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VILENKIN–FOURIER SERIES IN VARIABLE LEBESGUE SPACES

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Abstract. Let $S_n f$ be the nth partial sum of the Vilenkin-Fourier series of $f \in L^1(G)$. For $1 < p_- \leqslant p_+ < \infty$, we characterize all exponent $p(\cdot)$ such that if $f \in L^{p(\cdot)}(G)$, $S_n f$ converges to f in $L^{p(\cdot)}(G)$.

1. Introduction

Let $\{p_i\}_{i \geqslant 0}$ be a sequence of integers, $p_i \geqslant 2$. Let $G = \Pi_{i=0}^{\infty} \mathbb{Z}_{p_i}$ be the direct product of cyclic groups of order p_i, and μ the Haar measure on G normalized by $\mu(G) = 1$. Each element of G can be considered as a sequence $\{x_i\}$, with $0 \leqslant x_i < p_i$. Set $m_0 = 1$, $m_k = \Pi_{i=0}^{k-1} p_i$, $k = 1, 2, \ldots$. There is a well-known and natural measure preserving identification between group G and closed interval $[0,1]$. This identification consists in associating with each $\{x_i\} \in G$, $0 \leqslant x_i < p_i$, the point $\sum_{i=0}^{\infty} x_i m_{i+1}^{-1}$. If we disregard the countable set of p_i-rationals, this mapping is one-one, onto and measure-preserving.

For each $x = \{x_i\} \in G$, define $\phi_k(x) = \exp(2\pi i x_k / p_k)$, $k = 0, 1, \ldots$. The set $\{\psi_n\}$ of characters of G consists of all finite product of ϕ_k, which we enumerate in the following manner. Express each nonnegative integer n as a finite sum $n = \sum_{i=0}^{\infty} \alpha_k m_k$, with $0 \leqslant \alpha_k < p_k$, and define $\psi_n = \Pi_{i=0}^{\infty} \phi_k^{\alpha_k}$. The functions ψ_n form a complete orthonormal system on G. For the case $p_i = 2$, $i = 0, 1, \ldots$, G is the dyadic group, ϕ_k are Rademacher functions and ψ_n are Walsh functions. In general, the system $\{\psi_n\}$ is a realization of the multiplicative Vilenkin system. In this paper, there is no restriction on the orders $\{p_i\}$.

For $f \in L^1(G)$, let $S_n f$, $n = 0, 1, \ldots$, be the nth partial sum of the Vilenkin-Fourier series of f. When the orders p_i of cyclic groups are bounded Watari [19] showed that for $f \in L^p(G)$, $1 < p < \infty$,

$$\lim_{n \to \infty} \int_G |S_n f - f|^p d\mu = 0.$$

Young [17], Schipp [14] and Simon [15] showed independently that results concerning mean convergence of partial sums of the Vilenkin-Fourier series are still valid even if the orders p_i are unbounded.

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Let $\{G_k\}$ be the sequence of subgroups of G defined by

$$G_0 = G, \ \ G_k = \Pi_{i=0}^{k-1}\{0\} \times \Pi_{i=k}^{\infty}\mathbb{Z}_{p_i}, \ \ k = 1,2,\ldots.$$

On the closed interval $[0,1]$, cosets of G_k are intervals of the form $[jm_k^{-1},(j+1)m_k^{-1}]$, $j = 0,1,\ldots,m_k-1$. By $\mathscr{F}$ we denote the set of generalized intervals. This set is the collection of all translations of intervals $[0, jm_{k+1}^{-1}]$, $k = 0,1,\ldots$ $j = 1,\ldots,p_k$. Note that a set I belongs to $\mathscr{F}$ if (1) for some $x \in G$ and k, $I \subset x+G_k$, (ii) I is a union of cosets of $G_{k+1,}$ and (iii) if we consider $x+G_k$ as a circle, I is an interval. Let $\mathscr{F}_{-1} = \{G\}$. For $k = 0,1,\ldots,$ let $\mathscr{F}_k$ be the collection of all $I \in \mathscr{F}$ such that I is a proper subset of a coset of G_k, and is a union of cosets of G_{k+1}. The collections $\mathscr{F}_k$ are disjoint, and $\mathscr{F} = \cup_{k=-1}^{\infty}\mathscr{F}_k$. For $I \in \mathscr{F}$, we define the set $3I \in \mathscr{F}$ as follows. If $I = G$, let $3I = G$. For $I \in \mathscr{F}_k$, $k = 0,1,\ldots,$ there is $x \in G$ such that $I \subset x+G_k$. If $\mu(I) \geqslant \frac{\mu(G_k)}{3}$, let $3I = x+G_k$. If $\mu(I) < \frac{\mu(G_k)}{3}$, consider $x+G_k$ as a circle. Then I is an interval in this circle. Define $3I \in \mathscr{F}_k$ to be the interval in this circle which contains I at its center and has measure $\mu(3I) = 3\mu(I)$. In all cases, for $I \in \mathscr{F}$, $\mu(3I) \leqslant 3\mu(I)$.

We say that w is a weight function on G if w is measurable and $0 < w(x) < \infty$ a.e. Gosselin [7] (case $\sup_i p_i < \infty$) and Young [18] (no restriction on the orders p_i) characterized all weight functions w such that if $f \in L_w^p(G)$, $1 < p < \infty$, $S_n f$ converges to f in $L_w^p(G)$. Here $L_w^p(G)$ denotes the space of measurable functions on G such that $\|f\|_{p,w} = (\int_G |f|^p w d\mu)^{1/p} < \infty$.

DEFINITION 1.1. (see [18]) (i) We say that w satisfies $A_p(G)$ condition, $1 < p < \infty$, if

$$[w]_{A_p} = \sup_{I\in\mathscr{F}} \left(\frac{1}{\mu(I)}\int_I w\,d\mu\right)\left(\frac{1}{\mu(I)}\int_I w^{-1/(p-1)}\,d\mu\right)^{p-1} < \infty. \tag{1.1}$$

(ii) We say that w satisfies $A_1(G)$ condition if

$$[w]_{A_1} = \sup_{I\in\mathscr{F}} \frac{1}{\mu(I)}\int_I w\,d\mu\,(\operatorname{essinf}_I w(x))^{-1} < \infty.$$

For the case where the orders of cyclic groups are bounded, Gosselin [7] defined $A_p(G)$ condition, as the one where (1.1) condition holds for all I that are cosets of G_k, $k = 0,1,2,\ldots$. For this case A_p conditions, defined by Young and Gosselin, are equivalent (see [18]).

