

and the inequality (3) follows by passing to the limit as $\varepsilon\to0+$.

$(ii)\Longrightarrow(i)$ Indeed, the inequality (3) assures the fulfillment of the Jensen inequality by the function f'' on every compact interval included in $\operatorname{int}I$, which is known to imply the convexity of f'. See [23], Exercise 1, p. 63.

$(i)\Longrightarrow(iii)$ The proof of inequality (4) can be done in the same manner by using Remark 1.10.5, p. 61, in [23]. The implication $(iii)\Longrightarrow(i)$ follows from Exercise 2, p. 63 in [23]. $\square$

EXAMPLE 1. Applying Theorem 4 in the case of the function $f(x)=\log(1+x)$, $x\geqslant0$, we are led to the double inequality

$$\frac{b-a}{1+\frac{a+b}{2}}\leqslant\log\frac{1+b}{1+a}\leqslant\left(\frac{b-a}{4}\right)\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{2}{1+\frac{a+b}{2}}\right),$$

valid for all $0\leqslant a\leqslant b$. This provides a rational estimate of $\log(1+x)$, better than the estimate offered by Maclaurin's expansion.

EXAMPLE 2. If $f:[a,b]\to\mathbb{R}$ is a 3-times differentiable function with $M=\sup_{x\in[a,b]}f'''(x)<\infty$, then $Mx^3/6-f$ is a continuous 3-convex function. For example, in the case of the sine function, this works for $M=1$, which implies that

$$\frac{(a+b)^2}{2}-\cos\frac{a+b}{2}\leqslant\frac{a^2+ab+b^2}{6}-\frac{\sin b-\sin a}{b-a}$$
$$\leqslant\frac{1}{4}\left(\frac{a^2+ab+b^2}{2}-\cos a-\cos b-2\cos\left(\frac{a+b}{2}\right)\right).$$

Similarly, when $m=\inf_{x\in[a,b]}f'''(x)>-\infty$, then the function $f-mx^3/6$ is 3-convex and the conclusion of Theorem 4 applies to it.

Another application of Theorem 2 refers to the "support" of a 3-convex functions.

If f is a continuous 3-convex function defined on an interval I, then f' is a convex function on $\operatorname{int}I$ and we can apply to it the theory of subdifferentiability of convex functions. According to [23], Theorem 1.6.2, p. 36, the subdifferential $\partial f'(a)$ (of f' at a point a interior to I) equals the interval $[f''_-(a),f''_+(a)]$ and

$$f'(x)\geqslant f'(a)+y(x-a)\quad\text{for all } x\in\operatorname{int}I \text{ and } y\in\partial f'(a).$$

Notice that $\partial f'(a)=\{f''(a)\}$ when f is twice differentiable at a.

Therefore

$$f(x)\geqslant f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2}f''_+(a)\quad\text{if } x\in I \text{ and } x\geqslant a \tag{5}$$

and

$$f(x)\leqslant f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2}f''_-(a)\quad\text{if } x\in I \text{ and } x\leqslant a. \tag{6}$$

As a consequence,

$$f(x)=\sup\left\{f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2}y : a\in\operatorname{int}I,\ a<x,\ y\in\partial f'(a)\right\}$$

for all $x\in I$ different from the left endpoint of I.

Geometrically, $g=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2}f''_+(a)$ (respectively $y=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2}f''_-(a)$) represents the unique parabola tangent to the graph of f at a and which has the same right-hand (left-hand) derivative of second order at a. These "tight tangent parabolas" represent analogs of the line supports from the theory of convex functions. Notice that a tight tangent parabola at a point a is above the graph of f on $I\cap(-\infty,a]$ and under the graph on $I\cap[a,\infty)$.

PROBLEM 1. Suppose that f is a function continuous on I and differentiable on $\operatorname{int}I$, which admit at every point $a\in\operatorname{int}I$ a tight tangent parabola. Is f necessarily 3-convex?

The answer to this problem seems to be positive as suggests Bullen's analogue for 3-convex functions, of the familiar fact that the graph of a convex function lies always beneath its chords:

THEOREM 5. *A continuous function f is 3-convex on $[a,b]$ if and only if for every quadratic function Q that coincides with f at α, β and γ (where $a<\alpha<\beta<\gamma<b$) we have*

$$Q(t) \geqslant f(t) \quad \text{if } x \in [a,\alpha] \cup [\beta,\gamma]$$

and

$$f(t) \geqslant Q(t) \quad \text{if } x \in [\alpha,\beta] \cup [\gamma,b].$$

See [7], Theorem 5, p. 85 and also Theorem 10, p. 88.

The existence of tight tangent parabolas raises naturally the problem of an analogue of Fenchel duality in the case of continuous 3-convex functions.

PROBLEM 2. Does there exist an analogue of Fenchel conjugate in the case of continuous 3-convex functions?

In connection with the last problem, notice that a function which is both convex and 3-convex, may have a Fenchel conjugate which is not 3-convex. See the case of the exponential function, whose Fenchel conjugate is the function $f^*(x) = x\log x - x$ for $x > 0$ and $f^*(0) = 0$.

