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GENERALIZED INTEGRATION OPERATORS FROM WEIGHTED BERGMAN SPACES INTO GENERAL FUNCTION SPACES

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Abstract. This article studies the boundedness of the inclusion mapping from weighted Bergman spaces A^p_α into a class of tent type space $\mathcal{T}_s^{p,n}(\mu)$. As an application, the boundedness, compactness and essential norm of generalized integral operators $T_g^{n,k}$ and $S_g^{n,0}$ from A^p_α to general function spaces are also investigated.

1. Introduction

We denote by $\mathbb{D}$ and $\partial\mathbb{D}$ the unit disk and its boundary in the complex plane $\mathbb{C}$, respectively. Let $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. For $0<p<\infty$ and $\alpha>-1$, the weighted Bergman space A^p_α is the set of all $f\in H(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\alpha}=(\alpha+1)\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^\alpha dA(z)<\infty.$$

When $\alpha=0$, we denote A^p_α by A^p. The Bloch space $\mathscr{B}$ is the space of all $f\in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{B}}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty.$$

The little Bloch space $\mathscr{B}_0$, is the set consisting of all $f\in H(\mathbb{D})$ such that $\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)|=0$. Let H^∞ denote the space of all bounded analytic functions with the supremum norm $\|f\|_{H^\infty}=\sup_{z\in\mathbb{D}}|f(z)|$.

Let $0<p,s<\infty$, $-2<q<\infty$. The general function space $F(p,q,s)$, which was introduced by Zhao in [42], consists of all $f\in H(\mathbb{D})$ such that

$$\|f\|^p_{F(p,q,s)}=|f(0)|^p+\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q(1-|\sigma_a(z)|^2)^s dA(z)<\infty.$$

Here $\sigma_a(z)=\frac{a-z}{1-\bar{a}z}$. $F(p,q,s)$ is a Banach space under the norm $\|\cdot\|_{F(p,q,s)}$ when $p\geqslant 1$. It is easy to see that $F(p,p,0)$ is just the Bergman space. When $p=2$ and $q=0$, it gives the Q_s space. Especially, Q_1 is the $BMOA$ space, the space of analytic

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functions in the Hardy space whose boundary functions have bounded mean oscillation. Also, it is known that $F(p,q,s)$ contains only constant functions if $s+q \leqslant -1$. For some other results on the space their generalizations and operators on them see also [15, 26, 28, 38].

Let $g \in H(\mathbb{D})$. The Volterra integral operator T_g and its companion operator I_g with symbol g are defined by

$$T_g f(z) = \int_0^z g'(w) f(w) dw, \qquad I_g f(z) = \int_0^z g(w) f'(w) dw, \quad f \in H(\mathbb{D}),$$

respectively. The multiplication operator M_g is defined by $M_g f(z) = f(z)g(z)$. It is easy to see that

$$M_g f(z) = f(0)g(0) + I_g f(z) + T_g f(z).$$

Pommerenke [23] introduced the operator T_g and showed that T_g is bounded on H^2 if and only if g belongs to the space $BMOA$. For some generalizations on H^p spaces see [1, 2, 6, 8, 14, 29]. For some results on the Bergman-type spaces see [3, 8, 16]. Further results about Volterra integral operators on analytic function spaces on the unit disk, as well as the unit ball and unit polydisk in $\mathbb{C}^n$ can be found [11, 12, 13, 15, 17, 18, 19, 21, 22, 24, 26, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39] as well as in their respective references.

We define the Carleson box, denoted by $S(I)$, based on the arc I which is a subset of $\partial\mathbb{D}$, as follows:

$$S(I) = \left\{ z \in \mathbb{D} : 1 - |I| \leqslant |z| < 1 \ \text{ and } \ \frac{z}{|z|} \in I \right\}.$$

If I equals the unit circle $\partial\mathbb{D}$, we let $S(I)$ be the entire unit disk $\mathbb{D}$. For a positive Borel measure μ on $\mathbb{D}$ and $0 < s < \infty$, we say that μ is an s-Carleson measure if

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty.$$

The classical Carleson measure is obtained when $s = 1$.

Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and nonnegative integers, respectively. Let $0 < p, s < \infty$, $n \in \mathbb{N}_0$ and μ be a positive Borel measure on $\mathbb{D}$. We define $\mathcal{T}_s^{p,n}(\mu)$ as the set of functions $f \in H(\mathbb{D})$ satisfying the condition (see [25])

$$\sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} \left| f^{(n)}(z)(1-|z|^2)^n \right|^p d\mu(z) < \infty.$$

Suppose $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ satisfy $0 \leqslant k < n$, and $g \in H(\mathbb{D})$. Chalmoukis introduced the operator $T_g^{n,k}$ in [7], defined by

$$T_g^{n,k} f(z) = I^n (f^{(k)}(z) g^{(n-k)}(z)), \qquad f \in H(\mathbb{D}),$$

where $If(z) = \int_0^z f(w)dw$. In particular, $T_g^{1,0} f = T_g f$ for any $f \in H(\mathbb{D})$.

Qian and the author of this paper introduced and studied the operator $S_g^{n,0}$ in [25], where

$$S_g^{n,0}f(z) = I^n(f^{(n)}(z)g(z)).$$

In particular, $S_g^{1,0}f = I_g f$.

Chalmoukis investigated the boundedness of the operator $T_g^{n,k}$ on Hardy spaces H^p in [7]. Specifically, he demonstrated that $T_g^{n,k}: H^p \to H^p$ is bounded if and only if $g \in \mathscr{B}$ when $k \geqslant 1$. Meanwhile, $T_g^{n,k}: H^p \to H^q$ is bounded if and only if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\frac{1}{q}-\frac{1}{p}+n-k}|g^{(n-k)}(z)| < \infty$$

when $0<p<q<\infty$. In [9], Du, Li, and Qu studied the boundedness, compactness, and essential norm of the operator $T_g^{n,k}$ on weighted Bergman spaces induced by doubling weights. Further details related to the operator $T_g^{n,k}$, see [7, 9, 25].

The purpose of this paper is to establish that the inclusion mapping $I_d : A_\alpha^p \to \mathscr{T}_s^{p,n}(\mu)$ is bounded if and only if

$$\sup_{I\subset\partial\mathbb{D}} \frac{\int_{S(I)}(1-|z|^2)^{pn}d\mu(z)}{|I|^{pn+2+\alpha+s}} < \infty. \tag{1.1}$$

This result is then applied to characterize the boundedness of $T_g^{n,k}$ and $S_g^{n,0}$, which act from A_α^p to $F(p,p+\alpha,s)$. Additionally, we investigate the essential norm and compactness of $T_g^{n,k}$ and $S_g^{n,0}$ when act from A_α^p to $F(p,p+\alpha,s)$.

Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces, with $T: X\to Y$ being a bounded linear operator. The essential norm of $T: X\to Y$, denoted as $\|T\|_{e,X\to Y}$, can be defined as:

$$\|T\|_{e,X\to Y} = \inf_K\{\|T-K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

It is easy to observe that $T: X\to Y$ is compact if and only if $\|T\|_{e,X\to Y}=0$.

Throughout this paper, we say that $f\lesssim g$ if there exists a constant C such that $f\leqslant Cg$. The symbol $f\approx g$ means that $f\lesssim g\lesssim f$.

2. Boundedness of $I_d : A_\alpha^p \to \mathscr{T}_s^{p,n}(\mu)$

In this section, our objective is to investigate the boundedness of the inclusion mapping $I_d : A_\alpha^p \to \mathscr{T}_s^{p,n}(\mu)$. To accomplish this task, we will introduce several lemmas that will be utilized throughout this paper.

LEMMA 1. [28, Theorem 3.2] *Let $-2<q<\infty$, $0<s<\infty$, $1<p<\infty$ and $n\in\mathbb{N}$. Then the following statements are equivalent.*

(i) $f\in F(p,q,s)$;

(ii)

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f^{(n)}(z)|^p(1-|z|^2)^{p(n-1)+q}(1-|\sigma_a(z)|^2)^s dA(z) < \infty;$$

(iii)

$$\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f^{(n)}(z)|^p(1-|z|^2)^{p(n-1)+q+s}dA(z)<\infty.$$