THEOREM 1.2. ([18]) *Let w be a weight function on G. For $1 < p < \infty$, the following statements are equivalent:*

(i) $w \in A_p(G)$,

(ii) There is a constant C, depending only on w and p, such that for every $f \in L_w^p(G)$, we have

$$\int_G |S_n f|^p w\,d\mu \leqslant C\int_G |f|^p w\,d\mu,$$

(iii) For every $f \in L_w^p(G)$, we have

$$\lim_{n\to\infty} \int_G |S_n f - f|^p w d\mu = 0.$$

In this paper we characterize all exponents $p(\cdot)$ such that if $f \in L^{p(\cdot)}(G)$, then partial sums $S_n f$ of the Vilenkin-Fourier series of $f \in L^{p(\cdot)}(G)$ converge to f with $L^{p(\cdot)}$-norm. Now we give a definition of variable Lebesgue space. Let $p(\cdot) : G \to [1,\infty)$ be a measurable function. The variable Lebesgue space $L^{p(\cdot)}(G)$ is the set of all measurable functions f such that for some $\lambda > 0$,

$$\rho_{p(\cdot)}(f/\lambda) = \int_G (|f(x)|/\lambda)^{p(x)} d\mu < \infty.$$

$L^{p(\cdot)}(G)$ is a Banach function space equipped with the Luxemburg norm

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leqslant 1\}.$$

We use the notations $p_-(I) = \mathrm{essinf}_{x\in I} p(x)$ and $p_+(I) = \mathrm{esssup}_{x\in I} p(x)$ where $I \subset G$. If $I = G$ we simply use the following notation p_-, p_+. The function $p'(\cdot)$ denotes the conjugate exponent function of $p(\cdot)$, i.e., $1/p(x) + 1/p'(x) = 1$ $(x \in G)$. In this paper the constants C, c are absolute constants and may be different in different contexts and χ_A denotes the characteristic function of set A.

Very recently the convergence of partial sums of the Walsh-Fourier series in $L^{p(\cdot)}([0,1))$ space was investigated by Jiao et al. [8]. We denote by $C_d^{\log}$ the set of all functions $p(\cdot) : [0,1) \to [1\infty)$, for which there exists a positive constant C such that

$$|I|^{p_-(I)-p_+(I)} \leqslant C$$

for all dyadic intervals $I = [k2^{-n}, (k+1)2^{-n})$ $(k,n \in \mathbb{N}\ \ 0 \leqslant k < 2^n)$, here $|I|$ denotes the Lebesgue measure of I. Note that this condition may be interpreted as a dyadic version of log-Hölder continuity condition of $p(\cdot)$ (or on dyadic group). The log-Hölder condition is a very common condition for solving various problems of harmonic analysis in $L^{p(\cdot)}(\mathbb{R}^n)$ (see [2], [5]).

THEOREM 1.3. ([8]) *Let $p(\cdot) \in C_d^{\log}$ with $1 < p_- \leqslant p_+ < \infty$. If $f \in L^{p(\cdot)}([0,1))$, then for partial sums $S_n f$ of the Walsh-Fourier series of $f \in L^{p(\cdot)}([0,1))$ we have*

$$\sup_{n\in\mathbb{N}} \|S_n f\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}.$$

Since Walsh polynomials are dense in $L^{p(\cdot)}([0,1))$, Theorem 1.3 implies that $S_n f$ converges to the original function in $L^{p(\cdot)}([0,1))$-norm (for more details see [8] and the recent book [13], chapter 9).

In order to extend techniques and results of constant exponent case to the setting of variable Lebesgue spaces, a central problem is to determine conditions on an exponent $p(\cdot)$ under which the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$

(see monographs Cruz-Uribe and Fiorenza [2] and Diening et.al. [5]). We now define the Hardy-Littlewood maximal function that is appropriate for the study of Vilenkin-Fourier series. For $f \in L^1(G)$, let

$$Mf(x) = \sup_{x \in I, I \in \mathscr{F}} \frac{1}{\mu(I)} \int_I |f| d\mu.$$

This maximal function was introduced first by P. Simon in [16]. He showed that the maximal operator is bounded in $L^p(G)$, $1 < p < \infty$ and is of weak type $(1,1)$. Young [18] obtained the following analogue of Muckenhoupt's theorem [11].

THEOREM 1.4. *Let w be a weight function on G. For $1 < p < \infty$, the following two statements are equivalent:*

(i) $w \in A_p(G)$,

(ii) There is a constant C, depending only on w and p, such that for every $f \in L^p_w(G)$, we have

$$\int_G (Mf)^p w d\mu \leqslant C \int_G |f|^p w d\mu.$$

In case $p = 1$ the following two statements are also equivalent:

(iii) $w \in A_1(G)$,

(iv) There is constant C, depending only on w, such that for every $f \in L^1(G)$

$$\int_{\{Mf > y\}} w d\mu \leqslant C y^{-1} \int_G |f| w d\mu, \;\; y > 0.$$

DEFINITION 1.5. We say that the exponent $p(\cdot)$, $1 < p_- \leqslant p_+ < \infty$ satisfies the condition $\mathscr{A}(G)$, if there is a constant C such that for every $I \in \mathscr{F}$,

$$\frac{1}{\mu(I)} \|\chi_I\|_{p(\cdot)} \|\chi_I\|_{p'(\cdot)} \leqslant C. \tag{1.2}$$

The condition (1.2) plays exactly the same role for averaging operators in variable Lebesgue spaces as the Muckenhoupt A_p conditions for weighted Lebesgue spaces (see [9], [10], for Euclidian setting). We show that the $\mathscr{A}(G)$ condition is necessary and sufficient for the $L^{p(\cdot)}(G)$ boundedness of Hardy-Littlewood maximal function. One of the main result of the present paper is the following theorem.

THEOREM 1.6. *Assume for the exponent $p(\cdot)$ we have $1 < p_- \leqslant p_+ < \infty$. Then the following two statements are equivalent:*

(i) $p(\cdot) \in \mathscr{A}(G)$,

(ii) There is a constant C, depending only on $p(\cdot)$ such that for every $f \in L^{p(\cdot)}(G)$, we have

$$\|Mf\|_{p(\cdot)} \leqslant C \|f\|_{p(\cdot)}.$$

By the symmetry of the definition, $p(\cdot) \in \mathscr{A}(G)$ if and only if $p'(\cdot) \in \mathscr{A}(G)$ and from Theorem 1.6 we have that, even though, M is not a linear operator, the boundedness of M implies the "dual" inequality.

COROLLARY 1.7. *Let for exponent* $p(\cdot)$ *we have* $1 < p_- \leqslant p_+ < \infty$. *Then the maximal operator* M *is bounded on* $L^{p(\cdot)}(G)$ *if and only if* M *is bounded on* $L^{p'(\cdot)}(G)$.

We prove the following theorem (in the Euclidean setting see [2], Theorem 4.37 and [5], Theorem 5.7.2).

THEOREM 1.8. *Let for the exponent* $p(\cdot)$ *we have* $1 < p_- \leqslant p_+ < \infty$. *Then the following statements are equivalent:*

(i) Maximal operator M *is bounded on* $L^{p(\cdot)}(G)$,

(ii) There exists r_0, $0 < r_0 < 1$, *such that if* $r_0 < r < 1$, *then maximal operator* M *is bounded on* $L^{rp(\cdot)}(G)$.