REMARK 4. The support-type properties of 3-convex functions were investigated also by Wasowicz who proved the following result: If $f : [a,b] \to \mathbb{R}$ is a 3-convex function, then for every point $c \in (a,b)$ there exist quadratic functions p and q such that

$$p(a) = f(a),\ p(c) = f(c) \text{ and } p \leqslant f \text{ on } [a,b]$$

and

$$q(c) = f(c),\ p(b) = f(b) \text{ and } q \geqslant f \text{ on } [a,b].$$

See [40], Corollaries 10 and 11.

Another application of Theorem 2 (iv) is provided by the inequality (10), proved in the next section.

5. The Hermite-Hadamard inequality in the context of 3-convex functions

According to Choquet's theory, the meaning of the Hermite-Hadamard inequality for convex functions on intervals is that of a double estimate for the integral mean of every convex function $f : [a,b] \to \mathbb{R}$ with respect to a Borel probability measure μ on $[a,b]$, precisely,

$$f(\mathrm{bar}(\mu)) \leqslant \int_a^b f(x)\mathrm{d}\mu \leqslant \frac{b-\mathrm{bar}(\mu)}{b-a} \cdot f(a) + \frac{\mathrm{bar}(\mu)-a}{b-a} \cdot f(b). \tag{7}$$

Here $\operatorname{bar}(\mu)$ represents the barycenter of μ, that is, the unique point p in $[a,b]$ such that

$$f(p) = \int_a^b f(x)\,\mathrm{d}\mu(x) \tag{8}$$

for every continuous affine function $f:[a,b]\to\mathbb{R}$. One can easily check that $\operatorname{bar}(\mu) = \int_a^b x\,\mathrm{d}\mu(x)$, the moment of the first order of μ.

See [23] (and also [20] and [22]).

When μ is an absolutely continuous probability measure of the form wdx, where the weight $w \geqslant 0$ is continuous and symmetric about the vertical line $x = (a+b)/2$, that is,

$$w(x) = w(a+b-x) \quad \text{for all } x \in [a,b],$$

then $\operatorname{bar}(\mu) = (a+b)/2$ and the inequality (7) becomes

$$f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)w(x)\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}, \tag{9}$$

This is Fejér's variant of the classical Hermite-Hadamard inequality, also known as the Hermite-Hadamard-Fejér inequality. See [23], the remark after Exercise 6, p. 64.

REMARK 5. The inequality *(7)* also works outside the framework of probability measure, for examples for the so called Hermite-Hadamard measures, an example being $3(x^2-1/6)\mathrm{d}x$ on $[-1,1]$. For details, see [11] and [23], Section *7.5*, pp. 322–324.

Theorem 2 (iv) allows us to derive from the Hermite-Hadamard-Fejér inequality some consequences for the 3-convex functions.

For this, consider the case of a differentiable 3-convex function $f:[a,b]\to\mathbb{R}$ and of a continuous real weight w which admits a primitive $W \geqslant 0$, symmetric about the vertical line $x = (a+b)/2$. Three such examples are: 1) $w(x) = a+b-2x$ on $[a,b]$ (with the primitive $W(x) = (x-a)(b-x)$); 2) $2nx^{2n-1}$ on $[-a,a]$ (with the primitive x^{2n}); 3) $\cos x$ on $[0,\pi]$ (with the primitive $\sin x$).

Then

$$\begin{aligned}\int_a^b f(x)w(x)\mathrm{d}x &= fW|_a^b - \int_a^b f'(x)W(x)\mathrm{d}x \\ &= f(b)W(b) - f(a)W(a) - \int_a^b f'(x)W(x)\mathrm{d}x,\end{aligned}$$

and the Hermite-Hadamard-Fejér inequality leads to

$$\begin{aligned}&-\frac{f'(a)+f'(b)}{2}\int_a^b W(x)\mathrm{d}x \\ &\leqslant \int_a^b f(x)w(x)\mathrm{d}x - (f(b)W(b)-f(a)W(a)) \leqslant -f'\left(\frac{a+b}{2}\right)\int_a^b W(x)\mathrm{d}x.\end{aligned} \tag{10}$$

In the particular case where $w(x) = a+b-2x$ and $W(x) = (x-a)(b-x)$ we have

$$\int_a^b W(x)\mathrm{d}x = \frac{1}{6}(b-a)^3$$

and the inequality (10) becomes

$$\frac{(b-a)^3}{6}f'\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)(2x-a-b)\mathrm{d}x \leqslant \frac{(b-a)^3}{12}\left(f'(a)+f'(b)\right).$$

We pass now to the existence of an analogue of the Hermite-Hadamard inequality for the 3-convex functions.

As in the case of usual convex function it is useful to consider the following 3-convex ordering on the set $\mathrm{Prob}([a,b]$, of all Borel probability measures on $[a,b]$:

$$\nu \prec_{3cvx} \mu \text{ if and only if } \int_a^b f(x)\,\mathrm{d}\nu(x) \leqslant \int_a^b f(x)\,\mathrm{d}\mu(x)$$
$$\text{for all continuous and 3-convex functions } f:[a,b]\to\mathbb{R}.$$

Some important necessary and sufficient conditions for higher order convex ordering are available in the papers of Denuit, Lefevre and Shaked [9], Rajba [31] and Szostok [36].

The relation $\prec_{3cvx}$ is indeed a partial order relation. Clearly, it is transitive and reflexive; the fact that $\nu \prec_{3cvx} \mu$ and $\mu \prec_{3cvx} \nu$ imply $\mu = \nu$ comes from the fact that the linear space generated by the continuous 3-convex functions is dense in $C([a,b])$.