REMARK 1. Let

$$\|f\|^p_{F(p,q,s,1)}=\sum_{j=0}^{n-1}|f^{(j)}(0)|^p+\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f^{(n)}(z)|^p(1-|z|^2)^{p(n-1)+q}(1-|\sigma_a(z)|^2)^s dA(z),$$

$$\|f\|^p_{F(p,q,s,2)}=\sum_{j=0}^{n-1}|f^{(j)}(0)|^p+\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f^{(n)}(z)|^p(1-|z|^2)^{p(n-1)+q+s}dA(z).$$

From the proof of Theorem 3.2 of [28], we see that

$$\|f\|_{F(p,q,s)}\approx\|f\|_{F(p,q,s,1)}\approx\|f\|_{F(p,q,s,2)}.$$

LEMMA 2. [43, Theorem 4.28] *Let $-1<\alpha<\infty$, $1<p<\infty$ and $n\in\mathbb{N}$. Then $f\in A^p_\alpha$ if and only if*

$$\int_{\mathbb{D}}|f^{(n)}(z)|^p(1-|z|^2)^{pn+\alpha}dA(z)<\infty.$$

Moreover,

$$\|f\|^p_{A^p_\alpha}\approx\sum_{j=0}^{n-1}|f^{(j)}(0)|^p+\int_{\mathbb{D}}|f^{(n)}(z)|^p(1-|z|^2)^{pn+\alpha}dA(z).$$

LEMMA 3. [43, Theorem 5.4] *If f is analytic in $\mathbb{D}$ and $n\geqslant 2$, then $f\in\mathscr{B}$ if and only if the function $\left(1-|z|^2\right)^n f^{(n)}(z)$ is bounded in $\mathbb{D}$. Moreover, there exists a constant $C>0$ such that*

$$C^{-1}\|f\|_{\mathscr{B}}\leqslant\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^n\left|f^{(n)}(z)\right|\leqslant C\|f\|_{\mathscr{B}}$$

for all f with $f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0$.

The following result comes from [27, Theorem 4.1.2]. When $\beta=2$, it was proved in [40]. When $n=1$, it was proved in [41].

LEMMA 4. *Let $1<p<\infty$, $1<\beta<\infty$ and $n\in\mathbb{N}$. Then $g\in\mathscr{B}$ if and only if*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g^{(n)}(z)|^p(1-|z|^2)^{pn-2}(1-|\sigma_a(z)|^2)^\beta dA(z)<\infty.$$

LEMMA 5. *Let $-1<\alpha<\infty$, $0<s<\infty$ and $1<p<\infty$. Then*

$$\|f\|^p_{F(p,p+\alpha,s)}\lesssim\|f\|^p_{A^p_\alpha}.$$

Proof. By Lemma 2, we get

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p+\alpha}(1-|\sigma_a(z)|^2)^s dA(z)\\ \leqslant&\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p+\alpha}dA(z)\\ \lesssim&\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^{\alpha}dA(z),\end{aligned}$$

as desired. □

The proof of the following result is standard. See, for example [25, Lemma 2.4]. We omit the proof here.

LEMMA 6. *Let $-1<\alpha<\infty$, $0<s<\infty$, $1<p<\infty$ and $n\in\mathbb{N}_0$. If $f\in F(p,p+\alpha,s)$, then*

$$|f^{(n)}(z)|\lesssim\frac{\|f\|_{F(p,p+\alpha,s)}}{(1-|z|^2)^{\frac{2+\alpha}{p}+n}},\quad z\in\mathbb{D}.$$

LEMMA 7. [5, Lemma 2.1] *Let μ be a positive measure on $\mathbb{D}$ and $0<s<\infty$. Then μ is a bounded s-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^s d\mu(z)<\infty.$$

Now we are in a position to state and prove our main result in this section.