Hereafter, we will denote by $\mathscr{S}$ a family of pairs of non-negative, measurable functions. Given p, $1 \leqslant p < \infty$ if for some $w \in A_p(G)$ we write

$$\int_G f(x)^p w(x) d\mu \leqslant C \int_G g(x)^p w(x) d\mu, \quad (f,g) \in \mathscr{S},$$

then we mean that this inequality holds for all pairs $(f,g) \in \mathscr{S}$ such that the left hand side is finite, and that the constant C may depend on p and $[w]_{A_p}$. If we write

$$\|f\|_{p(\cdot)} \leqslant C_{p(\cdot)} \|g\|_{p(\cdot)}, \quad (f,g) \in \mathscr{S},$$

then we mean that this inequality holds for all pairs $(f,g) \in \mathscr{S}$ such that the left-hand side is finite and the constant may depend on $p(\cdot)$.

Using this convention we can state the Rubio de Francia extrapolation theorem in the following manner.

THEOREM 1.9. *Suppose for some* $p_0 \geqslant 1$ *the family* $\mathscr{S}$ *is such that for all* $w \in A_1(G)$

$$\int_G f(x)^{p_0} w(x) d\mu \leqslant C \int_G g(x)^{p_0} w(x) d\mu, \quad (f,g) \in \mathscr{S}.$$

If for the exponent $p(\cdot)$, *we have* $p_0 < p_- \leqslant p_+ < \infty$ *and the maximal operator* M *is bounded on* $L^{(p(\cdot)/p_0)'}(G)$, *then*

$$\|f\|_{p(\cdot)} \leqslant C_{p(\cdot)} \|g\|_{p(\cdot)}, \quad (f,g) \in \mathscr{S}.$$

Firstly, Theorem 1.9 was proved in [4] (Theorem 1.3) for variable exponent Lebesgue spaces on $\mathbb{R}^n$ and maximal operator M defined on cubes (balls) in $\mathbb{R}^n$, with sides parallel to the coordinate axes. In [3] the Rubio de Francia extrapolation theorem is proved for general Function spaces, using A_1 weights and maximal operator M defined by any Muckenhoupt basis (see Definition 3.1 in [3]). By Theorem 1.4 the set of generalized intervals $\mathscr{F}$ is a Muckenhoupt basis. Considering the following equality $\left(L^{p(\cdot)}(G)\right)^{1/p_0} = L^{p(\cdot)/p_0}(G)$, Theorem 1.9 is direct consequence of Theorem 4.6 from [3].

Now, we can formulate the main result of the present paper.

THEOREM 1.10. *Let for exponent $p(\cdot)$ we have $1 < p_- \leqslant p_+ < \infty$. Then the following statements are equivalent:*

(i) $p(\cdot) \in \mathscr{A}(G)$,

(ii) There is a constant C, depending only on $p(\cdot)$, such that for partial sums S_nf of the Vilenkin-Fourier series of $f \in L^{p(\cdot)}(G)$ we have

$$\sup_{n\in\mathbb{N}} \|S_nf\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}.$$

(iii) Partial sums S_nf of the Vilenkin-Fourier series of $f \in L^{p(\cdot)}(G)$ converge to the original function in $L^{p(\cdot)}$ space.

2. Preliminaries

The fundamental properties of $A_p(G)$ weights were investigated by Gosselin [7] and later by Young [18] (in this paper there is no restriction on the orders p_i). We formulate some properties of these weights (see [18]).

Note that if $w \in A_p(G)$, then $L^p_w(G) \subset L^1(G)$. We also mention that if $w \in A_p(G)$, $1 \leqslant p < \infty$, and $p < q < \infty$ then $w \in A_q(G)$. A important property of $A_p(G)$ weights is the reverse Hölder inequality.

PROPOSITION 2.1. ([18]) *Let $w \in A_p(G)$, $1 < p < \infty$. Then there exist $s > 1$ and a constant C such that for any $I \in \mathscr{F}$,*

$$\left(\frac{1}{\mu(I)}\int_I w^s d\mu\right)^{1/s} \leqslant \frac{C}{\mu(I)}\int_I w d\mu.$$

The following proposition is a consequence of the reverse Hölder inequality.

PROPOSITION 2.2. ([18]) *(i) Suppose $w \in A_p(G)$, $1 < p < \infty$. Then there exists $1 < s < p$ such that $w \in A_s(G)$.*

(ii) Suppose $w \in A_p(G)$, $1 < p < \infty$, then $w \in A_\infty(G)$.

DEFINITION 2.3. ([18]) Let $I_0 \in \mathscr{F}$. We say that a weight w (i.e. a nonnegative integrable function) satisfies $A_\infty(I_0)$ condition if for any $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that for any generalized interval $I \subset I_0$ and for any measurable subset $E \subset I$, $\mu(E) \leqslant \varepsilon\mu(I)$ implies $w(E) \leqslant \delta w(I)$ (for any measurable set A, $w(A) = \int_A w d\mu$ and $w_A = \frac{1}{\mu(A)}\int_A w d\mu$).

It is well known fact that the class A_∞ in Euclidian case can be defined in many equivalent ways. The most classical definition is due to Muckenhoupt [12]. It is said that a locally integrable function $w : \mathbb{R}^n \to [0,\infty)$ is in A_∞ class if for each $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that $|E| \leqslant \varepsilon|Q| \Rightarrow w(E) \leqslant \delta w(Q)$ holds, whenever Q is a d-dimensional cube and E is its arbitrary measurable subset of Q. Note that w satisfies the above condition if and only if it belongs A_p class for some $p \in (1,\infty)$. Coifman and Fefferman [1] proposed another approach based on verifying the following inequality

$$\frac{w(E)}{w(Q)} \leqslant C\left(\frac{|E|}{|Q|}\right)^{\varepsilon},$$

where Q,E are as before, while $C,\varepsilon > 0$ are constants depending only w. Note that the two conditions lead to same class of weights. For More detailed information we refer the reader to [6].

To prove the main result we need analogous result for Vilenkin group. It should be noted that we give the proof which we had not found in literature.

PROPOSITION 2.4. *Let $w \in A_\infty(I_0)$, where $I_0 \in \mathscr{F}$. There exist positive constants $C,\varepsilon > 0$ such that for any generalized interval $I \subset I_0$ and measurable subset $E \subset I$,*

$$\frac{w(E)}{w(I)} \leqslant C\left(\frac{\mu(E)}{\mu(I)}\right)^{\varepsilon}. \tag{2.1}$$

For proving the result we need modified form of the Calderón-Zygmund decomposition lemma (see [17], Lemma 2).

LEMMA 2.5. *Given an interval $I \in \mathscr{F}$ and a function $f \in L^1(G)$, then for $t \geqslant |f|_I$, there exists a collection I_j of disjoint generalized intervals $I_j \subset I$ such that*

$$t < \frac{1}{\mu(I_j)}\int_{I_j}|f|d\mu \leqslant 3t,\ \ \forall I_j,$$

and for almost every $x \in I \backslash \cup_j I_j$, $|f(x)| \leqslant t$.