REMARK 6. For every $\mu \in \mathrm{Prob}([a,b]$ one can choose a minimal Borel probability measure λ such that $\lambda \prec_{3cvx} \mu$ (and the same is true for the maximal measures). Indeed, $\mathrm{Prob}([a,b]$ can be identified with a weak star convex and compact subset of the dual space of $C([a,b])$, precisely with $\{x^* \in C([a,b]) : x^* \geqslant 0 \text{ and } x^*(1)=1\}$, so that every net of measures minorizing μ admit a convergent subnet in the weak star topology. Thus the existence of minimal Borel probability measures majorized by μ follows from Zorn's lemma.

REMARK 7. Given a Borel probability measure μ on $[a,b]$ whose support includes more than two points, *no* Dirac measure δ_p can be found such that

$$\delta_p(f) = f(p) \leqslant \int_a^b f(x)\,\mathrm{d}\mu(x)$$

for all continuous 3-convex functions. Indeed, checking this for the functions $\pm x$ and $\pm x^2$ we should have

$$p = \int_a^b x\mathrm{d}\mu(x) \text{ and } p^2 = \int_a^b x^2\mathrm{d}\mu(x),$$

which is not possible because the equality occurs in the Cauchy-Schwarz inequality if and only if one the two functions involved is a scalar multiple of the other.

The last two remarks lead naturally to the problem of characterizing the minimal Borel probability measures with respect to the ordering $\prec_{3cvx}$.

PROBLEM 3. Given a Borel probability measure m on $[a,b]$, find numbers $\lambda,\mu,\nu \in [0,1]$ such that for every continuous 3-convex function $f:[a,b]\to\mathbb{R}$ we have

$$\lambda f((1-\mu)a+\mu b)+(1-\lambda)f((1-\nu)a+\nu b)\leqslant \frac{1}{b-a}\int_a^b f(x)\mathrm{d}m(x). \tag{3J}$$

A result due to Bessenyei and Páles [3], Theorem 3.4, combined with a careful inspection of the argument of Lemma 3.2 in [36], shows that this problem has a unique solution, provided that the support of m contains at least 3 points. In the particular case when m equals $(1/(b-a))\,\mathrm{d}x$, this solution corresponds to

$$\lambda = 1/4,\ \mu = 0 \text{ and } \nu = 2/3,$$

and their result reads as follows:

THEOREM 6. *For a continuous function $f:I\to\mathbb{R}$, the following statements are equivalent:*

(i) f is 3-convex;

(ii) for all $a,b\in I$ with $a<b$,

$$\frac{1}{4}f(a)+\frac{3}{4}f\left(\frac{a+2b}{3}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x;$$

(iii) for all $a,b\in I$ with $a<b$,

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leqslant\frac{3}{4}f\left(\frac{2a+b}{3}\right)+\frac{1}{4}f(b).$$

In what follows we will refer to the extremal probability measures $\frac{1}{4}\delta_a+\frac{3}{4}\delta_{(a+3b)/3}$ and $\frac{3}{4}\delta_{(2a+b)/3}+\frac{1}{4}\delta_b$ as the 3-*condensation* of $(1/(b-a))\mathrm{d}x$ and respectively the 3-*dispersion* of $(1/(b-a))\mathrm{d}x$. The non symmetric form of these probability measures seems to be a consequence of Bullen's Theorem 5.

Notice that

$$\begin{aligned} f\left(\frac{a+b}{2}\right)&\leqslant\frac{1}{4}f(a)+\frac{3}{4}f\left(\frac{a+2b}{3}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\\ &\leqslant\frac{3}{4}f\left(\frac{2a+b}{3}\right)+\frac{1}{4}f(b)\leqslant\frac{f(a)+f(b)}{2}\end{aligned}$$

for all functions f which are both convex and 3-convex.

Theorem 6 outlines the following property of rigidity of the continuous 3-convex functions:

REMARK 8. If $f:[a,b]\to\mathbb{R}$ is a continuous 3-convex function such that $f(a)\geqslant 0$ and $f\left(\frac{a+2b}{3}\right)\geqslant 0$, then its integral mean value is also greater than or equal to 0. This imposes that the values of f in the interval $[(a+2b)/3,b]$ cannot be "too" negative (though it can be negative as shows the case of the function $-x^2+\sqrt{x}$ defined on $[0,3/2]$).

Similarly, if $f\left(\frac{2a+b}{3}\right) \leqslant 0$ and $f(b) \leqslant 0$, then the integral mean value of f is also less than or equal to 0 (and the values of f in the interval $[a,(2a+b)/3]$ cannot be "too" positive).

REMARK 9. According to Theorem 2.1 in [31], one can extend Theorem 6 to the context of signed measures (as is the case of the Hermite-Hadamard measures for convex functions). Unfortunately, very few examples illustrating the theory are known so far.

We end this section by mentioning few open problems related to Theorem 6.

PROBLEM 4. What is the statistical meaning of the 3-condensation of $(1/(b-a))\mathrm{d}x$? The same in the case of the 3-dispersion of $(1/(b-a))\mathrm{d}x$.