THEOREM 1. *Let $1<p<\infty$, $-1<\alpha<\infty$, $0<s<\infty$ and $n\in\mathbb{N}_0$. Let μ be a positive Borel measure on $\mathbb{D}$. Then the inclusion mapping $I_d:A^p_\alpha\to\mathscr{T}_s^{p,n}(\mu)$ is bounded if and only if (1.1) holds.*

Proof. Assume first that (1.1) holds. Let $d\nu(z)=(1-|z|^2)^{pn}d\mu(z)$. We observe that

$$\sup_{I\subset\partial\mathbb{D}}\frac{\nu(S(I))}{|I|^{pn+2+\alpha+s}}<\infty,$$

which, in combination with [43, Theorem 7.4], implies that the inclusion mapping $I_d:A^p_{pn+\alpha+s}\to L^p(d\nu)$ is a bounded operator. For any arc $I\subset\partial\mathbb{D}$, let ξ be the center point of I and $w=(1-|I|)\xi$. From [10, p. 232], we see that

$$|1-\overline{w}z|\approx 1-|w|^2\approx|I|,\quad z\in S(I). \tag{2.0}$$

Let $f\in A^p_\alpha$. By Lemma 2, we see that

$$f^{(n)}\in A^p_{pn+\alpha}\subset A^p_{pn+s+\alpha}.$$

Let $l > 2s$. Using the above results and Lemma 5, we obtain

$$
\begin{aligned}
&\frac{1}{|I|^s}\int_{S(I)} |f^{(n)}(z)|^p(1-|z|^2)^{pn}d\mu(z)\\
\approx&(1-|w|^2)^{l-s}\int_{S(I)}\left|\frac{f^{(n)}(z)}{(1-\overline{w}z)^{l/p}}\right|^p d\nu(z) \qquad (\text{using } (2.0))\\
\lesssim&(1-|w|^2)^{l-s}\int_{\mathbb{D}}\frac{|f^{(n)}(z)|^p}{|1-\overline{w}z|^l}(1-|z|^2)^{np+s+\alpha}dA(z)\\
&\quad (I_d : A^p_{pn+\alpha+s} \to L^p(d\nu) \text{ is bounded})\\
\leqslant&\int_{\mathbb{D}}|f^{(n)}(z)|^p(1-|z|^2)^{pn+\alpha}\frac{(1-|z|^2)^s(1-|w|^2)^s}{|1-\overline{w}z|^{2s}}dA(z) \qquad (l>2s)\\
\lesssim&\|f\|^p_{F(p,p+\alpha,s)} \lesssim \|f\|^p_{A^p_\alpha},
\end{aligned}
$$

which implies the desired result.

Conversely, assume that the inclusion mapping $I_d : A^p_\alpha \to \mathscr{T}_s^{p,n}(\mu)$ is bounded. Using this assumption and taking $f(z) = z^n \in A^p_\alpha$, we obtain

$$\int_{\mathbb{D}}(1-|z|^2)^{pn}d\mu(z) < \infty.$$

For any arc $I \subset \partial\mathbb{D}$, let ξ be the center point of I and $w = (1-|I|)\xi$. Take

$$f_w(z) = \frac{(1-|w|^2)}{\overline{w}^n(1-\overline{w}z)^{1+\frac{2+\alpha}{p}}}. \tag{2.1}$$

By Lemma 3.10 of [43], we see that $f_w \in A^p_\alpha$. By (2.0),

$$|f_w^{(n)}(z)|^p \approx \frac{1}{|I|^{pn+2+\alpha}}, \quad z \in S(I).$$

By the assumption that the inclusion mapping $I_d : A^p_\alpha \to \mathscr{T}_s^{p,n}(\mu)$ is bounded, we have

$$
\begin{aligned}
\frac{1}{|I|^s}\int_{S(I)}|f_w^{(n)}(z)|^p(1-|z|^2)^{pn}d\mu(z) &\lesssim \|I_d f_w\|^p_{\mathscr{T}_s^{p,n}(\mu)}\\
&\lesssim \|I_d\|^p\|f_w\|^p_{A^p_\alpha} \lesssim \|f_w\|^p_{A^p_\alpha} < \infty,
\end{aligned}
$$

which implies that

$$\sup_{I\subset\partial\mathbb{D}}\frac{\int_{S(I)}(1-|z|^2)^{pn}d\mu(z)}{|I|^{pn+2+\alpha+s}} < \infty.$$

So, (1.1) holds. The proof is complete. $\square$

3. Boundedness

In this section, we provide some characterizations for the boundedness of $T_g^{n,k}$ and $S_g^{n,0}$ from A_α^p to $F(p,p+\alpha,s)$.