Proof of Proposition 2.4. Fix a generalized interval $I \subset I_0$ and for integer $k \geqslant 0$ define the sequence $t_k = 10^k w_I = 10^k t_0$. Using Lemma 2.5 For each k we may find Calderón-Zygmund generalized intervals I_j^k of w in following manner. First construct Calderón-Zygmund generalized intervals I_j^0 relative to I at height t_0 (Calderón-Zygmund generalized intervals of rang 0). Denote $\Omega_0 = \cup I_j^0$. For any fixed I_j^0 interval find Calderón-Zygmund generalized intervals (of rang 1) of w and height t_1. Denote by I_j^1 the intervals of rang 1 and $\Omega_1 = \cup I_j^1$. Note that $\Omega_1 \subset \Omega_0 \subset I$. In this manner we may construct collection I_j^k Calderón-Zygmund generalized intervals and the set Ω_k with properties:

a) $\Omega_{k+1} \subset \Omega_k,\ k = 0,1,2,\dots,$

b) $t_k < w_{I_j^k} \leqslant 3t_k,\ k = 0,1,2,\dots,$

c) $w(x) \leqslant t_k,\ x \in I \backslash \Omega_k$.

Note that from the construction for any i there exists j such that $I_i^{k+1} \subset I_j^k$.

Then

$$\begin{aligned}\mu(\Omega_{k+1} \cap I_j^k) &= \sum_{I_i^{k+1} \subset I_j^k} \mu(I_i^{k+1}) < t_{k+1}^{-1} \sum_{I_i^{k+1} \subset I_j^k} w(I_i^{k+1}) \\ &\leqslant t_{k+1}^{-1} w(I_j^k) \leqslant \frac{3t_k}{t_{k+1}}\mu(I_j^k) = \frac{3}{10}\mu(I_j^k).\end{aligned}$$

Hence, by $A_\infty(I_0)$ condition with $\varepsilon = 3/10$, there exists $\delta > 0$ such that $w(\Omega_{k+1} \cap I_j^k) \leqslant \delta w(I_j^k)$, and if we sum over all j, we obtain $w(\Omega_{k+1}) \leqslant \delta w(\Omega_k)$ and consequently we have that $w(\Omega_k) \leqslant \delta^{k+1} w(I)$.

For almost every $x \in I \setminus \Omega_k$, $w(x) \leqslant t_k$. For fixed ε

$$\begin{aligned}
\frac{1}{\mu(I)} \int_I w(x)^{1+\varepsilon} d\mu &= \frac{1}{\mu(I)} \int_{I \setminus \Omega_0} w(x)^{1+\varepsilon} d\mu + \frac{1}{\mu(I)} \sum_{k=0}^{\infty} \int_{\Omega_k \setminus \Omega_{k+1}} w(x)^{1+\varepsilon} d\mu \\
&\leqslant \frac{t_0^\varepsilon}{\mu(I)} \int_{I \setminus \Omega_0} w(x) d\mu + \frac{1}{\mu(I)} \sum_{k=0}^{\infty} t_{k+1}^\varepsilon w(\Omega_k) \\
&\leqslant \frac{t_0^\varepsilon}{\mu(I)} \int_{I \setminus \Omega_0} w(x) d\mu + \frac{1}{\mu(I)} \sum_{k=0}^{\infty} 10^{(k+1)\varepsilon} t_0^\varepsilon \delta^{k+1} w(I).
\end{aligned}$$

Fix $\varepsilon > 0$ so that $10^\varepsilon \delta < 1$, we obtain that last term is bounded by

$$t_0^\varepsilon \frac{1}{\mu(I)} \int_I w(x) d\mu + C\mu(I)^{-1} t_0^\varepsilon w(I) \leqslant C \left(\frac{1}{\mu(I)} \int_I w(x) d\mu \right)^{1+\varepsilon}.$$

Hence, given $\varepsilon > 0$ the weight satisfies Reverse Hölder inequality.

Finally if we use Hölder's inequality for $w(E) = \int_E w(x) d\mu$ and Reverse Hölder's inequality for $1+\varepsilon$ we get (2.1). $\square$

For $0 < r < \infty$ define $M_r f(x) = M(|f|^r)(x)^{1/r}$. For brevity, hereafter we will write f_I instead of $\int_I f d\mu / \mu(I)$.

As a consequence of the reverse Hölder inequality we get that if $w \in A_p(G)$ for some p, then there exists $s > 1$ such that $M_s w(x) \leqslant CMw(x)$. We need a sharper version of this inequality.

PROPOSITION 2.6. *Given $w \in A_1(G)$, if $s_0 = 1 + \frac{1}{8[w]_{A_1}}$, then for $1 < s \leqslant s_0$ and for almost every x,*

$$M_s w(x) \leqslant 4Mw(x) \leqslant 4[w]_{A_1} w(x). \tag{2.2}$$

This type of estimates is well known in Euclidian setting. For the sake of completeness we will give a proof for the Vilenkin group.

We need an inequality that is the reverse of the weak $(1,1)$ inequality for maximal operator M.

LEMMA 2.7. *Given a function $f \in L^1(G)$, for every interval $I \in \mathscr{F}$ and $t \geqslant |f|_I$,*

$$\mu(\{x \in I : Mf(x) > t\}) \geqslant \frac{1}{3t} \int_{\{x \in I : |f(x)| > t\}} |f(x)| d\mu.$$

Proof. $t \geqslant |f|_I$; if $t \geqslant \|f\|_{L^\infty}$, then this result is true. Otherwise, by Lemma 2.5, let I_i be the Calderón-Zygmund intervals of f relative to I and t. For every $x \in I_i$

$$Mf(x) \geqslant \frac{1}{\mu(I_i)} \int_{I_i} |f| d\mu > t.$$

Since $|f(x)| \leqslant t$ for almost every $x \in I \setminus \cup_i I_i$, we have

$$\mu(\{x \in I : Mf(x) > t\}) \geqslant \sum_j \mu(I_j) \geqslant \frac{1}{3t} \sum_j \int_{I_j} |f| d\mu$$
$$\geqslant \frac{1}{3t} \int_{\{x \in I : |f(x)| > t\}} |f(x)| d\mu. \quad \square$$

Proof of Proposition 2.6. Let $\varepsilon = (8[w]_{A_1})^{-1}$, $s_0 = 1 + \varepsilon$, and fix an interval I and $x_0 \in I$. To prove the first inequality of (2.2) it is sufficient to show that

$$\frac{1}{\mu(I)} \int_I w(x)^{s_0} d\mu \leqslant 4Mw(x_0)^{s_0}.$$

We have that

$$\begin{aligned}
\frac{1}{\mu(I)} \int_I w(x)^{s_0} d\mu &= \frac{1}{\mu(I)} \int_I w(x)^{\varepsilon} w(x) d\mu \\
&= \varepsilon(\mu(I))^{-1} \int_0^{\infty} t^{\varepsilon-1} w(\{x \in I : w(x) > t\}) dt \\
&= \varepsilon(\mu(I))^{-1} \int_0^{Mw(x_0)} t^{\varepsilon-1} w(\{x \in I : w(x) > t\}) dt \\
&\quad + \varepsilon(\mu(I))^{-1} \int_{Mw(x_0)}^{\infty} t^{\varepsilon-1} w(\{x \in I : w(x) > t\}) dt.
\end{aligned}$$

