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Φ–LIKE ANALYTIC FUNCTIONS ASSOCIATED WITH A VERTICAL DOMAIN

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Abstract. In this article, using the principle of subordination we introduce a new class of Φ-like functions associated with a vertical strip domain and provided some interesting deviations or adaptation which are helpful in unification and extension of various studies of analytic functions. Furthermore, we illustrated the impact of vertical strip domain on various conic region. Inclusion relations, geometrical interpretation, coefficient estimates, inverse function coefficient estimates and solution to the Fekete-Szegő problem of the defined class are discussed. Applications of our main results are given as corollaries.

1. Introduction

Denote by $\mathcal{A}$ the class of analytic functions in $\mathbb{U} = \{z : |z| < 1\}$ having a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}. \tag{1}$$

Let Φ be analytic on $f(\mathbb{U})$ with $\Phi(0) = 0$ and $\mathrm{Re}[\Phi'(0)] > 0$. Brickman in [1] (also see [13, 16]) defined a class called Φ-like functions, a complete generalization of various subclasses of starlike and spiral-like functions, as below:

$$\mathcal{S}^*_\Phi = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{\Phi[f(z)]}\right) > 0, \quad (z \in \mathbb{U}) \right\}.$$

Note that $f \in \mathcal{S}^*_\Phi$ in spite of being a complete generalization, are essentially univalent in $\mathbb{U}$. Let $\mathcal{P}$ denote the family consisting of functions having a series representation of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

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analytic in $\mathbb{U}$ and satisfying the condition $\operatorname{Re}(p(z)) > 0$. For

$$\theta = \frac{1-\alpha}{\beta-\alpha}\pi \in (0,\pi) \tag{2}$$

and

$$\lambda = \exp\left(i\frac{1-\alpha}{\beta-\alpha}\pi\right) = \cos\theta + i\sin\theta. \tag{3}$$

We now define a function

$$\Psi[\alpha,\beta;h(z)] = 1 + i\frac{\beta-\alpha}{\pi}\log\left[\frac{h(z)(1-\lambda^2)+(1+\lambda^2)}{2}\right], \tag{4}$$

with $h(z) \in \mathcal{P}$ and λ is defined as in (3). The primary motivation is to define a new class of Φ-like functions and to discuss certain analytic characterization subordinate to $\Psi[\alpha,\beta;h(z)]$ as in (4).

1.1. Crescent shaped domain

The function $h(z) = z + \sqrt{1+z^2}$ maps $\mathbb{U}$ onto lune shaped domain. If we let $h(z) = z + \sqrt{1+z^2}$ in (4), then the function maps $\mathbb{U}$ onto a starlike domain conformally (for all admissible values of α and β) and is of the form

$$\begin{aligned}\Psi[\alpha,\beta;h(z)] = 1 &+ \frac{i\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)(\beta-\alpha)}{2\pi}z + \frac{i\left(1-e^{\frac{4i\pi(1-\alpha)}{(\beta-\alpha)}}\right)(\beta-\alpha)}{8\pi}z^2 \\ &- \frac{i\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)^2\left(2+e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)(\beta-\alpha)z^3}{24\pi} \\ &- \frac{i\left(3-4e^{\frac{4i\pi(1-\alpha)}{(\beta-\alpha)}}+e^{\frac{8i\pi(1-\alpha)}{(\beta-\alpha)}}\right)(\beta-\alpha)z^4}{64\pi} + O[z]^5.\end{aligned}$$

On the impact of $\Psi[\alpha,\beta;h(z)]$ the crescent shaped domain becomes leaf-like convex domain symmetrical about the real axis, for a choice of the parameter, $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$ (see Figure 1b).

1.2. Vertical domain

If we let $h(z) = \frac{1+z}{1-z}$ in (4), then function $\Psi[\alpha,\beta;h(z)]$ reduces to

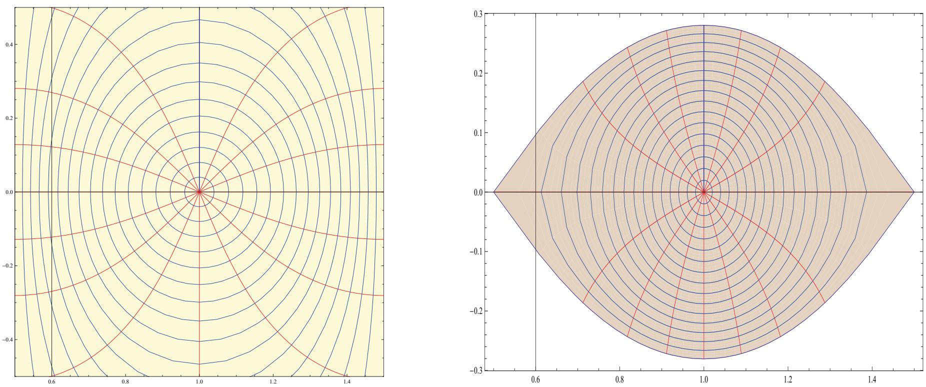
$$p_{\alpha,\beta}(z) = 1 + i\frac{\beta-\alpha}{\pi}\log\left(\frac{1-\lambda^2 z}{1-z}\right). \tag{5}$$

