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ON THE DEMOCRACY INEQUALITY FOR HAAR SYSTEMS AND SOME GEOMETRIC AND ANALYTICAL PROPERTIES IN HERZ SPACES ON DYADIC SETTINGS

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Abstract. In this paper we explore geometric condition to obtain that the Democracy inequality of Haar systems on Herz spaces implies that these spaces are Lebesgue spaces in the setting of spaces of homogeneous type. For this purpose, we give previously a construction of dyadic Herz spaces and prove some analytic properties.

1. Introduction

The Herz spaces in the Euclidean setting of $\mathbb{R}^n$, initially introduced by Herz in [7] in the context of the study of Berstein-type theorems and Lipschitz spaces, were characterized by an equivalent norm in a later work by Johnson in [9]. Today, this characterization is used as the definition of Herz spaces and may be apt to extend the definition of these spaces to the context of other metrics in $\mathbb{R}^n$. Indeed, Ragusa in [11] defines the Herz spaces associated with parabolic metrics in $\mathbb{R}^n$ and studies their applications in this context to obtain regularity results of weak solutions for parabolic differential equations in divergence form. With this in mind we can approach in two ways the extension to general measure metric spaces. One of them is following [11], that is, considering balls and crowns associated with the underlying metric in the measure metric space. On the other hand, as we will state in Section 3 of this work, we can develop a dyadic theory for Herz spaces. In particular, this dyadic context arises naturally when Haar wavelets are considered. In the case of Herz spaces in $\mathbb{R}^n$, results related with problems of non linear approximation have been obtained in the work of Isuki and Sawano in [8] where they consider in $\mathbb{R}^n$, with the usual metric, the following norm given in [9]. For $1 < p,q < \infty$, the cube $Q_0 = [-1,1]^n$ and the crowns $C_j = \left[-2^j,2^j\right]^n \setminus \left[-2^{j-1},2^{j-1}\right]^n$ for $j \in \mathbb{N}$, the Herz Spaces in $\mathbb{R}^n$, $\mathscr{K}_{\mathbf{p},\mathbf{q}}(\mathbb{R}^{\mathbf{n}})$, are given by the set of all measurable functions f for which the norm

$$\|f\|_{\mathscr{K}_{p,q}} = \left(\|f\chi_{Q_0}\|_p^q + \sum_{l\in\mathbb{N}} \|f\chi_{C_l}\|_p^q \right)^{\frac{1}{q}}$$

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is finite, where $\|f\|_p = (\int_{\mathbb{R}^n} |f|^p d\mu)^{1/p}$ is the p-norm of the Lebesgue spaces. By its definition, is $p = q$ we have $\mathscr{K}_{p,q}(\mathbb{R}^n) = \mathscr{K}_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. Thus the Herz spaces are generalizations of the classical Lebesgue spaces.

If we consider the usual dyadic family $D = \cup_{j\in\mathbb{Z}} D^j$ of the dyadic cubes in $\mathbb{R}^n$, that is the cubes $Q^j_{\mathbf{k}} = \prod_{i=1}^n I^j_{k_i} \in D^j$ where $I^j_{k_i} = [k_i 2^{-j}, (k_i+1)2^{-j})$ with $j \in \mathbb{Z}$ and $\mathbf{k} = (k_1,\dots,k_n) \in \mathbb{Z}^n$, it is not difficult to see that the crowns and the cube $[-1,1]^n$ in the above definition of $\mathscr{K}_{p,q}$-norm can be seen like unions of dyadic usual cubes in $\mathbb{R}^n$. This simple observation permit us to extend this space to the contexts of measure metrics spaces with a dyadic point of view. In fact, since we work on dyadic structure the usual norm $K_{p,q}$ is equivalent to a finite sum of dyadic norms over quadrants in the euclidean context (see Section 3). Thus we will generalize this in the setting of spaces of homogeneous type.