THEOREM 2. *Let $g\in H(\mathbb{D})$, $1<p<\infty$, $-1<\alpha<\infty$, $0<s<\infty$, $n\in\mathbb{N}$, $k\in\mathbb{N}_0$ such that $0\leqslant k<n$. Then $T_g^{n,k}:A_\alpha^p\to F(p,p+\alpha,s)$ is bounded if and only if $g\in\mathscr{B}$. Moreover,*

$$\|T_g^{n,k}\|_{A_\alpha^p\to F(p,p+\alpha,s)}\approx\sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)|.$$

Proof. Assume that $g\in\mathscr{B}$. We first consider the case $k=0$. Since $g\in\mathscr{B}$ and $s+\alpha+2>1$, by Lemma 4 we see that

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g^{(n)}(z)|^p(1-|z|^2)^{pn-2}(1-|\sigma_a(z)|^2)^{s+\alpha+2}dA(z)<\infty,$$

which implies that

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{s+2+\alpha}d\mu_g(z)<\infty,$$

where $d\mu_g(z)=|g^{(n)}(z)|^p(1-|z|^2)^{pn+s+\alpha}dA(z)$. Using Lemma 7, we see that μ_g is an $(s+2+\alpha)$-Carleson measure. Let $f\in A_\alpha^p$. From Theorem 1, we can easily deduce that

$$\begin{aligned}
&\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|(T_g^{n,0}f)^{(n)}(z)|^p(1-|z|^2)^{pn+\alpha+s}dA(z)\\
=&\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^p|g^{(n)}(z)|^p(1-|z|^2)^{pn+\alpha+s}dA(z)\\
=&\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f(z)|^pd\mu_g(z)\\
\lesssim&\|f\|_{A_\alpha^p}^p.
\end{aligned}\tag{3.1}$$

Now we consider the case $k\geqslant1$. From Lemmas 1, 3 and 5 we get

$$\begin{aligned}
&\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|(T_g^{n,k}f)^{(n)}(z)|^p(1-|z|^2)^{pn+\alpha+s}dA(z)\\
=&\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f^{(k)}(z)g^{(n-k)}(z)|^p(1-|z|^2)^{pn+\alpha+s}dA(z)\\
\lesssim&\|g\|_{\mathscr{B}}^p\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f^{(k)}(z)|^p(1-|z|^2)^{pk+\alpha+s}dA(z)\\
\lesssim&\|g\|_{\mathscr{B}}^p\|f\|_{F(p,p+\alpha,s)}^p\\
\lesssim&\|g\|_{\mathscr{B}}^p\|f\|_{A_\alpha^p}^p.
\end{aligned}\tag{3.2}$$

From (3.1) and (3.2), we see that $T_g^{n,k}: A_\alpha^p \to F(p, p+\alpha, s)$ is bounded.

Conversely, assume that $T_g^{n,k}: A_\alpha^p \to F(p, p+\alpha, s)$ is bounded. For $w \in \mathbb{D}$, we define

$$G_w(z) = \frac{(1-|w|^2)}{(1-\overline{w}z)^{1+\frac{2+\alpha}{p}}}.$$

It is easy to check that $G_w \in A_\alpha^p$ using Lemma 3.10 of [43]. Moreover,

$$G_w^{(k)}(z) = \prod_{i=1}^{k}\left(i+\frac{2+\alpha}{p}\right)\frac{\overline{w}^k(1-|w|^2)}{(1-\overline{w}z)^{1+k+\frac{2+\alpha}{p}}},$$

$$G_w^{(k)}(w) = \prod_{i=1}^{k}\left(i+\frac{2+\alpha}{p}\right)\frac{\overline{w}^k}{(1-|w|^2)^{k+\frac{2+\alpha}{p}}}.$$

Using Lemma 6, we obtain that

$$\frac{\|T_g^{n,k}G_w\|_{F(p,p+\alpha,s)}}{(1-|w|^2)^{n+\frac{2+\alpha}{p}}} \gtrsim |(T_g^{n,k}G_w)^{(n)}(w)| \gtrsim \frac{|w|^k|g^{(n-k)}(w)|}{(1-|w|^2)^{k+\frac{2+\alpha}{p}}}.$$