The function $p_{\alpha,\beta}(z)$ maps $\mathbb{U}$ onto a vertical domain, which is related to the class of analytic functions associated with the vertical domain. Note that the function $p_{\alpha,\beta}(z)$ is a conformal mapping of $\overline{\mathbb{U}} \setminus \{1, \lambda^{-2}\}$ with

$$p_{\alpha,\beta}(\mathbb{U}) = \{w \in \mathbb{C} : \alpha < \operatorname{Re}(w) < \beta\} = \Omega_{\alpha,\beta}.$$

The study by Kuroki and Owa in [9] brought a renewed interest to study the class of functions associated with vertical domain. Recently several authors defined various subclasses of analytic functions related to a vertical domain, see [2, 3, 4, 12, 10, 11, 17, 18, 19, 21, 20]. Further it is well known that

$$\begin{aligned}\Psi[\alpha,\beta;h(z)] &= 1+\sum_{n=1}^{\infty}\frac{\beta-\alpha}{n\pi}\,i\left[1-e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right]z^n \\ &= 1+\frac{i\left(1-e^{\frac{2i\pi(1-\alpha)}{\beta-\alpha}}\right)(\beta-\alpha)}{\pi}z+\frac{i\left(1-e^{\frac{4i\pi(1-\alpha)}{\beta-\alpha}}\right)(\beta-\alpha)}{2\pi}z^2+O[z]^3.\end{aligned}$$



(a) $h(z)=\frac{1+z}{1-z}$ (b) *if* $h(z)=z+\sqrt{1+z^2}$

Figure 1: *Images of unit disc under* $\Psi[\frac{1}{2},\frac{3}{2};h(z)]$

Figure 1a shows a sketch of the region if $h(z)=\frac{1+z}{1-z}$, $\alpha=\frac{1}{2}$ and $\frac{3}{2}$ in (4).

Motivated by the study of Brickman [1] and various studies of analytic function on vertical domain, we now introduce the following family of analytic functions.

DEFINITION 1. For $\mu\geqslant 0$, $0\leqslant\alpha<1<\beta$, $\Psi[\alpha,\beta;h(z)]$ be defined as in (4) and $f\in\mathcal{A}$ is said to be in $\mathcal{K}(\alpha,\beta;\mu;h)$ if it satisfies

$$\frac{\mu z^2 f''(z)+zf'(z)}{\Phi[f(z)]}\prec\Psi[\alpha,\beta;h(z)], \tag{6}$$

where Φ be analytic on $f(\mathbb{U})$ with $\Phi(0)=0$, $\Phi'(0)=1$ and $h(z)\in\mathcal{P}$ be assumed as

$$h(z)=1+L_1z+L_2z^2+L_3z^3+\ldots,\ z\in\mathbb{U},\ L_1>0. \tag{7}$$

REMARK 1. Note that normalization of Φ in Definition 1 excludes the generalization of spirallike function.

REMARK 2. If we let $\Phi[y(x)] = (1-\mu)y(x) + \mu xy'(x)$ and $h(z) = \frac{1+z}{1-z}$ in Definition 1, then

$$\begin{aligned}\mathcal{K}(\alpha,\beta;\mu;h) &\equiv \mathcal{SK}(\mu;\alpha,\beta)\\ &= \left\{ f \in \mathcal{A};\, \alpha < \operatorname{Re}\left(\frac{\mu z^2 f''(z) + zf'(z)}{(1-\mu)f(z) + \mu zf'(z)} \right) < \beta \right\}.\end{aligned}$$

REMARK 3. If we let $\Phi(w) = w$ and $h(z) = \frac{1+z}{1-z}$ in Definition 1, then

$$\begin{aligned}\mathcal{K}(\alpha,\beta;\mu;h) &\equiv \mathcal{K}(\mu;\alpha,\beta)\\ &= \left\{ f \in \mathcal{A};\, \alpha < \operatorname{Re}\left(\frac{zf'(z)}{f(z)} + \mu \frac{z^2 f''(z)}{f(z)} \right) < \beta \right\}.\end{aligned}$$

The class $\mathcal{K}(\mu;\alpha,\beta)$ was studied by Sun et al. in [19].