For the first term we have

$$\varepsilon(\mu(I))^{-1} \int_0^{Mw(x_0)} t^{\varepsilon-1} w(\{x \in I : w(x) > t\}) dt$$
$$\leqslant \varepsilon(\mu(I))^{-1} w(I) \int_0^{Mw(x_0)} t^{\varepsilon-1} dt = \frac{1}{\mu(I)} \int_I w(y) d\mu \cdot Mw(x_0)^{\varepsilon} \leqslant Mw(x_0)^{1+\varepsilon}.$$

Using Lemma 2.7 we obtain

$$\begin{aligned}
&\varepsilon(\mu(I))^{-1} \int_{Mw(x_0)}^{\infty} t^{\varepsilon-1} w(\{x \in I : w(x) > t\}) dt \\
&= \varepsilon(\mu(I))^{-1} \int_{Mw(x_0)}^{\infty} t^{\varepsilon-1} \int_{\{x \in I : w(x) > t\}} d\mu dt \\
&\leqslant 3\varepsilon(\mu(I))^{-1} \int_0^{\infty} t^{\varepsilon} \mu(\{x \in I ; Mw(x) > t\}) dt \\
&= \frac{3\varepsilon}{1+\varepsilon} \frac{1}{\mu(I)} \int_I Mw(x)^{1+\varepsilon} d\mu \\
&\leqslant \frac{3\varepsilon [w]_{A_1}^{1+\varepsilon}}{1+\varepsilon} \frac{1}{\mu(I)} \int_I w(x)^{1+\varepsilon} d\mu.
\end{aligned}$$

From above estimates we get

$$\frac{1}{\mu(I)}\int_I w(x)^{1+\varepsilon}d\mu \leqslant Mw(x_0)^{1+\varepsilon}+\frac{3\varepsilon[w]_{A_1}^{1+\varepsilon}}{1+\varepsilon}\frac{1}{\mu(I)}\int_I w(x)^{1+\varepsilon}d\mu.$$

Since for all $x \geqslant 1,\ x^{1/8x} \leqslant 2,$ we have

$$\frac{3\varepsilon[w]_{A_1}^{1+\varepsilon}}{1+\varepsilon} \leqslant \frac{3}{8}[w]_{A_1}^{-1}[w]_{A_1}^{1+(8[w]_{A_1})^{-1}} \leqslant \frac{3}{4}$$

and consequently the first inequality in (2.2) is valid. The second inequality in (2.2) is clear. $\square$

3. Proof of Theorem 1.6

Given a generalized interval $I \in \mathscr{F}$ define the averaging operator A_I by

$$A_I f(x) = \frac{1}{\mu(I)}\int_I f d\mu\, \chi_I(x).$$

PROPOSITION 3.1. *Given a exponent $p(\cdot),\ 1 < p_- \leqslant p_+ < \infty$, there exists a constant $C > 0$ such that for any interval $I \in \mathscr{F}$*

$$\|A_I f\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}$$

if and only if $p(\cdot) \in \mathscr{A}(G)$.

The proof of Proposition 3.1 is essentially the same as for averaging operator defined by cubes for Euclidean setting (see for example [2], Proposition 4.47).

Lemma 3.2 shows that the condition $p(\cdot) \in \mathscr{A}(G)$ is actually sufficient for modular inequality. Analogous estimate for the case $L^{p(\cdot)}(\mathbb{R}^n)$ was obtained by Kopaliani [9]. The proof in [9] is based on some concepts from convex analysis. Lerner in [10] gave a different and simple proof. In this paper our approach is based on the adaptation of Lerner's proof [10].

LEMMA 3.2. *Given exponent $p(\cdot)$ such that $1 < p_- \leqslant p_+ < \infty$, suppose $p(\cdot) \in \mathscr{A}(G)$. Let $f \in L^{p(\cdot)}(G)$. If there exists an interval $I \in \mathscr{F}$ and constants $c_1, c_2 > 0$ such that $|f|_I \geqslant c_1$ and $\|f\|_{p(\cdot)} \leqslant c_2$, where $c_1, c_2 > 0$, then there exists a constant c depending only on $p(\cdot), c_1, c_2$ such that*

$$\int_I (|f|_I)^{p(x)}d\mu \leqslant c\int_I |f(x)|^{p(x)}d\mu.$$

Proof. Using the condition $p_+ < \infty$ we may consider only the case $c_1 = c_2 = 1$. Since $p'_+ < \infty$, there exists $\alpha > 0$ such that

$$\int_I \alpha^{p'(y)-1}d\mu = \int_Q |f(x)|d\mu. \tag{3.1}$$

Since $|f|_I \geqslant 1$, we have $\alpha \geqslant 1$. By generalized Hölder inequality

$$\int_I f(x)d\mu \leqslant 2\|f\|_{p(\cdot)}\|\chi_I\|_{p'(\cdot)}$$

we get $\int_I \alpha^{p'(y)-1}d\mu \leqslant 2\|\chi_I\|_{p'(\cdot)}$ and consequently,

$$\alpha \leqslant c/\|\chi_I\|_{p'(\cdot)}. \tag{3.2}$$

Given this value α, we have that

$$\begin{aligned}\int_I (|f|_I)^{p(x)}d\mu &= \int_I \left(\frac{1}{\mu(I)}\int_I \alpha^{p'(y)-1}d\mu\right)^{p(x)} d\mu \\ &= \left(\frac{1}{\mu(I)}\int_I \left(\frac{1}{\mu(I)}\int_I \alpha^{p'(y)-p'(x)}d\mu\right)^{p(x)-1} d\mu\right)\int_I \alpha^{p'(y)}d\mu.\end{aligned} \tag{3.3}$$

For each $x \in I$ partition I into $E_1(x) = \{y \in I : p'(y) > p'(x)\}$ and $E_2(x) = I\backslash E_1(x)$. Using (3.2) and the estimate $\alpha \geqslant 1$, we obtain

$$\begin{aligned}\int_I \alpha^{p'(y)-p'(x)}d\mu &= \int_{E_1(x)} \alpha^{p'(y)-p'(x)}d\mu + \int_{E_2(x)} \alpha^{p'(y)-p'(x)}d\mu \\ &\leqslant c(\|\chi_I\|_{p'(\cdot)})^{p'(x)} + \mu(I).\end{aligned}$$