Thus,

$$\sup_{|w|>1/2} |g^{(n-k)}(w)|(1-|w|^2)^{n-k} < \infty.$$

It is obvious that

$$\sup_{|w|\leqslant 1/2} |g^{(n-k)}(w)|(1-|w|^2)^{n-k} < \infty.$$

Therefore,

$$\sup_{w\in\mathbb{D}} |g^{(n-k)}(w)|(1-|w|^2)^{n-k} < \infty,$$

which implies that $g \in \mathscr{B}$ by Lemma 3. The proof is complete. $\square$

THEOREM 3. *Let $1 < p < \infty$, $-1 < \alpha < \infty$, $0 < s < \infty$, $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ such that $0 \leqslant k < n$. Then $S_g^{n,0}: A_\alpha^p \to F(p, p+\alpha, s)$ is bounded if and only if $g \in H^\infty$. Moreover,*

$$\|S_g^{n,0}\|_{A_\alpha^p \to F(p,p+\alpha,s)} \approx \|g\|_{H^\infty}. \tag{3.3}$$

Proof. We first assume that $S_g^{n,0}: A_\alpha^p \to F(p, p+\alpha, s)$ is bounded. For $b \in \mathbb{D}$ and $r > 0$, let $\mathbb{D}(b, r)$ denote the Bergman metric disk centered at b with radius r. From [43], we see that

$$\frac{(1-|b|^2)^2}{|1-\overline{b}z|^4} \approx \frac{1}{(1-|z|^2)^2} \approx \frac{1}{(1-|b|^2)^2}, \quad z \in \mathbb{D}(b,r). \tag{3.4}$$

For any $w\in\mathbb{D}\setminus\{0\}$, let f_w be defined in (2.1). We have that $f_w\in A^p_\alpha$ by Lemma 3.10 of [43]. Using (3.4), we obtain

$$|f_w^{(n)}(z)|^p\approx\frac{1}{(1-|z|)^{np+2+\alpha}},\ \ z\in\mathbb{D}(w,r).$$

Therefore,

$$\begin{aligned}\infty>&\|S_g^{n,0}f\|^p_{F(p,p+\alpha,s)}\\ =&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(S_g^{n,0}f_w)^{(n)}(z)|^p(1-|z|^2)^{pn+\alpha}(1-|\sigma_a(z)|^2)^s dA(z)\\ =&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f_w^{(n)}(z)g(z)|^p(1-|z|^2)^{pn+\alpha}(1-|\sigma_a(z)|^2)^s dA(z)\\ \gtrsim&\int_{\mathbb{D}(w,r)}|f_w^{(n)}(z)g(z)|^p(1-|z|^2)^{pn+\alpha}(1-|\sigma_w(z)|^2)^s dA(z)\\ \gtrsim&\int_{\mathbb{D}(w,r)}|g(z)|^p(1-|z|^2)^{-2}dA(z)\\ \gtrsim&|g(w)|^p,\end{aligned}$$

which implies that $g\in H^\infty$.

Conversely, suppose that $g\in H^\infty$. Let $f\in A^p_\alpha$. Then by Lemma 1 we obtain

$$\begin{aligned}\|S_g^{n,0}f\|^p_{F(p,p+\alpha,s)}\approx&\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|(S_g^{n,0}f)^{(n)}(z)|^p(1-|z|^2)^{pn+\alpha+s}dA(z)\\ =&\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f^{(n)}(z)g(z)|^p(1-|z|^2)^{pn+\alpha+s}dA(z)\\ \lesssim&\|g\|^p_{H^\infty}\sup_{I\subset\partial\mathbb{D}}\frac{1}{|I|^s}\int_{S(I)}|f^{(n)}(z)|^p(1-|z|^2)^{pn+\alpha+s}dA(z)\\ \lesssim&\|g\|^p_{H^\infty}\|f\|^p_{F(p,p+\alpha,s)}\\ \lesssim&\|g\|^p_{H^\infty}\|f\|^p_{A^p_\alpha}.\end{aligned}$$

Therefore, $S_g^{n,0}:A^p_\alpha\to F(p,p+\alpha,s)$ is bounded. From the above proof, we see that (3.3) holds. The proof is complete. □

4. Essential norm

In this section, we investigate the essential norm of $T_g^{n,k}$ and $S_g^{n,0}$ from A^p_α into $F(p,p+\alpha,s)$. The proof of the following result can be proved similarly as [25, Lemma 5.1]. We omit the proof here.