PROBLEM 5. (The Fejér analog of Theorem 6) Characterize the continuous mappings $w:[a,b]\to[0,\infty)$ such that at least one of the following two inequalities

$$\left[\frac{1}{4}f(a)+\frac{3}{4}f\left(\frac{a+2b}{3}\right)\right]\int_a^b w(x)\mathrm{d}x \leqslant \int_a^b f(x)w(x)\mathrm{d}x$$

and

$$\int_a^b f(x)w(x)\mathrm{d}x \leqslant \left[\frac{3}{4}f\left(\frac{2a+b}{3}\right)+\frac{1}{4}f(b)\right]\int_a^b w(x)\mathrm{d}x$$

holds for all continuous 3-convex functions $f:[a,b]\to\mathbb{R}$.

After this paper was accepted for publication, we were informed by Professor Tomasz Szostok that Problem 5 can be answered even in a more general setting. See [37] for details.

6. 3-convexity and the Hornich-Hlawka functional inequality

A straightforward consequence of Theorem 2 is the fact that every nonnegative and continuous 3-convex function verifies the Hornich-Hlawka functional inequality:

PROPOSITION 1. If $f:[0,A]\to\mathbb{R}$ is a continuous 3-convex function, then

$$f(x)+f(y)+f(z)+f(x+y+z) \geqslant f(x+y)+f(y+z)+f(z+x)+f(0) \tag{11}$$

for all points $x,y,z\in[0,A]$ such that $x+y+z\leqslant A$; if in addition $f(0)\geqslant 0$, then

$$f(x)+f(y)+f(z)+f(x+y+z) \geqslant f(x+y)+f(y+z)+f(z+x). \tag{12}$$

The Hornich-Hlawka functional inequality is not characteristic to the 3-convex functions. Indeed, as was noticed by Sendov and Zitikis [35], Theorem 4.2 (see also [24], Theorem 7), the Hornich-Hlawka functional inequality (12) also works in the framework of completely monotone functions. Recall that a function $f:[0,\infty)\rightarrow\mathbb{R}_+$ is *completely monotone* if it is continuous on $[0,\infty)$, indefinitely differentiable on $(0,\infty)$ and

$$(-1)^n f^{(n)}(x)\geqslant 0 \quad \text{for all } x>0 \text{ and } n\geqslant 0.$$

Some simple examples are e^{-x}, $1/(1+x)$, and $(1/x)\log(1+x)$. Notice that every completely monotone function is nonincreasing, convex and 3-concave.

The result of Proposition 1 can be considerably improved by adding additional hypothesis.

THEOREM 7. *Suppose that* $f:[0,A]\rightarrow\mathbb{R}$ *is a continuous* 3*-convex function which is also nondecreasing and concave. Then*

$$f(|x|)+f(|y|)+f(|z|)+f(|x+y+z|)\geqslant f(|x+y|)+f(|y+z|)+f(|z+x|)+f(0)$$

for all $x,y,z\in[-A,A]$ *with* $|x|+|y|+|z|\leqslant A$.

Theorem 7 is implicit in a paper due to Ressel, who formulated his result in terms of differences of higher order. See [32], Theorem 1 and formula (5). For the convenience of the reader we will include here a full argument.

Notice first that the inequality stated in Theorem 7 is invariant under the permutations of the elements x,y,z and also to the symmetry $(x,y,z)\rightarrow(-x,-y,-z)$. As a consequence, the proof of Theorem 7 can be reduced to the following two cases:

Case 1: the elements x,y and z have the same sign, in which case we may reduce ourselves to the situation where

$$x\geqslant y\geqslant z\geqslant 0;$$

Case 2: two of the elements x,y,z are nonnegative, while the third is nonpositive, in which case the proof reduces to the situation where

$$x\geqslant y\geqslant 0\geqslant z.$$

Case 1 is covered by the assertion (iii) of Theorem 2. Case 2 can be split into four subcases:

Case $2a$: $x\geqslant y\geqslant 0\geqslant z$ and $|z|\geqslant x+y$;
Case $2b$: $x\geqslant y\geqslant 0\geqslant z$ and $x+y\geqslant|z|\geqslant x,y$;
Case $2c$: $x\geqslant y\geqslant 0\geqslant z$ and $x\geqslant|z|\geqslant y$;
Case $2d$: $z\leqslant 0\leqslant y\leqslant x$ and $x\geqslant y\geqslant|z|$.

The assertion of Theorem 7 in Case $2a$ makes the objective of Lemma 4, while the other cases (Case $2b$, Case $2c$ and Case $2d$) are settled by Lemma 5.

LEMMA 4. *Suppose that $f:[0,A]\to\mathbb{R}$ is a continuous function such that*

$$f(x)+f(y)+f(z)+f(x+y+z)\geqslant f(x+y)+f(y+z)+f(z+x)+f(0)$$

for all $x,y,z\geqslant 0$ with $x+y+z\leqslant A$. Then f also verifies the functional inequality

$$f(|x|)+f(|y|)+f(|z|)+f(|x+y+z|)\geqslant f(|x+y|)+f(|y+z|)+f(|z+x|)+f(0)$$

for all triplets $x,y,z\in[-A,A]$ of which two elements are nonnegative and their sum does not exceed the absolute value of the third element.