REMARK 4. If $\mu = 0$ in $\mathcal{K}(\mu;\alpha,\beta)$, then $\mathcal{K}(0;\alpha,\beta) \equiv \mathcal{S}^*(\alpha,\beta)$ of $\mathcal{A}$ which satisfies the inequality

$$\alpha < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta, \qquad (z \in \mathbb{U};\, 0 \leqslant \alpha < \beta < 1).$$

The class $\mathcal{S}^*(\alpha,\beta)$ studied by Kuroki and Owa [9]. Kargar et al. [3] introduced an analytic function $\mathcal{B}_\alpha(z)$ and the vertical strip $\mathbb{D}_\alpha$, as follows:

$$\mathcal{B}_\alpha(z) = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right), \qquad \left(z \in \mathbb{U};\, \frac{\pi}{2} \leqslant \alpha < \pi\right),$$

and

$$\mathbb{D}_\alpha = \left\{ w : \frac{\alpha-\pi}{2\sin\alpha} < \operatorname{Re}(w) < \frac{\alpha}{2\sin\alpha} \right\},$$

respectively. The function $\mathcal{B}_\alpha(z)$ is convex and univalent in $\mathbb{U}$. In addition, $\mathcal{B}_\alpha(z)$ maps U onto $\mathbb{D}_\alpha$, or onto the convex hull of three points (one of which may be that point at infinity) on the boundary of $\mathbb{D}_\alpha$. It can be easily seen that for appropriate choice of λ, (4) reduces to the function $\mathcal{B}_\alpha(z)$.

Recently, the problems related to coefficient estimates including the Fekete-Szegő inequalities are core of attraction for many well-known mathematicians see for example [6, 7, 15]. We are essentially motivated by these recent research going on, in the following sections we give certain geometrical interpretation and determine coefficient estimates, Fekete-Szegő problem for $f \in \Psi[\alpha,\beta;h(z)]$. Moreover we obtained coefficient estimates and Fekete-Szegő result for f^{-1} also we gave applications of our main results as corollaries.

2. Coefficient estimates

We begin with the following lemma.

LEMMA 1. [14, Theorem VII] (also see [5]) *Let* $f(z)=\sum_{k=1}^{\infty} a_k z^k$ *be analytic in* $\mathbb{U}$ *and* $g(z)=\sum_{k=1}^{\infty} b_k z^k$ *be analytic and convex univalent in* $\mathbb{U}$. *If* $f(z)\prec g(z)$, *then* $|a_k|\leqslant |b_1|$ *for* $k=1,2,\ldots$.

LEMMA 2. *Let the function* $\Psi[\alpha,\beta;h(z)]=1+i\frac{\beta-\alpha}{\pi}\log\left[\frac{h(z)(1-\lambda^2)+(1+\lambda^2)}{2}\right]$ *be convex univalent in* $\mathbb{U}$ *where the function* h *is defined as in* (7). *If* $\ell(z)=1+\sum_{k=1}^{\infty}\ell_k z^k$ *is analytic in* $\mathbb{U}$ *and holds the following condition*

$$\ell(z)\prec 1+i\frac{\beta-\alpha}{\pi}\log\left[\frac{h(z)(1-\lambda^2)+(1+\lambda^2)}{2}\right], \tag{8}$$

then

$$|\ell_k|\leqslant \frac{L_1(\beta-\alpha)}{\pi}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right), \quad (k\geqslant 1;\, L_1>0). \tag{9}$$

Proof. If the function h has the power series expansion (7), then

$$\begin{aligned}\Psi[\alpha,\beta;h(z)]=1&+\frac{i\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)}{2\pi}z\\&+\frac{i\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)(\beta-\alpha)}{2\pi}\left[L_2-\frac{L_1^2}{4}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)\right]z^2\\&+\frac{i\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)(\beta-\alpha)}{2\pi}\left[L_3-\frac{L_1L_2}{2}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)\right.\\&\left.+\frac{L_1^3}{12}\left(1-2e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}+e^{\frac{4i\pi(1-\alpha)}{(\beta-\alpha)}}\right)\right]z^3+O[z]^4,\ \ z\in\mathbb{U}.\end{aligned} \tag{10}$$

Since the subordination relation is invariant regarding a translation, the assumption (8) is equivalent to

$$\ell(z)-1\prec\Psi[\alpha,\beta;h(z)]-1.$$

Also, because the convexity of $\Psi[\alpha,\beta;h(z)]$ implies the convexity of $\Psi[\alpha,\beta;h(z)]-1$, from Lemma 1 it follows the conclusion (9). □

The function $\Psi[\alpha,\beta;h(z)]$ need not be convex univalent in $\mathbb{U}$. However, it easy to find a function $h(z)$ so that $\Psi[\alpha,\beta;h(z)]$ is convex univalent in $\mathbb{U}$ (see Subsection 1.2). Now we obtain coefficient inequality for the class $\mathcal{K}(\alpha,\beta;\mu;h)$ when $\Psi[\alpha,\beta;h(z)]$ is convex univalent in $\mathbb{U}$.