LEMMA 8. *Let $1<p<\infty$, $-1<\alpha<\infty$, $0<s<\infty$, $n\in\mathbb{N}$ and $k\in\mathbb{N}_0$ such that $0\leqslant k<n$. If $0<r<1$ and $g\in\mathscr{B}$, then $T_{g_r}^{n,k}:A^p_\alpha\to F(p,p+\alpha,s)$ is compact.*

For $0<r<1$, $z\in\mathbb{D}$ and $f\in\mathscr{B}$, set $f_r(z)=f(rz)$. Let $\operatorname{dist}_{\mathscr{B}}(f,\mathscr{B}_0)$ denote the distance from the Bloch function to the little Bloch space, that is,

$$\operatorname{dist}_{\mathscr{B}}(f,\mathscr{B}_0)=\inf_{g\in\mathscr{B}_0}\|f-g\|_{\mathscr{B}}.$$

The following result can be found in [4].

LEMMA 9. *If $g\in\mathscr{B}$, then*

$$\limsup_{|z|\to 1^-}(1-|z|^2)|g'(z)|\approx \operatorname{dist}_{\mathscr{B}}(g,\mathscr{B}_0)\approx \limsup_{r\to 1^-}\|g-g_r\|_{\mathscr{B}}.$$

THEOREM 4. *Let $g\in H(\mathbb{D})$, $1<p<\infty$, $-1<\alpha<\infty$, $0<s<\infty$, $n\in\mathbb{N}$ and $k\in\mathbb{N}_0$ such that $0\leqslant k<n$. If $T_g^{n,k}:A_\alpha^p\to F(p,p+\alpha,s)$ is bounded, then*

$$\|T_g^{n,k}\|_{e,A_\alpha^p\to F(p,p+\alpha,s)}\approx \limsup_{|z|\to 1^-}(1-|z|^2)|g'(z)|\approx \operatorname{dist}_{\mathscr{B}}(g,\mathscr{B}_0).$$

Proof. Let $0<r<1$. By Lemma 8, $T_{g_r}^{n,k}:A_\alpha^p\to F(p,p+\alpha,s)$ is compact. Then by Theorem 2,

$$\|T_g^{n,k}\|_{e,A_\alpha^p\to F(p,p+\alpha,s)}\leqslant \|T_g^{n,k}-T_{g_r}^{n,k}\|=\|T_{g-g_r}^{n,k}\|\approx\|g-g_r\|_{\mathscr{B}}.$$

Using Lemma 9, we have

$$\|T_g^{n,k}\|_{e,A_\alpha^p\to F(p,p+\alpha,s)}\lesssim \limsup_{r\to 1^-}\|g-g_r\|_{\mathscr{B}}\approx \operatorname{dist}_{\mathscr{B}}(g,\mathscr{B}_0).$$

On the other hand, suppose $\{z_j\}$ is a sequence in $\mathbb{D}$ such that $\lim_{j\to\infty}|z_j|=1$. Let G_{z_j} be defined as in the proof of Theorem 2 for each j. Then $\{G_{z_j}\}$ is a bounded sequence in A_α^p, and as $j\to\infty$, it converges uniformly to zero on every compact subset of $\mathbb{D}$. Let $K:A_\alpha^p\to F(p,p+\alpha,s)$ be a compact operator. Since A_α^p is a reflexive space we have that $\lim_{j\to\infty}\|KG_{z_j}\|_{F(p,p+\alpha,s)}=0$ (see [20]). From the proof of Theorem 2, we also have

$$\begin{aligned}\|T_g^{n,k}-K\| &\gtrsim \limsup_{j\to\infty}\|(T_g^{n,k}-K)G_{z_j}\|_{F(p,p+\alpha,s)}\\ &\gtrsim \limsup_{j\to\infty}\Big(\|T_g^{n,k}G_{z_j}\|_{F(p,p+\alpha,s)}-\|KG_{z_j}\|_{F(p,p+\alpha,s)}\Big)\\ &\approx \limsup_{j\to\infty}\|T_g^{n,k}G_{z_j}\|_{F(p,p+\alpha,s)}\\ &\gtrsim \limsup_{j\to\infty}(1-|z_j|^2)^{n-k}|g^{(n-k)}(z_j)|.\end{aligned}$$