Proof. It suffices to consider the case where $x,y\geqslant 0\geqslant z$ and $x+y\leqslant |z|$. Then, $|z|-x=|x+z|$, $|z|-y=|y+z|$ and $|z|-x-y=|x+y+z|$. According to our hypothesis, applied to x, y and $|z|-x-y$, we have

$$f(x)+f(y)+f(|z|-x-y)+f(|z|)\geqslant f(x+y)+f(|z|-y)+f(|z|-x)+f(0),$$

equivalently,

$$\begin{aligned}f(|x|)+f(|y|)+f(|z|)+f(|x+y+z|)&\\ \geqslant f(|y+z|)+f(|x+z|)+f(|x+y|)+f(0).&\qquad\square\end{aligned}$$

LEMMA 5. *If $f:[0,A]\to\mathbb{R}$ is a nondecreasing and concave function, then*

$$f(|x|)+f(|y|)+f(|z|)+f(|x+y+z|)\geqslant f(|x+y|)+f(|y+z|)+f(|z+x|)+f(0),$$

for all $x,y,z\in[-A,A]$ such that $z\leqslant 0\leqslant y\leqslant x$.

When the domain of f is $\mathbb{R}_+$ and f is nonnegative, then the property of concavity implies the property of being nondecreasing. See [23], Exercise 4, p. 31.

Proof. The range $z\leqslant 0\leqslant y\leqslant x$ can be split into the following cases:

Case $2b$: $z\leqslant 0\leqslant y\leqslant x$ and $x,y\leqslant |z|\leqslant x+y$. Then $0\leqslant x+y+z\leqslant |z|\leqslant x+y$, so by Lemma 3 it follows that

$$f(x+y)+f(0)\leqslant f(|z|)+f(x+y+z).$$

On the other hand, $|x+z|=|z|-x$, $|y+z|=|z|-y$ and $|x+y+z|=x+y+z$. Since f is nondecreasing, we have $f(|z|-x)\leqslant f(y)$ and $f(|z|-y)\leqslant f(|x|)$. Therefore

$$\begin{aligned}f(|x+y|)&+f(|y+z|)+f(|z+x|)+f(0)\\ &=f(x+y)+f(0)+f(|z|-y)+f(|z|-x)\\ &\leqslant f(x+y+z)+f(|z|)+f(x)+f(y)\\ &=f(|z|)+f(|x+y+z|)+f(|x|)+f(|y|).\end{aligned}$$

Case $2c$: $z \leqslant 0 \leqslant y \leqslant x$ and $y \leqslant |z| \leqslant x$. Applying Lemma 3 for $0 \leqslant y \leqslant x \leqslant x+y$ and taking into account that f is nondecreasing we obtain

$$\begin{aligned} f(|x+y|)+f(|y+z|)+f(|z+x|)+f(0) & \\ &= f(x+y)+f(0)+f(|z|-y)+f(x-|z|) \\ &\leqslant f(x)+f(y)+f(|z|)+f(x+y+z) \\ &= f(|x|)+f(|y|)+f(|z|)+f(|x+y+z|). \end{aligned}$$

Case $2d$: $z \leqslant 0 \leqslant y \leqslant x$ and $|z| \leqslant y \leqslant x$. Similar to Case $2c$. Applying Lemma 3 for $0 \leqslant |z| \leqslant x+y-|z| \leqslant x+y$ and using the fact that f is nondecreasing we obtain

$$\begin{aligned} f(|x+y|)+f(|y+z|)+f(|z+x|)+f(0) & \\ &= f(x+y)+f(0)+f(y-|z|)+f(x-|z|) \\ &\leqslant f(x+y-|z|)+f(|z|)+f(y-|z|)+f(x-|z|) \\ &\leqslant f(x+y+z)+f(|z|)+f(y)+f(x) \\ &= f(|x|)+f(|y|)+f(|z|)+f(|x+y+z|). \end{aligned}$$

The proof of Lemma 5 is now complete. □

Lemma 5, fails in the case of nondecreasing and 3-convex functions which are not concave. To check this, consider the restriction of the cubic function x^3 to $[0,\infty)$ and the triplet $x=y=1$ and $z=-1$.

Some example illustrating Theorem 7 and Corollary 1 in the case of the Bernstein functions $x/(1+x)$, $\log(1+x)$ and the identity of $[0,\infty))$ are indicated below:

(RHH) the *rational form of the Hornich-Hlawka inequality,*

$$\frac{|x|^\alpha}{1+|x|^\alpha}+\frac{|y|^\alpha}{1+|y|^\alpha}+\frac{|z|^\alpha}{1+|z|^\alpha}+\frac{|x+y+z|^\alpha}{1+|x+y+z|^a} \geqslant \frac{|x+y|^a}{1+|x+y|^a}+\frac{|y+z|^a}{1+|y+z|^a}+\frac{|z+x|^a}{1+|z+x|^a};$$

(MHH) the *multiplicative form of the Hornich-Hlawka inequality,*

$$(1+|x|)(1+|y|)(1+|z|)(1+|x+y+z|) \geqslant (1+|x+y|)(1+|y+z|)(1+|z+x|);$$

(HH^α) the *fractional power form of the Hornich-Hlawka inequality*:

$$|x|^\alpha+|y|^\alpha+|z|^\alpha+|x+y+z|^\alpha \geqslant |x+y|^\alpha+|y+z|^\alpha+|z+x|^\alpha.$$

Here $\alpha \in (0,1]$ is a parameter.