On the other hand in non linear approximation theory is relevant the notion of democratic system (see [10]). Recall that a normalized system $\beta = \{\phi_i\}_{i\in I}$ with I a denumerable set is democratic in a Banach space $\mathbb{B}$ if there exist a positive constant C such that the following Democracy inequality

$$\|\sum_{\phi\in F_1}\phi\| \leqslant C\|\sum_{\phi\in F_2}\phi\|$$

holds for every finite subsets F_1 and F_2 of β with the same cardinal, that is $|F_1| = |F_2|$. Temlyakov and Konyagin proved that this property is very important to characterize good properties of approximation algorithms associated to systems β (see [10]). The democracy property has been extensively studied in particular for the case of Haar wavelets in the Euclidean setting (see [4], [5] and [12]). In the sequel we shall write $\mathbb{Z}_0^+$ to denote the set of all non negative integers and $m(A)$ to denote the Lebesgue measure of a measurable subset $A \subseteq \mathbb{R}^n$. Recall that the usual Haar system in $\mathbb{R}^n$, $\mathbf{H}$, is the system of all functions $h(x) = \prod_{i=1}^n h_I^{\varepsilon_i}(x_i)$ where $(\varepsilon_1,\dots,\varepsilon_n) \in \{0,1\}^n \setminus (0,\dots,0)$, $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, $h_I^0 = \frac{\chi_I}{m(I)^{1/2}}$ and $h_I^1(x_i) = h_I(x_i) = \alpha_I(\chi_{I^-}(x_i) - \chi_{I^+}(x_i))$ are the Haar functions in $\mathbb{R}$ where we write I^- and I^+ to denote the left and right dyadic subinterval, respectively, of the dyadic interval $I = [k2^{-j}, (k+1)2^{-j})$ with $j,k \in \mathbb{Z}$ and α_I is such that h_I is normalized in L^2. This Haar system has the following universal property on certain Banach spaces $\mathbb{B}$:

If $\mathbf{H}$ *is democratic in the Banach space* $\mathbb{B}$ *then* $\mathbb{B}$ *is a Lebesgue spaces.*

In particular, this holds for $\mathbb{B} = L^{p,q}(\mathbb{R}^n)$ with$1 < p,q < \infty$ the Lorentz spaces (see [5]), $\mathbb{B} = L^{\varphi}(\mathbb{R}^n)$ the Orlicz spaces (see [4]) and in general for $\mathbb{B}$ invariant rearrangement spaces on $[0,1]$ (see [12]).

This universal property does not hold in general when we consider the abstract setting of spaces of homogeneous type. For example in [2] the authors have shown that this universal property failed for the Lorentz spaces in spaces of homogeneous type. More precisely they have given an example of a space of homogeneous type such that the Haar systems is democratic on the Lorentz spaces $L^{p,q}$ on its metric measure space for all $1 < p,q < \infty$ and $L^{p,q}$ is not any Lebesgue space. In the Euclidean case of $\mathbb{R}^n$, Izuki and Sawano proved in [8] the following result for the Haar system truncated to zero resolution level given by $\hat{\mathbf{H}} = \mathbf{H_0} \cup \{\frac{\chi_Q}{m(Q)^{1/2}} : Q \in D^0\}$ where $\mathbf{H_0}$ is the set of all function $h \in \mathbf{H}$ defined above with $j \in \mathbb{Z}_0^+$.

THEOREM 1. *The Haar system truncated to zero resolution, $\hat{\mathbf{H}}$, is democratic in $\mathscr{K}_{p,q}(\mathbb{R}^n)$ with $1 < p,q < \infty$ if and only if $p = q$.*

Notice that this result is in line with the above universal property considering the Banach spaces $\mathscr{K}_{\mathbf{p},\mathbf{q}}(\mathbb{R}^{\mathbf{n}})$ and the Haar system truncated to zero resolution $\hat{\mathbf{H}}$.