THEOREM 1. *Let* $\Psi[\alpha,\beta;h(z)]$ *be convex univalent in* $\mathbb{U}$*. If* $f\in\mathcal{K}(\alpha,\beta;\mu;h)$*, then*

$$|a_2|\leqslant\frac{L_1(\beta-\alpha)}{\pi\,|2\,[1+\mu]-\Lambda_2|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right).$$

and for $k=2,3,4,\ldots$

$$\begin{aligned}|a_k|\leqslant{}&\frac{L_1(\beta-\alpha)}{\pi\,|k\,[1+\mu(k-1)]-\Lambda_k|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\\&\times\prod_{j=2}^{k-1}\left[1+\frac{|L_1,\Lambda_j|(\beta-\alpha)}{\pi\left|j\,[1+\mu(j-1)]-\Lambda_j\right|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\right].\end{aligned}\tag{11}$$

Proof. By the definition of $\mathcal{K}(\alpha,\beta;\mu;h)$, we have

$$\frac{\mu z^2f''(z)+zf'(z)}{\Phi[f(z)]}=p(z)\prec1+i\frac{\beta-\alpha}{\pi}\log\left[\frac{h(z)(1-\lambda^2)+(1+\lambda^2)}{2}\right],\tag{12}$$

where $p(z)\in\mathcal{P}$. Suppose

$$\Phi[f(z)]=z+\sum_{k=2}^{\infty}B_kz^k\quad(B_1=1),$$

equation (12) can be equivalently rewritten as

$$z+\sum_{k=2}^{\infty}(k\,[1+\mu(k-1)]\,a_k-B_k)\,z^k=\left[z+\sum_{k=2}^{\infty}B_kz^k\right]\left[\sum_{k=1}^{\infty}p_kz^k\right],$$

where the coefficients B_k are in terms of a_k which depends on the definition of Φ. On equating the coefficient of z^k, we get

$$(k\,[1+\mu(k-1)]\,a_k-B_k)=p_{k-1}+p_{k-2}B_2+\cdots+p_1B_{k-1}.$$

On computation, we have

$$|k\,[1+\mu(k-1)]\,a_k-B_k|\leqslant\sum_{j=1}^{k-1}|B_{k-j}||p_j|.\tag{13}$$

From (12), it implies that $p(z)\prec\Psi[\alpha,\beta;h(z)]$. By using Lemma 2, we have $|p_j|\leqslant\frac{L_1(\beta-\alpha)}{\pi}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)$, $(\forall j\geqslant1)$. Now using the upper bound of $|p_j|$ in the inequality (13), we have (for $B_1=1$)

$$|k\,[1+\mu(k-1)]\,a_k-B_k|\leqslant\frac{L_1(\beta-\alpha)}{\pi}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\sum_{j=1}^{k-1}|B_j|.\tag{14}$$

Now we consider the case that the function $\Phi[f(z)]$ is linear in $f(z)$ with no restriction on the order of derivatives (That is, by linear we meant that $\Phi[f(z)]$ does not contain $[f(z)]^2, [f(z)]^3, \ldots$). Then without loss of generality, we can assume that $B_k = \Lambda_k a_k$, where Λ_k is complex. Hence we can rewrite (14) as

$$|a_k| \leqslant \frac{L_1(\beta-\alpha)}{\pi\,|k[1+\mu(k-1)]-\Lambda_k|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\sum_{j=1}^{k-1}|\Lambda_j a_j|. \tag{15}$$

Taking $k=2$, in (15), we get

$$|a_2| \leqslant \frac{L_1(\beta-\alpha)}{\pi\,|2[1+\mu]-\Lambda_2|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right).$$

Now to the establish the inequality for $k=3,4,\ldots$ we have to prove that

$$\begin{aligned}
&\frac{L_1(\beta-\alpha)}{\pi\,|k[1+\mu(k-1)]-\Lambda_k|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\sum_{j=1}^{k-1}|\Lambda_j a_j| \\
&\leqslant \frac{L_1(\beta-\alpha)}{\pi\,|k[1+\mu(k-1)]-\Lambda_k|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right) \\
&\quad\times\prod_{j=2}^{k-1}\left[1+\frac{|L_1\Lambda_j|(\beta-\alpha)}{\pi\left|j[1+\mu(j-1)]-\Lambda_j\right|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\right].
\end{aligned} \tag{16}$$