The natural analogue of Theorem 7, for more that 3 numbers does not hold. Indeed, the function

$$\sum_{i=1}^4 |x_i| - \sum_{1\leqslant i<j\leqslant 4} |x_i+x_j| + \sum_{1\leqslant i<j<k\leqslant 4} |x_i+x_j+x_k| - |x_1+x_2+x_3+x_4|$$

takes both positive and negative values as the variables $x_1,\dots,x_4$ run over $\mathbb{R}$.

However, an extension of Theorem 7 to the case of $n>3$ variables is still possible by using an inductive scheme due to Vasić and Adamović [39]. We state here a slightly modified version of their result as appeared in [19], Theorem 2, p. 528:

THEOREM 8. *Suppose that φ is a real-valued function defined on a commutative additive semigroup $\mathscr{S}$ such that*

$$\sum_{k=1}^{3}\varphi(x_k)+\varphi\left(\sum_{k=1}^{3}x_k\right)\gtrless\sum_{1\leqslant i<j\leqslant 3}\varphi(x_i+x_j)$$

for all $x_1,x_2,x_3\in\mathscr{S}$. Then for each pair $\{k,n\}$ of integers with $2\leqslant k<n$ we also have

$$\binom{n-2}{k-1}\sum_{k=1}^{n}\varphi(x_k)+\binom{n-2}{k-2}\varphi\left(\sum_{k=1}^{n}x_k\right)\gtrless\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\varphi\left(\sum_{j=1}^{k}x_{i_j}\right),$$

whenever $x_1,\dots,x_n\in\mathscr{S}$.

This result yields the following generalization of Theorem 7:

THEOREM 9. *If $f:\mathbb{R}_+\to E$ is a continuous 3-convex function which is also nondecreasing and concave, then*

$$\binom{n-2}{k-1}\sum_{k=1}^{n}f(|x_k|)+\binom{n-2}{k-2}f\left(\left|\sum_{k=1}^{n}x_k\right|\right)$$
$$\geqslant\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}f\left(\left|\sum_{j=1}^{k}x_{i_j}\right|\right)+(k-1)\binom{n-1}{k}f(0),$$

for all pairs $\{k,n\}$ of integers with $2\leqslant k<n$ and all strings $x_1,\dots,x_n$ of real numbers.

Proof. Apply Theorem 8 to $f(|\cdot|)-f(0)$, taking into account Theorem 7 and the formula

$$\binom{n-2}{k-1}+\binom{n-2}{k-2}=\binom{n-1}{k-1}.\quad\square$$

7. The case of vector-valued functions

The concept of n-convexity can be extended in a straightforward way to the case of functions with values in an ordered Banach space by using the same definition based on divided differences.

Recall that an *ordered Banach space* is any Banach space E endowed with the ordering $\leqslant$ associated to a closed convex cone E_+ via the formula

$$x\leqslant y \text{ if and only if } y-x\in E_+,$$

such that

$$E = E_+ - E_+, \quad (-E_+) \cap E_+ = \{0\},$$

and

$$0 \leqslant x \leqslant y \text{ in } E \text{ implies } \|x\| \leqslant \|y\|.$$

The basic facts concerning the theory of ordered Banach spaces are made available by the book of Schaefer and Wolff [33]. See [21] for a short overview centered on two important particular cases: $\mathbb{R}^n$, the n-dimensional Euclidean space endowed with the coordinate-wise ordering, and $\mathrm{Sym}(n,\mathbb{R})$ the ordered Banach space of all $n \times n$-dimensional symmetric matrices with real coefficients endowed with the operator norm

$$\|A\| = \sup_{\|x\| \leqslant 1} |\langle Ax, x\rangle|$$

and the Löwner ordering,

$$A \leqslant B \text{ if and only if } \langle A\mathbf{x}, \mathbf{x}\rangle \leqslant \langle B\mathbf{x}, \mathbf{x}\rangle \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

Here the operator norm can be replaced by any Schatten norm, in particular with the *Frobenius norm*,

$$\|A\|_F = \left(\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2\right)^{1/2},$$

provided that $A = (a_{ij})_{i,j=1}^n$. The Frobenius norm is associated to the trace inner product

$$\langle A, B\rangle = \mathrm{trace}(AB).$$

The positive cone of $\mathbb{R}^n$ is the first orthant $\mathbb{R}^n_+$, while the positive cone of $\mathrm{Sym}(n,\mathbb{R})$ is the set $\mathrm{Sym}^+(n,\mathbb{R})$ consisting of all positive semi-definite matrices.

REMARK 10. The study of vector-valued functions can be reduced to that of real-valued functions. Indeed, in any ordered Banach space E, any inequality of the form $u \leqslant v$ is equivalent to $x^*(u) \leqslant x^*(v)$ for all $x^* \in E^*_+$. See [21].

As a consequence, a function $f : I \to E$ is respectively nondecreasing, convex or n-convex if and only if $x^* \circ f$ has this property whenever $x^* \in E^*$ is a positive functional. For $E = \mathbb{R}^n$, this remark concerns the components of f.

Remark 10 easily yields that most of the results in the preceding sections extends verbatim to the vector-valued framework. In particular, so are Theorem 2, Proposition 1, Theorem 7, Theorem 4 and Theorem 9.