Since we have at our disposal, in the setting of measure metric spaces, the structure of dyadic cubes (see [3]) and the Haar systems associated (see [1]), we propose to approach the following questions:

1. Can we define Herz spaces associated to dyadic structure in the context of space of homogeneous type? What basic analytical properties do these spaces have?

2. Does the Haar system satisfy the Democracy inequality in the Herz space in the context of space of homogeneous type? Does the Theorem 1 holds in space of homogeneous type?

For the first point we use the dyadic structure of space of homogeneous type given by [3]. In particular we define such spaces and prove that they are Banach spaces. Also we show that these Herz spaces contain some proper dense subspace. For the second point we show that the Haar system can be democratic in the Herz spaces $K_{p,q}$ with $p \neq q$ in contrast with the results in [8] in $\mathbb{R}^n$. Moreover we explore geometric conditions on space of homogeneous type such that we can recover the above universal property respect to the democracy in the euclidean setting. More precisely the following statement, that we give here in a informal way contains the main results of this work that we will state and prove in Section 3 and Section 4 (see Section 2 to precise definition of dyadic family $\mathscr{D}$ and Haar system $\mathscr{H}$ in spaces of homogeneous type and see Section 3 for the precise definition of Herz spaces $\mathscr{K}_{p,q}(X)$ in such setting of measure metric spaces):

Let (X,d,μ) be a space of homogeneous type, $\mathscr{D}$ a dyadic family and $\mathscr{H}$ a Haar system associated to $\mathscr{D}$. Then

1. *The Herz spaces $\mathscr{K}_{p,q}(X)$ are Banach spaces for every $1 < p,q < \infty$.* (See Theorem 6).

2. *The space of all function in $\mathscr{K}_{p,q}(X)$ with dyadic support is dense in $\mathscr{K}_{p,q}(X)$ for all $1 < p,q < \infty$.* (See Proposition 7).

3. *There exist a space of homogeneos type (X,d,μ) such that the Haar system truncated to zero level is democratic in Herz spaces $\mathscr{K}_{p,q}(X)$ for every $1 < p,q < \infty$.* (See Proposition 8).

4. *For the space (X,d,μ) in the above item we get that the Herz space $\mathscr{K}_{p,q}(X)$ is not a Lebesgue space for any $1 < p,q < \infty$* (see Proposition 9).

5. *Let (X,d,μ) be a space of homogeneous type with the geometric property of concentration. If $1 < p,q < \infty$ then the Democracy inequality of Haar system*

truncated to zero level in the Herz space $\mathscr{K}_{p,q}(X)$ implies that $\mathscr{K}_{p,q}(X)$ is a Lebesgue space. (See Theorem 11).

This paper is organized as follow. In Section 2 we state the properties of dyadic families and Haar systems in spaces of homogeneous type. In Section 3 we introduce the Herz spaces associated to dyadic families in the setting of spaces of homogeneous type and we state their basic analytical properties. In Section 4 we study the Democracy inequality for Haar systems in Herz spaces. In particular we shows that in the general setting of measure metric spaces such property does not imply that the Herz space is a Lebesgue space. We explore geometric properties on the space of homogeneous type (X,d,μ) such that the universal property mentioned above holds.

2. On space of homogeneous type and Haar wavelet in Lebesgue spaces

In this section we introduce the basic objects of dyadic families and Haar system in the general context of space of homogeneous type that we consider in this work. Let us recall that a space of homogeneous type is a triple (X,d,μ) where X is a set, d is a quasi-metric on X and μ is a measure defined on the Borel σ-algebra that satisfies the following doubling condition: there exists a positive constant C such that

$$0<\mu(B(x,2r))\leqslant C\mu(B(x,r))<\infty$$

for all points $x\in X$ and every positive real number r, where $B(x,r)$ is the ball with center $x\in X$ and radius r. We take from [1] the following definition of dyadic family.