In view of $p(\cdot) \in A(G)$, we have

$$\begin{aligned}&\frac{1}{\mu(I)}\int_I \left(\frac{1}{\mu(I)}\int_I \alpha^{p'(y)-p'(x)}d\mu\right)^{p(x)-1} d\mu \\ &\leqslant c\frac{1}{\mu(I)}\int_I \left(\frac{1}{\mu(I)}(\|\chi_I\|_{p'(\cdot)})^{p'(x)}+1\right)^{p(x)-1} d\mu \\ &\leqslant c + c\frac{1}{\mu(I)}\int_I \left(\frac{1}{\mu(I)}(\|\chi_I\|_{p'(\cdot)})^{p'(x)}\right)^{p(x)-1} d\mu \\ &\leqslant c + c\int_I \left(\frac{\|\chi_I\|_{p'(\cdot)}}{\mu(I)}\right)^{p(x)} d\mu \\ &\leqslant c + c\int_I \left(\frac{1}{\|\chi_I\|_{p(\cdot)}}\right)^{p(x)} d\mu \leqslant c.\end{aligned} \tag{3.4}$$

Further,

$$\begin{aligned}\int_I \alpha^{p'(y)}d\mu &= 2\alpha\int_I |f(x)|d\mu - \int_I \alpha^{p'(y)}d\mu \\ &\leqslant 2\alpha\int_{\{y\in I:\, 2\alpha|f(y)|>\alpha^{p'(y)}\}} |f(y)|d\mu \\ &\leqslant c\int_I |f(y)|^{p(y)}d\mu.\end{aligned} \tag{3.5}$$

From (3.3), (3.4) and (3.5) we obtain desired estimate. $\square$

COROLLARY 3.3. *Let $1 < p_- \leqslant p_+ < \infty$ and $p(\cdot) \in \mathscr{A}(G)$. Suppose that $\xi_1 \leqslant t \leqslant \xi_2/\|\chi_I\|_{p(\cdot)}$, where $\xi_1, \xi_2 > 0$ and $I \in \mathscr{F}$. Then $t^{p(x)} \in A_\infty(I)$ with A_∞ constant depending only on $p(\cdot), \xi_1, \xi_2$.*

Proof. Let $I' \subset I$, where $I', I \in \mathscr{F}$ and $E \subset I'$ be any measurable subset with $\mu(E) > \mu(I')/2$. Define $f = t\chi_E$. Then

$$|f|_{I'} = \frac{1}{\mu(I')}\int_{I'} t\chi_E(x)d\mu = t\frac{\mu(E)}{\mu(I')} \geqslant \frac{\xi_1}{2},$$

$$\|f\|_{p(\cdot)} = t\|\chi_E\|_{p(\cdot)} \leqslant \xi_2 \frac{\|\chi_E\|_{p(\cdot)}}{\|\chi_I\|_{p(\cdot)}} \leqslant \xi_2.$$

Therefore, f satisfies the hypotheses of Lemma 3.2 with $c_1 = \xi_1/2, c_2 = \xi_2$ and there exists a constant c depending only on $p(\cdot), \xi_1, \xi_2$ such that

$$\frac{1}{2^{p_+}}\int_{I_0} t^{p(\cdot)} d\mu \leqslant c\int_E t^{p(\cdot)} d\mu,$$

which proves that $t^{p(x)} \in A_\infty(I)$. $\square$

Proof of Theorem 1.6. The part $(ii) \Rightarrow (i)$ of Theorem 1.6 follows immediately from Proposition 3.1 and from the fact that $|f|_I\chi_I(x) \leqslant Mf(x)$ for any interval $I \in \mathscr{F}$.

Implication $(i) \Rightarrow (ii)$. Suppose $f \in L^{p(\cdot)}(G)$ and $\|f\|_{p(\cdot)} \leqslant 1$. It is sufficient to proof that there exists a positive constant C (independent of f) such that for any nonnegative function $g \in L^{p'(\cdot)}(G)$, with $\|g\|_{p'(\cdot)} \leqslant 1$

$$\int_G Mf(x)g(x)d\mu \leqslant C. \tag{3.6}$$

For each positive integer k set

$$\Omega_k = \{x \in G : Mf(x) > 3^k\}.$$

Note that

$$\int_{G\backslash\Omega_1} Mf(x)g(x)d\mu \leqslant C. \tag{3.7}$$

Define $D_k = \Omega_k\backslash\Omega_{k+1}$. Let F_k be an arbitrary compact subset of D_k. We will prove that

$$\int_{\cup F_k} Mf(x)g(x)d\mu \leqslant C. \tag{3.8}$$

By simple limiting argument from (3.8) and from (3.7) we obtain (3.6).

Let $\mu(F_k) > 0$. There exists a finite collection of generalized intervals $I_\alpha, \alpha \in A_k$, $F_k \subset \cup_{\alpha\in A_k} I_\alpha$, such that $|f|_{I_\alpha} > 3^k$, $\alpha \in A_k$ and for all fixed α, there exists $x_\alpha \in I_\alpha$ such that $Mf(x_\alpha) \leqslant 3^{k+1}$. Note that if I_{α_1} and I_{α_2} belong to distinct $\mathscr{F}_l$'s and are not disjoint ($\mu(I_{\alpha_1} \cap I_{\alpha_2}) > 0$) then one is a subset of the other. Consequently without loss of generality we may assume that in collection $I_\alpha, \alpha \in A_k$ if $\mu(I_{\alpha_1} \cap I_{\alpha_2}) > 0$

for some α_1 and α_2, then I_{α_1} and I_{α_2} belong to the same $\mathscr{F}_l$'s (for some l). By Vitali covering lemma, we may select from collection $I_\alpha, \alpha \in A_k$ the finite collection of pairwise disjoint intervals $\{I_j^k\}$ $j \in \{1,\dots,N_k\}$ such that $F_k \subset \cup_j 3I_j^k$.

Without loss of generality we may assume that $\mu(F_k) > 0$ for all $k \geqslant 1$. Define the sets $E_1^k = 3I_1^k \cap F_k,$ $E_j^k = (3I_j^k \setminus \cup_{s<j} 3I_s^k) \cap F_k,$ $j > 1$. Note that the sets E_j^k are pairwise disjoint and $\cup_j E_j^k = F_k$.

Define

$$Tg(x) = \sum_{k=1}^{\infty} \sum_j \left(\frac{1}{\mu(I_j^k)} \int_{E_j^k} g\,d\mu \right) \chi_{I_j^k}(x).$$

Using the above definition, we get

$$\begin{aligned}\int_{\cup_k F_k} (Mf)(x)g(x)d\mu &\leqslant 3^{k+1} \sum_{k=1}^{\infty} \sum_j \int_{E_j^k} g\,d\mu \leqslant 3 \sum_{k=1}^{\infty} \sum_j f_{I_j^k} \int_{E_j^k} g\,d\mu \\ &= 3 \int_G fTg \leqslant 6\|f\|_{p(\cdot)} \|Tg\|_{p'(\cdot)},\end{aligned}$$

and consequently for proving (3.8), it is sufficient to show that $\|Tg\|_{p'(\cdot)} \leqslant C$.

Note that $I_j^k \subset \Omega_k = \cup_{l=0}^{\infty} D_{k+l}$ and hence $Tg = \sum_{l=0}^{\infty} T_l g$, where

$$T_l g(x) = \sum_{k=1}^{\infty} \sum_j a_{j,k}(g) \chi_{I_j^k \cap D_{k+l}}(x), \quad (l = 0,1,\dots)$$

where $\alpha_{j,k}(g) = \frac{1}{\mu(I_j^k)} \int_{E_j^k} g\,d\mu$.