Combining Remark 10 with Lemma 1 one obtains the following practical test of 3-convexity for the vector-valued differentiable functions:

THEOREM 10. *Suppose that f is a continuous function defined on an interval I and taking values in an ordered Banach space E. If f is three times differentiable on the interior of I, then f is a 3-convex function if and only if $f''' \geqslant 0$.*

An example illustrating Theorem 10 is provided by the function

$$f:\mathbb{R}_+\to \mathrm{Sym}(n,\mathbb{R}),\quad f(t)=-e^{-tA},$$

associated to a positive semi-definite matrix $A\in \mathrm{Sym}(n,\mathbb{R})$. This function is of class C^∞ and its first three derivatives are given by the formulas

$$f'(t)=Ae^{-tA},\quad f''(t)=-A^2e^{-tA},\quad f'''(t)=A^3e^{-tA}.$$

This shows that f is nondecreasing, concave and 3-convex (according to the ordering of $\mathrm{Sym}(n,\mathbb{R})$). The fact that A^3e^{-tA} is positive semidefinite follows from the fact that the product of positive semi-definite matrices that commute with each other is a matrix of the same type.

According to Theorem 7,

$$e^{-|r|A}+e^{-|s|A}+e^{-|t|A}+e^{-|r+s+t|A}\leqslant I+e^{-|r+s|A}+e^{-|s+t|A}+e^{-|t+r|A}$$

for all $r,s,t\in\mathbb{R}$. Here I is the identity matrix. In the 1-dimensional case, this reduces to the inequality

$$e^{-\alpha|x|}+e^{-\alpha|y|}+e^{-\alpha|z|}+e^{-\alpha|x+y+z|}\leqslant 1+e^{-\alpha|x+y|}+e^{-\alpha|y+z|}+e^{-\alpha|x+z|},$$

which works for all $x,y,z\in\mathbb{R}$ and $\alpha>0$. This last inequality can be extended to the framework of real symmetric matrices:

THEOREM 11. *Suppose that $f:[0,\infty)\to\mathbb{R}$ is a continuous 3-convex function which is also nondecreasing and concave. Then*

$$f(|A|)+f(|B|)+f(|C|)+f(|A+B+C|)$$
$$\geqslant f(|A+B|)+f(|B+C|)+f(|C+A|)+f(0)I_n,$$

whenever A,B,C are three real symmetric matrices of order n that commute with each other.

Here $|A|=\left(A^2\right)^{1/2}$ denotes the modulus of A and I_n is the unit matrix of order n.

Proof. Notice first that every finite family of self-adjoint matrices that commute with each other admits an orthonormal basis consisting of vectors that are eigenvectors of each these matrices. See Mirsky [18], Theorem 10.6.8, p. 322. This reduces the proof of the theorem to the case where all the matrices A, B and C are diagonal. Or, if

$$A=\begin{pmatrix}\lambda_1(A) & & 0\\ & \ddots & \\ 0 & & \lambda_n(A)\end{pmatrix},$$

then

$$f(A)=\begin{pmatrix}f(\lambda_1(A)) & & 0\\ & \ddots & \\ 0 & & f(\lambda_n(A))\end{pmatrix},$$

so that the conclusion of the theorem follows from Theorem 7. $\square$

REMARK 11. Theorem 11 also works in the context of commuting self-adjoint compact operators defined on an infinite dimensional Hilbert space provided that $f : [0,\infty) \to \mathbb{R}$ is continuous, nondecreasing, concave 3-convex and $f(0) = 0$. We do not know whether the commutativity condition is necessary or not.

Acknowledgement. The authors would like to thank the reviewer for a number of valuable suggestions and for pointing out some minor typos in an earlier version of this paper.