DEFINITION 1. Let (X,d,μ) be a space of homogeneous type. We say that $\mathscr{D}=\bigcup_{j\in\mathbb{Z}}\mathscr{D}^j$ is a *dyadic family* on X if each $\mathscr{D}^j$ is a family of open measurable subsets Q of X such that

(d.1) For each $j\in\mathbb{Z}$ the cubes in $\mathscr{D}^j$ are pairwise disjoint.

(d.2) For each $j\in\mathbb{Z}$ the family $\mathscr{D}^j$ covers all X in the sense that $\mu(X\setminus\bigcup_{Q\in D^j}Q)=0$.

(d.3) If $Q\in\mathscr{D}^j$ and $i<j$, then there exists a unique $\tilde{Q}\in\mathscr{D}^i$, such that $Q\subseteq\tilde{Q}$.

(d.4) If $Q\in\mathscr{D}^j$ and if $Q'\in\mathscr{D}^i$ with $i\leqslant j$, then $Q\subset Q'$ or $Q\cap Q'=\emptyset$.

(d.5) There exist $\delta\in(0,1)$ and two constant, a_1 and a_2, such that for each $Q\in\mathscr{D}^j$ there exists a point $x\in Q$ for which $B_d(x,a_1\delta^j)\subseteq Q\subseteq B_d(x,a_2\delta^j)$.

In the sequel we consider the following notion of quadrant associated to a dyadic family as given in [1].

DEFINITION 2. Let (X,d,μ) be a space of homogeneous type and $\mathscr{D}$ a dyadic family. Let Q a fixed dyadic cube in $\mathscr{D}$. We call *quadrant* of X containing Q to the set

$$\mathfrak{C}(Q)=\bigcup_{\{Q'\in\mathscr{D}:Q'\supseteq Q\}}Q'.$$

In [1] the authors prove the main properties of the quadrants. They are contained in the following statement.

LEMMA 1. *Let (X,d,μ) be a space of homogeneous type and $\mathscr{D}$ a dyadic family. The family of quadrants satisfies the following properties:*

(c.1) For each quadrant $\mathfrak{C}$, we have that $(\mathfrak{C},d,\mu)$ is a space of homogeneous type;

(c.2) if the intersection of two quadrants is nonempty, the quadrants coincide;

(c.3) there exists a purely geometric constant M such that $X=\cup_{i=1}^{M}\mathfrak{C}_i$, with $\mathfrak{C}_i$ quadrants of X;

(c.4) if $\mu(X)=\infty$ then for every quadrant $\mathfrak{C}$, we also have $\mu(\mathfrak{C})=\infty$.

Now we introduce the Haar wavelets in the setting of spaces of homogeneous type given in [1]. This system satisfies the sames properties that the usual Haar wavelets in $\mathbb{R}^n$. More precisely we have the following definition, where with $\tilde{\mathscr{D}}^j$ we denote the family of dyadic cubes $Q\in\mathscr{D}^j$ such that $\#(\mathscr{O}(Q))>1$ where $\mathscr{O}(Q)=\{R\in\mathscr{D}^{j+1}: R\subset Q\}$ with $\#(\mathscr{O}(Q))$ the number of elements of set $\mathscr{O}(Q)$.

DEFINITION 3. Let $\mathscr{D}$ be a dyadic family on the space of homogeneous type (X,d,μ). A system $\mathscr{H}$ of real borelian measurable simple functions h defined on X is a *Haar System* associated to $\mathscr{D}$ if satisfy:

(h.1) For each $h\in\mathscr{H}$ there exists a unique $j\in\mathbb{Z}$ and a cube $Q=Q(h)\in\tilde{D}^j$ such that $\{x\in X: h(x)\neq 0\}\subseteq Q$ and this property does not hold for any dyadic cube in $\mathscr{D}^{j+1}$.