It is obvious that above inequality is true for $k=3$. Let us assume that the inequality (16) holds for $k=m$. Now let $k=m+1$ in (15)

$$\begin{aligned}
|a_{m+1}| &\leqslant \frac{L_1(\beta-\alpha)}{\pi\,|(m+1)[1+\mu m]-\Lambda_{m+1}|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\sum_{j=1}^{m}|\Lambda_j a_j| \\
&= \frac{L_1(\beta-\alpha)}{\pi\,|(m+1)[1+\mu m]-\Lambda_{m+1}|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\left[\sum_{j=1}^{m-1}|\Lambda_j a_j|+|\Lambda_m a_m|\right] \\
&\leqslant \frac{L_1(\beta-\alpha)}{\pi\,|(m+1)[1+\mu m]-\Lambda_{m+1}|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right) \\
&\quad\times\prod_{j=2}^{m-1}\left[1+\frac{|L_1\Lambda_j|(\beta-\alpha)}{\pi\left|j[1+\mu(j-1)]-\Lambda_j\right|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\right] \\
&\quad+\frac{L_1(\beta-\alpha)}{\pi\,|(m+1)[1+\mu m]-\Lambda_{m+1}|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right) \\
&\quad\times\frac{L_1(\beta-\alpha)}{\pi\,|m[1+\mu(m-1)]-\Lambda_m|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right) \\
&\quad\times\prod_{j=2}^{m-1}\left[1+\frac{|L_1\Lambda_j|(\beta-\alpha)}{\pi\left|j[1+\mu(j-1)]-\Lambda_j\right|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\right] \\
&\quad\times\frac{L_1(\beta-\alpha)}{\pi\,|(m+1)[1+m\mu]-\Lambda_{m+1}|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right) \\
&\quad\times\prod_{j=2}^{m}\left[1+\frac{|L_1\Lambda_j|(\beta-\alpha)}{\pi\left|j[1+\mu(j-1)]-\Lambda_j\right|}\sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\right].
\end{aligned}$$

Hence we have established that the inequality (16) is true for $k=m+1$. Hence, the estimate (11) for $|a_k|$ $(k=3,4,5,\ldots)$ follows. Hence the proof of Theorem 1. $\square$

COROLLARY 1. [19] (Theorem 3.1) *If $f \in \mathcal{A}$ satisfies the inequality*

$$\alpha < \operatorname{Re}\left(\frac{zf'(z)}{f(z)} + \mu \frac{z^2 f''(z)}{f(z)}\right) < \beta, \quad (0 \leqslant \alpha < 1 < \beta),$$

then

$$|a_2| \leqslant \frac{2(\beta-\alpha)}{\pi(2\mu+1)} \sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right).$$

and for $k=2,3,4,\ldots$

$$|a_k| \leqslant \frac{2(\beta-\alpha)}{\pi(k-1)[1+k\mu]} \sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right) \prod_{j=2}^{k-1}\left[1+\frac{2(\beta-\alpha)}{\pi(j-1)[1+j\mu]} \sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\right].$$

Proof. If we let $h(z)=(1+z)/(1-z)$ and $\Phi(w)=w$ in Theorem 1, then $\Psi[\alpha,\beta;(1+z)/(1-z)]$ is univalent in $\mathbb{U}$ and maps $\mathbb{U}$ onto a convex domain. Substituting the value $L_1=2$ and $\Lambda_n=1$, $(\forall n \geqslant 1)$ in Theorem 1, we obtain the assertion of the Corollary. $\square$

COROLLARY 2. (see Corollary 3.2 in [19]) *If $f \in \mathcal{S}^*(\alpha,\beta)$, then*

$$|a_k| \leqslant \prod_{j=2}^{k}\left[\frac{k-2+|M_1|}{k-1}\right], \quad (k=2,3,\ldots).$$

where $M_1 = \frac{i(\beta-\alpha)}{\pi}\left(1-e^{\frac{2i\pi(1-\alpha)}{\beta-\alpha}}\right)$.

If we let $\Phi[y(x)]=(1-\mu)y(x)+\mu x y'(x)$ in Theorem 1, we have

COROLLARY 3. *Let $\Psi[\alpha,\beta;h(z)]$ be convex univalent in $\mathbb{U}$. If $f \in \mathcal{A}$ satisfies the inequality*

$$\frac{\mu z^2 f''(z)+zf'(z)}{(1-\mu)f(z)+\mu z f'(z)} \prec \Psi[\alpha,\beta;h(z)],$$

then

$$|a_2| \leqslant \frac{L_1(\beta-\alpha)}{\pi(1+\mu)} \sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right).$$

and for $k=2,3,4,\ldots$

$$|a_k| \leqslant \frac{L_1(\beta-\alpha)}{\pi(k-1)[1+\mu(k-1)]} \sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right) \prod_{j=2}^{k-1}\left[1+\frac{L_1(\beta-\alpha)}{\pi(j-1)} \sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)\right].$$

3. Fekete-Szegő inequality

Now we will find the solution to the Fekete-Szegő problem for $f \in \mathcal{K}(\alpha,\beta;\mu;h)$.