Hence,

$$\|T_g^{n,k}\|_{e,A_\alpha^p\to F(p,p+\alpha,s)}\gtrsim \limsup_{j\to\infty}(1-|z_j|^2)^{n-k}|g^{(n-k)}(z_j)|.$$

It follows from the arbitrariness of $\{z_j\}$ and Lemmas 3 and 9 that

$$\begin{aligned}\|T_g^{n,k}\|_{e,A_\alpha^p\to F(p,p+\alpha,s)} &\gtrsim \limsup_{|z|\to 1^-}(1-|z|^2)^{n-k}|g^{(n-k)}(z)| \\ &\approx \limsup_{|z|\to 1^-}(1-|z|^2)|g'(z)| \approx \mathrm{dist}_{\mathscr{B}}(g,\mathscr{B}_0).\end{aligned}$$

The proof is complete. □

The following result can be deduced by Theorem 4 directly.

COROLLARY 1. *Let $g\in H(\mathbb{D})$, $1<p<\infty$, $-1<\alpha<\infty$, $0<s<\infty$, $n\in\mathbb{N}$ and $k\in\mathbb{N}_0$ such that $0\leqslant k<n$. Then $T_g^{n,k}: A_\alpha^p\to F(p,p+\alpha,s)$ is compact if and only if $g\in\mathscr{B}_0$.*

THEOREM 5. *Let $g\in H(\mathbb{D})$, $1<p<\infty$, $-1<\alpha<\infty$, $0<s<\infty$ and $n\in\mathbb{N}$. If $S_g^{n,0}: A_\alpha^p\to F(p,p+\alpha,s)$ is bounded, then*

$$\|S_g^{n,0}\|_{e,A_\alpha^p\to F(p,p+\alpha,s)} \approx \limsup_{|z|\to 1^-}|g(z)|.$$

Proof. By Theorem 3 we have

$$\|S_g^{n,0}\|_{e,A_\alpha^p\to F(p,p+\alpha,s)} = \inf_K \|S_g^{n,0}-K\| \lesssim \limsup_{|z|\to 1^-}|g(z)|.$$

On the other hand, suppose $\{z_j\}$ is a sequence in $\mathbb{D}$ such that $\lim_{j\to\infty}|z_j|=1$. Let f_{z_j} be defined as in (2.1). Let $K: A_\alpha^p\to F(p,p+\alpha,s)$ be a compact operator. Then, we similarly have $\lim_{j\to\infty}\|Kf_{z_j}\|_{F(p,p+\alpha,s)}=0$. Hence,

$$\begin{aligned}\|S_g^{n,0}-K\| &\gtrsim \limsup_{j\to\infty}\|(S_g^{n,0}-K)f_{z_j}\|_{F(p,p+\alpha,s)} \\ &\gtrsim \limsup_{j\to\infty}\|S_g^{n,0}f_{z_j}\|_{F(p,p+\alpha,s)} - \limsup_{j\to\infty}\|Kf_{z_j}\|_{F(p,p+\alpha,s)} \\ &= \limsup_{j\to\infty}\|S_g^{n,0}f_{z_j}\|_{F(p,p+\alpha,s)}.\end{aligned}$$

Therefore, from the proof of Theorem 3,

$$\|S_g^{n,0}\|_{e,A_\alpha^p\to F(p,p+\alpha,s)} \gtrsim \limsup_{j\to\infty}\|S_g^{n,0}f_{z_j}\|_{F(p,p+\alpha,s)} \gtrsim \limsup_{j\to\infty}|g(z_j)|,$$

which implies that

$$\|S_g^{n,0}\|_{e,A_\alpha^p\to F(p,p+\alpha,s)} \gtrsim \limsup_{|z|\to 1^-}|g(z)|.$$

The proof is complete. □

From Theorem 5 we get the following result.

COROLLARY 2. *Let* $1 < p < \infty$, $-1 < \alpha < \infty$, $0 < s < \infty$ *and* $n \in \mathbb{N}$. *Then* $S_g^{n,0} : A_\alpha^p \to F(p, p+\alpha, s)$ *is compact if and only if* $g = 0$.

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