Let $\mathscr{I}_1 = \{(j,k) \ : \ \alpha_{j,k}(g) > 1\}$ and $\mathscr{I}_2 = \{(j,k) \ : \ \alpha_{j,k}(g) \leqslant 1\}$.

By condition $p \in \mathscr{A}(G)$ and Hölder inequality implies that for any interval $I \in \mathscr{F}$, $\|\chi_{3I}\|_{p(\cdot)} \leqslant C\|\chi_I\|_{p(\cdot)}$. We have

$$\begin{aligned}\alpha_{j,k}(g) &\leqslant \frac{2}{\mu(I_j^k)} \|\chi_{E_j^k}\|_{p(\cdot)} \|g\chi_{E_j^k}\|_{p'(\cdot)} \leqslant \frac{2}{\mu(I_j^k)} \|\chi_{3I_j^k}\|_{p(\cdot)} \\ &\leqslant \frac{C}{\|\chi_{3I_j^k}\|_{p'(\cdot)}} \leqslant \frac{C}{\|\chi_{I_j^k}\|_{p'(\cdot)}}.\end{aligned}$$

Let $(j,k) \in \mathscr{I}_1$. Then by Corollary 3.3 $\alpha_{j,k}(g)^{p'(x)} \in A_\infty(I_j^k)$ and by Lemma 3.2, (see, also (2.1))

$$\begin{aligned}\int_{I_j^k \cap D_{k+l}} \alpha_{j,k}(g)^{p'(x)} d\mu &\leqslant C \left(\frac{\mu(I_j^k \cap D_{k+l})}{\mu(I_j^k)} \right)^{\varepsilon} \int_{I_j^k} \alpha_{j,k}(g)^{p'(x)} d\mu \\ &\leqslant C \left(\frac{\mu(I_j^k \cap D_{k+l})}{\mu(I_j^k)} \right)^{\varepsilon} \int_{E_j^k} g(x)^{p'(x)} d\mu. \qquad (3.9)\end{aligned}$$

If $(j,k) \in \mathscr{I}_2$, then we have

$$\int_{I_j^k \cap D_{k+l}} \alpha_{j,k}(g)^{p'(x)} d\mu \leqslant \int_{I_j^k \cap D_{k+l}} \alpha_{j,k}(g) d\mu = \frac{\mu(I_j^k \cap D_{k+l})}{\mu(I_j^k)} \int_{E_j^k} g(x) d\mu. \tag{3.10}$$

We need estimate $\mu(I_j^k \cap D_{k+l})$ for $l \geqslant 2$. Let $x \in I_j^k$ and $I \in \mathscr{F}$ be an arbitrary interval such that $x \in I$. Observe that either $I \subset 3I_j^k$ or $I_j^k \subset 3I$. If the second inclusion holds, then $3I \cap D_k \neq \emptyset$ and hence

$$|f|_I \leqslant 3|f|_{3I} \leqslant 3 \cdot 3^{k+1} \leqslant 3^{k+l} \ (l \geqslant 2).$$

Therefore, if $|f|_I > 3^{k+l}$, then $I \subset 3I_j^k$. From this and from weak type property of M, we get

$$\mu(I_j^k \cap D_{k+l}) \leqslant \mu\{x \in I_j^k : M(f\chi_{3I_j^k})(x) > 3^{k+l}\} \leqslant \frac{C}{3^{k+l}} \int_{3I_j^k} |f| d\mu$$
$$\leqslant C\frac{\mu(I_j^k)}{3^{k+l}} |f|_{3I_j^k} \leqslant C\frac{3^{k+1}}{3^{k+l}} \mu(I_j^k) \leqslant \frac{C}{3^l} \mu(I_j^k). \tag{3.11}$$

By estimates (3.9), (3.10), (3.11), when $l \geqslant 2$ we obtain

$$\begin{aligned}\int_G (T_l g(x))^{p'(x)} d\mu &= \sum_{k=1}^{\infty} \sum_j \int_{I_j^k \cap D_{k+l}} \alpha_{j,k}(g)^{p'(x)} d\mu \\ &\leqslant C3^{-l\varepsilon} \sum_{(j,k)\in\mathscr{I}_1} \int_{E_j^k} g(x)^{p'(x)} d\mu + C3^{-l} \sum_{(j,k)\in\mathscr{I}_2} \int_{E_j^k} g(x) d\mu \\ &\leqslant C3^{-l\alpha} \left(\int_G g(x)^{p'(x)} d\mu + \int_G g(x) d\mu \right).\end{aligned}$$

Where $\alpha = \min\{1, \varepsilon\}$.

Using the fact that $\|g\|_1 \leqslant 2\|\chi_{G)}\|_{p'(\cdot)}$, and $\int_G g(x)^{p'(x)} d\mu \leqslant 1$ we obtain

$$\|T_l g\|_{p'(\cdot)} \leqslant C3^{-l\alpha/p'_+} \ (l \geqslant 2).$$

For $l = 0,1$ if we use a trivial estimate $\mu(I_j^k \cap D_{k+l}) \leqslant \mu(I_j^k)$, analogously will be obtained the estimate $\|T_l g\|_{p'(\cdot)} \leqslant C$. Finally we obtain

$$\|Tg\|_{p'(\cdot)} \leqslant \sum_{l=0}^{\infty} \|T_l g\|_{p'(\cdot)} \leqslant C. \quad \square$$

4. Proof of Theorem 1.8

The implication $(ii) \Rightarrow (i)$ is straightforward. Fix $r_0, r_0 < r < 1$, and let $s = 1/r$. by Hölder's inequality, we have that $Mf(x) \leqslant M(|f|^s)(x)^{1/s} = M_s f(x)$. Note that $\||f|^s\|_{p(\cdot)} = \|f\|^s_{sp(\cdot)}$ and

$$\|Mf\|_{p(\cdot)} \leqslant \|M(|f|^s)^{1/s}\|_{p(\cdot)} = \|M(|f|^s)\|^r_{rp(\cdot)} \leqslant C\||f|^s\|^r_{rp(\cdot)} = C\|f\|_{p(\cdot)}.$$

To prove that $(i) \Rightarrow (ii)$, we first construct a $A_1(G)$ weight using the Rubio de Francia iteration algorithm. Given $h \in L^{p(\cdot)}(G)$, define

$$\mathscr{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{L^{p(\cdot)}(G)}},$$

where for $k \geqslant 1$, $M^k = M \circ M \circ \cdots \circ M$ denotes k iterations of the Maximal operator M and $M^0 f = |f|$. The function $\mathscr{R}h(x)$ has the following properties:

(a) For all $x \in G$, $|h(x)| \leqslant \mathscr{R}h(x)$;

(b) $\mathscr{R}$ is bounded on $L^{p(\cdot)}(G)$ and $\|\mathscr{R}h\|_{p(\cdot)} \leqslant 2\|h\|_{p(\cdot)}$;

(c) $\mathscr{R}h \in A_1(G)$ and $[\mathscr{R}h]_{A_1} \leqslant 2\|M\|_{L^{p(\cdot)}(G)}$.