LEMMA 3. [8] *If* $\vartheta(z) = 1 + \sum_{k=1}^{\infty} \vartheta_k z^k \in \mathcal{P}$*, and* v *is complex number, then*

$$\left|\vartheta_2 - v\vartheta_1^2\right| \leqslant 2\max\{1;|2v-1|\},$$

and the result is sharp for the functions

$$\vartheta_1(z) = \frac{1+z}{1-z} \quad \text{and} \quad \vartheta_2(z) = \frac{1+z^2}{1-z^2}.$$

We find the following result when $\Phi(w)$ to be linear in w, with no restrictions in the order of derivatives.

THEOREM 2. *If* $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \in \mathcal{K}(\alpha,\beta;\mu;h)$*, then for all* $\rho \in \mathbb{C}$ *we have*

$$\left|a_3 - \rho a_2^2\right| \leqslant \frac{L_1(\beta-\alpha)\sin\left[\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right]}{2\pi\,|3(1+2\mu)-\Lambda_3|}\max\{1,|2\mathcal{Q}_1-1|\},$$

where $\mathcal{Q}_1$ *is given by*

$$\begin{aligned}\mathcal{Q}_1 = \frac{1}{2}\Bigg(1 - \frac{L_2}{L_1} + \frac{L_1\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)}{4} &- \frac{i\Lambda_2\left[3(1+2\mu)-\Lambda_3\right]}{2\pi\left[2(1+\mu)-\Lambda_2\right]}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)\\ &+ \frac{\rho\left[3(1+2\mu)-\Lambda_3\right]}{2\pi\left[2(1+\mu)-\Lambda_2\right]^2}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)\Bigg).\end{aligned} \tag{17}$$

The inequality is sharp for each $\rho \in \mathbb{C}$.

Proof. As $f \in \mathcal{K}(\alpha,\beta;\mu;h)$, by (6) we have

$$\frac{\mu z^2 f''(z) + z f'(z)}{\Phi[f(z)]} \prec \Psi[\alpha,\beta;h\{w(z)\}]. \tag{18}$$

Thus, let $\vartheta \in \mathcal{P}$ be of the form $\vartheta(z) = 1 + \sum_{k=1}^{\infty} \vartheta_n z^n$ and defined by

$$\vartheta(z) = \frac{1+w(z)}{1-w(z)}, \quad z \in \mathbb{U}.$$

On computation, we have

$$w(z) = \frac{1}{2}\vartheta_1 z + \frac{1}{2}\left(\vartheta_2 - \frac{1}{2}\vartheta_1^2\right)z^2 + \frac{1}{2}\left(\vartheta_3 - \vartheta_1\vartheta_2 + \frac{1}{4}\vartheta_1^3\right)z^3 + \cdots,\ z \in \mathbb{U}.$$

The right hand side of (18)

$$\begin{aligned}\Psi[\alpha,\beta;h\{w(z)\}] = 1 &+ \frac{i\vartheta_1}{4\pi}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)z \\ &+\frac{i}{4\pi}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha) \\ &\times\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{L_2}{L_1}+\frac{L_1\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)}{4}\right)\right]z^2+\ldots. \end{aligned} \tag{19}$$

Without loss of generality, we can assume that $\Phi[f(z)] = z+\sum_{n=2}^{\infty}B_n z^n$. Then left hand side of (18) will be of the form

$$\begin{aligned}\frac{\mu z^2 f''(z)+zf'(z)}{\Phi[f(z)]} &= 1+[2(1+\mu)a_2-B_2]z+\{[3(1+2\mu)a_3-B_3] \\ &\quad -\left[2(1+\mu)a_2B_2-B_2^2\right]\}z^2+\cdots. \end{aligned}\tag{20}$$

As in Theorem 11, we can let $B_n = \Lambda_n a_n$ $(\forall n = 2,3,\ldots)$ for some Λ_n which may be real or complex. From (20) and (19), we obtain

$$a_2 = \frac{i\vartheta_1}{4\pi[2(1+\mu)-\Lambda_2]}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha) \tag{21}$$

and

$$\begin{aligned}a_3 = \frac{iL_1\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)(\beta-\alpha)}{4\pi[3(1+2\mu)-\Lambda_3]}&\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{L_2}{L_1}+\frac{L_1\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)}{4}\right.\right. \\ &\left.\left.-\frac{i\Lambda_2[3(1+2\mu)-\Lambda_3]}{2\pi[2(1+\mu)-\Lambda_2]}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)\right)\right]. \end{aligned}\tag{22}$$