REFERENCES

[1] G. BENNETT, *Some forms of majorization*, Houston J. Math., **36** (2010), 1037–1066.
[2] M. BESSENYEI, ZS. PÁLES, *Higher-order generalizations of Hadamard's inequality*, Publ. Math. Debrecen, **61** (2002), 623–643.
[3] M. BESSENYEI, ZS. PÁLES, *Characterization of higher order monotonicity via integral inequalities*, P. Roy. Soc. Edinb. A, **140** (2010), 723–736.
[4] R.P. BOAS, JR., D.V. WIDDER, *Functions with positive differences*, Duke Math. J., **7** (1940), 496–503.
[5] R. BOJANIĆ, J. ROULIER, *Approximation of convex functions by convex splines and convexity preserving continuous linear operators*, Rev. Anal. Numér. Théor. Approx. **3** (1975), 143–150.
[6] Z. BRADY, *Inequalities and higher order convexity*, arXiv preprint arXiv:1108.5249, 2011.
[7] P. S. BULLEN, *A criterion for n-convexity*, Pacific J. Math., **36** (1971), 81–98.
[8] P. S. BULLEN, *An inequality of N. Levinson*, Univ. Beograd Publ. Elektrotehn., Fak. Ser. Mat. Fiz., **412/460** (1973), 109–112.
[9] M. DENUIT, C. LEFEVRE, M. SHAKED, *The s-convex orders among real random variables, with applications*, Math. Inequal. Appl., **1** (1998), 585–613.
[10] L. FEJÉR, *Über die Fourierreihen*, II, Math. Naturwiss. Anz Ungar. Akad. Wiss., **24** (1906), 369–390.
[11] A. FLOREA, C. P. NICULESCU, *A Hermite-Hadamard inequality for convex-concave symmetric functions*, Bull. Soc. Sci. Math. Roum., **50** (98) (2007), 149–156.
[12] S. G. GAL, *Shape Preserving Approximation by Real and Complex Polynomials*, Birkhäuser, Boston, 2008.
[13] E. HOPF, *Uber die Zusammenhänge zwischen gewissen höheren Differenzenquotienten reeller Funktionen einer reellen Variablen und deren Differenzierbarkeitseigenschaften*, Dissertation, Univ. Berlin, 1926.
[14] S. KHALID, J. PEČARIĆ, J. PEČARIĆ, *On Zipf-Mandelbrot entropy and 3-convex functions*, Adv. Oper. Theory, **4** (2019), 724–737.
[15] S. KHALID, J. PEČARIĆ, M. PRALJAK, 3*-convex functions and generalizations of an inequality of Hardy-Littlewood-Pólya*, Glasnik matematički, **48** (2013), 335–356.
[16] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities: Cauchy's Equation and Jensen's Inequality*, Springer Science & Business Media, 2009.
[17] D.-Ş. MARINESCU, M. MONEA, *Some inequalities for convex and 3-convex functions with applications*, Kragujevac J. Math., **39** (2015), 83–91.
[18] L. MIRSKY, *Introduction to Linear Algebra*, Oxford University Press, 1955.
[19] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Classical and new inequalities in analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
[20] C. P. NICULESCU, *The Hermite-Hadamard inequality for convex functions of a vector variable*, Math. Inequal. Appl., **5** (2002), 619–623.
[21] C. P. NICULESCU, O. OLTEANU, *From the Hahn-Banach extension theorem to the isotonicity of convex functions and the majorization theory*, Rev. R. Acad. Cienc. Exactas Fíis. Nat. Ser. A Mat. RACSAM, Serie A. Matemáticas, **114**, 4, (2020), 1–19.
[22] C. P. NICULESCU AND L.-E. PERSSON, *Old and new on the Hermite-Hadamard inequality*, Real Analysis Exchange, **29**, 2, (2003/2004), 663–685.

[23] C. P. NICULESCU, L.-E. PERSSON, *Convex Functions and their Applications. A Contemporary Approach*, 2nd ed., CMS Books in Mathematics vol. **23**, Springer-Verlag, New York, 2018.

[24] C. P. NICULESCU, S. SRA, *Functional inequalities for functions with positive differences on convex cones* Math. Results **78**, 6, (2023), article 217.

[25] J. E. PEČARIĆ, F. PROSCHAN, Y. L TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Mathematics in Science and Engineering vol. 187, Acad. Press., Inc., Boston, 1992.

[26] T. POPOVICIU, *Sur quelques propriétés des fonctions d'une ou de deux variables réelles*, Mathematica (Cluj), **8** (1934), 1–85.

[27] T. POPOVICIU, *Sur l'approximation des fonctions convexes d'ordre supérieur*, Mathematica (Cluj) **10** (1935), 49–54.

[28] T. POPOVICIU, *Les Fonctions Convexes*, Hermann Cie. Editeurs, Paris, 1944.

[29] T. POPOVICIU, *On some inequalities*, Gaz. Mat. Bucureşti, **51** (1946), 81–85, (Romanian).

[30] T. POPOVICIU, *Sur une inegalité de N. Levinson*, Mathematica (Cluj), **6** (1969), 301–306.

[31] T. RAJBA, *On a generalization of a theorem of Levin and Stečkin and inequalities of the Hermite-Hadamard type*, Math. Inequal. Appl., **20** (2017), 363–375.

[32] P. RESSEL, *The Hornich–Hlawka inequality and Bernstein functions*, J. Math. Inequal., **9** (2015), 883–888.

[33] H. H. SCHAEFER, M. P. WOLFF, *Topological Vector Spaces*, Graduate Texts in Mathematics vol. **3**, Springer Verlag, 1966.

[34] R. L. SCHILLING, R. SONG, Z. VONDRAČEK, *Bernstein Functions. Theory and Applications*, De Gruyter, Berlin, 2010.

[35] H. S. SENDOV, R. ZITIKIS, *The shape of the Borwein–Affleck–Girgensohn function generated by completely monotone and Bernstein functions*, J. Optim. Theory Appl., **160** (2014), 67–89.

[36] T. SZOSTOK, *Inequalities of Hermite-Hadamard type for higher order convex functions, revisited*, Communications on Pure & Applied Analysis, **20**, 2, (2021), 903–912.

[37] T. SZOSTOK, *A generalization of a theorem of Brass and Schmeisser*, preprint arXiv, 2023.

[38] GH. TOADER, *Some generalizations of Jessen's inequality*, Anal. Numér. Théor. Approx., **16** (1987), 191–194.

[39] P. M. VASIĆ, D. D. ADAMOVIĆ, *Sur un système infini d'inégalités fonctionnelles*, Publ. Inst. Math. Nouv. Sér., **9**, (23), (1969), 107–114.

[40] SZ. WASOWICZ, *Support-type properties of convex functions of higher order and Hadamard type inequalities*, J. Math. Anal. Appl., **332** (2007), 1229–1241.

(Received April 18, 2023)

Dan-Ştefan Marinescu
National College "Iancu de Hunedoara"
Hunedoara, Romania
e-mail: marinescuds@gmail.com

Constantin P. Niculescu
Department of Mathematics
University of Craiova
Craiova 200585, Romania
e-mail: constantin.p.niculescu@gmail.com