The proof of properties (a),(b),(c) are the same, as Euclidian setting (see [2], pp.157) and we omit it here. By property (c) and Proposition 2.6 there exists $s_0 > 1$ such that for all s, $1 < s < s_0$,

$$M_s(\mathscr{R}h)(x) \leqslant M_{s_0}(\mathscr{R}h)(x) \leqslant 8\|M\|_{L^{p(\cdot)}(G)} \mathscr{R}h(x).$$

Let $r_0 = 1/s_0$. Fix r such that $r_0 < r < 1$. Let $s = 1/r$.
By properties (a) and (b) we have

$$\begin{aligned}
\|Mf\|_{rp(\cdot)} &= \|(Mf)^{1/s}\|^s_{p(\cdot)} = \|M_s(|f|^r)\|^s_{p(\cdot)} \\
&\leqslant \|M_s(\mathscr{R}(|f|^r)\|^s_{p(\cdot)} \leqslant C\|M\|^s_{L^{p(\cdot)}(G)} \|\mathscr{R}(|f|^r)\|^s_{p(\cdot)} \\
&\leqslant C\||f|^r\|^s_{p(\cdot)} = C\|f\|_{rp(\cdot)}. \quad \square
\end{aligned}$$

5. Proof of Theorem 1.10

Since Vilenkin polynomials are dense in $L^{p(\cdot)}(G)$ $(1 \leqslant p_- \leqslant p_+ < \infty)$ the proof of equivalence of (ii) and (iii) is straightforward. The implications $(i) \Rightarrow (ii)$ follows from Rubio de Francia extrapolation theorem (Theorem 1.9), if we use Young's weighted estimates for partial sum $S_n f$ of the Vilenkin-Fourier series (Theorem 1.2), Theorem 1.6, Theorem 1.8 and corollary 1.7.

Proof of $(ii) \Rightarrow (i)$. Consider $I \in \mathscr{F}$. There is $x \in G$ such that I is a proper subset of $x + G_k$ and I is a union of cosets of G_{k+1}. First consider the case $\mu(I) \leqslant \mu(G_k)/2$. Take $\alpha_k = [\mu(G_k)/2\mu(I)]$, where $[a]$ is the largest integer less than or equal to a. We have $\alpha_k \geqslant 1$. Let $f \in L^{p(\cdot)}(G)$ be a nonnegative function with support in I. We use the following estimate (see [18], pp. 286–287): for $x \in I$,

$$\phi_k^{-(\alpha_k-1)/2}(x) S_{\alpha_k m_k}(f\phi_k^{(\alpha_k-1)/2})(x) \geqslant \frac{1}{2\pi\mu(I)} \int_I f(t)d\mu = \frac{1}{2\pi} A_I f(x).$$

We have

$$\|A_I f\|_{p(\cdot)} \leqslant C\|\phi_k^{-(\alpha_k-1)/2} S_{\alpha_k m_k}(f\phi_k^{-(\alpha_k-1)/2})\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}.$$

From this estimate we obtain in standard way (1.2) in case $\mu(I) \leqslant \mu(G_k)/2$ (see Proposition 3.1).

Consider the case $\mu(I) > \mu(G_k)/2$. Note that every coset of G_k is in $\mathscr{F}_{k-1}$ and $\mu(G_k) \leqslant \mu(G_{k-1})/2$ and consequently (1.2) holds for all cosets of G_k. We have

$$\|\chi_I\|_{p(\cdot)}\|\chi_I\|_{p'(\cdot)} \leqslant \|\chi_{x+G_k}\|_{p(\cdot)}\|\chi_{x+G_k}\|_{p'(\cdot)} \leqslant C\mu(G_k) \leqslant C\mu(I). \quad \square$$

REFERENCES

[1] R. COIFMAN AND C. FEFFERMAN, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.

[2] D. CRUZ-URIBE AND A. FIORENZA, *Variable Lebesgue Spaces, Foundations and Harmonic Analysis*, Birkhäuser, Basel (2013).

[3] D. CRUZ-URIBE, J. M. MARTELL AND C. PÉREZ, *Weights, extrapolation and theory of Rubio de Francia*, Operator Theory: Advances and Applications, 2015, Birkhäuser/Springer Basel AG, Basel, 2011.

[4] D. CRUZ-URIBE, A. FIORENZA, J. M. MARTELL AND C. PÉREZ, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fen. Math. **31** (1): 239–264, 2006.

[5] L. DIENING, P. HARJULEHTO, P. HÄSTÖ AND M. RŮŽIČKA, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, 2017. Springer, Heidelberg (2011).

[6] J. DUOANDIKOETXEA, F. J. MARTÍN-REYES AND S. OMBROSI, *On the A_∞ conditions for general bases*, Math. Z. **282** (2016), no. 3–4, 955–972.

[7] J. A. GOSSELIN, *A weighted norm inequality for Vilenkin-Fourier series*, Proc. Amer. Math. Soc. **49** (1975), 349–353.

[8] Y. JIAO, F. WEISZ, L. WU AND D. ZHOU, *Variable martingale Hardy spaces and their applications in Fourier analysis*, Dissertationes Math. **550** (2020) pp. 1–67.

[9] T. KOPALIANI, *Infimal convolution and Muckenhoupt $A_{p(\cdot)}$ condition in variable Lp spaces*, Arch. Math. (Basel), **89** (2): 185–192, 2007.

[10] A. K. LERNER, *On some questions related to the maximal operator on variable L^p spaces*, Trans. Amer. Math. Soc., **362** (8): 4229–4242, 2010.

[11] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–251.

[12] B. MUCKENHOUPT, *The equivalence of two conditions for weight functions*, Studia Math. **49** (1974), 101–106.

[13] L.-E. PERSSON, G. TEPHNADZE AND F. WEISZ, *Martingale Hardy spaces and summability of the one-dimensional Vilenkin-Fourier series*, Birkhäuser/Springer, 2022.

[14] F. SCHIPP, *On L^p-convergence of series with respect to product systems*, Analysis Mathematica, **2** (1976), 49–64.

[15] P. SIMON, *Verallgemeinerte Walsh-Fourierreihen*, II, Acta Math. Acad. Sci. Hungar. **27** (1976), 49–64.

[16] P. SIMON, *On a maximal function*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **21** (1978), 41–44.

[17] W.-S. YOUNG, *Mean convergence of generalized Walsh-Fourier series*, Trans. Amer. Math. Soc. **218** (1976), 311–320.

[18] W.-S. YOUNG, *Weighted norm inequalities for Vilenkin-Fourier series*, Trans. Amer. Math. Soc. **340** (1993), 273–291.
[19] C. WATARI, *On generalized Walsh Fourier series*, Tohoku Math. J. (2) **10**, 211–241 (1958).

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