Now to prove the Fekete-Szegő inequality for the class $\mathcal{K}(\alpha,\beta;\mu;h)$, we consider

$$\begin{aligned}\left|a_3-\rho a_2^2\right| = \Bigg|&\frac{iL_1\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)(\beta-\alpha)}{4\pi[3(1+2\mu)-\Lambda_3]}\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{L_2}{L_1}+\frac{L_1\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)}{4}\right.\right. \\ &\left.\left.-\frac{i\Lambda_2[3(1+2\mu)-\Lambda_3]}{2\pi[2(1+\mu)-\Lambda_2]}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)\right)\right] \\ &+\frac{\rho\vartheta_1^2}{16\pi^2[2(1+\mu)-\Lambda_2]^2}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)^2L_1^2(\beta-\alpha)^2\Bigg| \end{aligned}\tag{23}$$

$$= \left| \frac{iL_1\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)(\beta-\alpha)}{4\pi\left[3(1+2\mu)-\Lambda_3\right]} \left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1-\frac{L_2}{L_1}+\frac{L_1\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)}{4} \right.\right.\right.$$
$$\left. -\frac{i\Lambda_2\left[3(1+2\mu)-\Lambda_3\right]}{2\pi\left[2(1+\mu)-\Lambda_2\right]}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)\right]$$
$$\left.\left. +\frac{\rho\left[3(1+2\mu)-\Lambda_3\right]}{2\pi\left[2(1+\mu)-\Lambda_2\right]^2}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)\right)\right|$$
$$\leqslant \frac{L_1(\beta-\alpha)\sin\left[\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right]}{2\pi\left|3(1+2\mu)-\Lambda_3\right|}\left[2+\frac{|\vartheta_1|^2}{4}\left(\left|\frac{L_2}{L_1}-\frac{L_1\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)}{4}\right.\right.\right.$$
$$+\frac{i\Lambda_2\left[3(1+2\mu)-\Lambda_3\right]}{2\pi\left[2(1+\mu)-\Lambda_2\right]}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)$$
$$\left.\left.\left.-\frac{\rho\left[3(1+2\mu)-\Lambda_3\right]}{2\pi\left[2(1+\mu)-\Lambda_2\right]^2}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)\right|-2\right)\right] \tag{24}$$

Denoting

$$\mathcal{T} := \left|\frac{L_2}{L_1}-\frac{L_1\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)}{4}+\frac{i\Lambda_2\left[3(1+2\mu)-\Lambda_3\right]}{2\pi\left[2(1+\mu)-\Lambda_2\right]}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)\right.$$
$$\left.-\frac{\rho\left[3(1+2\mu)-\Lambda_3\right]}{2\pi\left[2(1+\mu)-\Lambda_2\right]^2}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)\right|,$$

if $\mathcal{T} \leqslant 2$, from (24) we obtain

$$\left|a_3-\rho a_2^2\right| \leqslant \frac{L_1(\beta-\alpha)\sin\left[\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right]}{\pi\left|3(1+2\mu)-\Lambda_3\right|}. \tag{25}$$

Further, if $\mathcal{T} \geqslant 2$ from (24) we deduce

$$\left|a_3-\rho a_2^2\right| \leqslant \frac{L_1(\beta-\alpha)\sin\left[\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right]}{\pi\left|3(1+2\mu)-\Lambda_3\right|}\left(\left|\frac{L_2}{L_1}-\frac{L_1\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)}{4}\right.\right. \tag{26}$$
$$+\frac{i\Lambda_2\left[3(1+2\mu)-\Lambda_3\right]}{2\pi\left[2(1+\mu)-\Lambda_2\right]}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)$$
$$\left.\left.-\frac{\rho\left[3(1+2\mu)-\Lambda_3\right]}{2\pi\left[2(1+\mu)-\Lambda_2\right]^2}\left(1-e^{\frac{2i\pi(1-\alpha)}{(\beta-\alpha)}}\right)L_1(\beta-\alpha)\right|\right). \tag{27}$$

Equality for (25) holds if $\vartheta_1 = 0$, $\vartheta_2 = 2$. From Lemma 3 we have $\vartheta(z^2) = \vartheta_2(z) = \dfrac{1+z^2}{1-z^2}$. Therefore, the extremal function of the class $\mathcal{K}(\alpha,\beta;\,\mu;h)$ is given by

$$\frac{\mu z^2 f''(z)+zf'(z)}{\Phi[f(z)]} = \Psi[\alpha,\beta;h(z^2)].$$

Similarly, the equality for (25) holds if $\vartheta_2 = 2$. Equivalently, by Lemma 3 we have $\vartheta(z) = \vartheta_1(z) = \dfrac{1+z}{1-z}$. Therefore, the extremal function in $\mathcal{K}(\alpha,\beta;\,\mu;h)$ is given by

$$\frac{\mu z^2 f''(z)+zf'(z)}{\Phi[f(z)]} = \Psi[\alpha,\beta;h_1(z)],$$

and the proof of the theorem is complete. $\square$

Letting $\Phi(w) = w$ and $h(z) = (1+z)/(1-z)$ in Theorem 2, we get

COROLLARY 4. [19] *Let $f \in \mathcal{K}(\mu;\,\alpha,\beta)$. Then, for a complex number ρ,*

$$|a_3-\rho a_2^2| \leqslant \frac{(\beta-\alpha)\sin\left[\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right]}{\pi(1+3\mu)} \max\left\{1, \left|\frac{X_2}{X_1} - \frac{2\rho(3\mu+1)-(2\mu+1)}{(2\mu+1)^2}X_1\right|\right\},$$

where $X_n = \frac{\beta-\alpha}{n\pi}\, i\left[1-e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right]$.

Letting $\mu = 0$ in Corollary 4, we have the following.

COROLLARY 5. *Let $f \in \mathcal{S}^*(\alpha,\beta)$. Then, for a complex number ρ,*

$$|a_3-\rho a_2^2| \leqslant \frac{(\beta-\alpha)\sin\left[\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right]}{\pi(1+3\mu)} \max\left\{1, \left|\frac{X_2}{X_1} - (2\rho-1)X_1\right|\right\},$$

where $X_n = \frac{\beta-\alpha}{n\pi}\, i\left[1-e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right]$.

4. Coefficient inequalities for the function f^{-1}

We let $\mathcal{S}$ to denote the class of functions univalent in $\mathbb{U}$. It is well-known from Koebe $1/4$-quarter theorem that every function f of the form (1) in $\mathcal{S}$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z, z \in \mathbb{U}$ and $f(f^{-1}(w)) = w$, $(|w| < r; r \geqslant 1/4)$ where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)z^2 - \left(5a_2^2 - 5a_2a_3 + a_4\right)w^3 + \cdots. \tag{28}$$

Since $f'(0) = 1 \neq 0$ for all $f \in \mathcal{K}(\alpha,\beta;\,\mu;h)$ and $f(0) = 0$, there exist an inverse function in some small disk with center at $w = 0$. Next result is valid only for the functions in $\mathcal{K}(\alpha,\beta;\,\mu;h)$ which are univalent.

THEOREM 3. *Let* $f \in \mathcal{K}(\alpha, \beta; \mu; h)$ *and let* f^{-1} *be the inverse of* f *defined by*

$$f^{-1}(w) = w + \sum_{k=2}^{\infty} b_k w^k, \quad (|w| < r; r \geqslant 1/4),$$

then

$$|b_2| \leqslant \frac{L_1(\beta - \alpha)}{\pi\,|2\,[1+\mu] - \Lambda_2|} \sin\left(\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right)$$

and

$$|b_3| \leqslant \frac{L_1(\beta - \alpha)\sin\left[\frac{\pi(1-\alpha)}{(\beta-\alpha)}\right]}{2\pi\,|3(1+2\mu) - \Lambda_3|} \max\{1, |2\mathcal{V}_1 - 1|\}$$

where $\mathcal{V}_1 = [\mathcal{Q}_1]_{with\ \rho=2}$, $\mathcal{Q}_1$ *is defined as in* (17).

Proof. From (1) and (28), we have

$$b_2 = -a_2 \quad \text{and} \quad b_3 = 2a_2^2 - a_3.$$

The estimate for $|b_2| = |a_2|$ follows immediately from (21). Letting $\rho = 2$ in (23), we get the estimate $|b_3|$. □

REMARK 5. For different choices of the function Φ and h in the Definition 1, the function class $\mathcal{K}(\alpha, \beta; \mu; h)$ reduces to well-known class like starlike, convex, and spirallike. So our main results have lots of applications, here we restricted ourselves to pointing out only few of them.

Conclusion

Very few articles have appeared on Φ-like functions, as challenges in handling a broad class of analytic functions is enormous and it is computationally tedious. In this present study, we have provided a very broad generalization for a class of starlike functions associated with the vertical domain. To make this study more versatile, we have discussed impact of the vertical domain on conic region. Further, we have provided lots scopes for future research pertaining to this study.

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