(h.2) For all $Q\in\tilde{D}=\cup_{j\in\mathbb{Z}}\tilde{\mathscr{D}}^j$ there exist exactly $M_Q=\#(\mathscr{O}(Q))-1$, $M_Q\geqslant 1$, functions $h\in\mathscr{H}$ such that $(h.1)$ is hold. We write $\mathscr{H}_Q$ to denote the set of these functions h.

(h.3) For each $h\in\mathscr{H}$ we have that $\int_X h\,d\mu=0$.

(h.4) For each $Q\in\tilde{D}$, if V_Q is the vector space of all funtions defined on cubes that are constant in each $Q'\in\mathscr{O}(Q)$, then the system $\left\{\frac{\chi_Q}{(\mu(Q))^{1/2}}\right\}\cup\mathscr{H}_Q$ is an orthonormal basis for V_Q.

In this work we shall consider the following truncated Haar system at zero level.

DEFINITION 4. Let $\mathscr{D}$ be a dyadic family on the space of homogeneous type (X,d,μ) and $\mathscr{H}$ a Haar System associated to $\mathscr{D}$. The *truncated Haar system at zero level* is

$$\mathscr{B}=\left\{\frac{\chi_R}{\mu(R)^{\frac{1}{2}}}, R\in\mathscr{D}^0\right\}\cup\mathscr{H}_0,$$

where $\mathscr{H}_0=\left\{h: Q(h)\in\mathscr{D}^j, j\geqslant 0\right\}$.

One of the main results on Haar systems in Lebesgue spaces on spaces of homogeneous type (X,d,μ) is contained in the following statement (see [1] for the proof). As usual we write $\langle f,g\rangle$ to denote $\int_X f(x)g(x)d\mu(x)$ and $\|f\|_p$ to denote the Lebesgue L^p-norm. Notice that we consider real valued functions.

PROPOSITION 2. *Let $\mathscr{D}$ be a dyadic family on the space of homogeneous type (X,d,μ) and $\mathscr{H}$ a Haar System associated to $\mathscr{D}$. Then $\mathscr{H}$ is an orthonormal basis of $L^2(X,\mu)$. Moreover, $\mathscr{H}$ is an unconditional basis of $L^p(X,\mu)$ with $1<p<\infty$ and there exist two positive constant C_1 and C_2 such that*

$$C_1\|f\|_p \leqslant \left\| \left(\sum_{h\in\mathscr{H}} |\langle f,h\rangle h|^2 \right)^{1/2} \right\|_p \leqslant C_2\|f\|_p$$

and

$$\left\| \sum_{R\in\mathscr{D}^0} \left\langle f, \frac{\chi_R}{\mu(R)^{\frac{1}{2}}} \right\rangle \frac{\chi_R}{\mu(R)^{\frac{1}{2}}} \right\|_p \leqslant \|f\|_p$$

for all function $f\in L^p(X,\mu)$.

In [1] the authors also proove that for each $f\in L^2$

$$f = P_0(f) + \sum_{j=0}^{\infty} P_{j+1}(f) - P_j(f),$$

where the convergence is in L^2-sense and $P_j(f) = \sum_{Q\in\mathscr{D}^j} \left\langle f, \frac{\chi(Q)}{\mu(Q)^{\frac{1}{2}}} \right\rangle \frac{\chi(Q)}{\mu(Q)^{\frac{1}{2}}}$.

On the other hand for $x\in X$ we have a unique dyadic cube $Q^j_{k(x)} \in \mathscr{D}^j$ such that $x\in Q^j_{k(x)}$ for all $j\in\mathbb{Z}$. Then for $N\in\mathbb{N}$ we consider

$$f^N(x) = P_0(f)(x) + \sum_{j=0}^{N} P_{j+1}(f)(x) - P_j(f)(x).$$

Then $f^N(x) = P_{N+1}(f)(x) = \dfrac{1}{\mu(Q^{N+1}_{k(x)})} \displaystyle\int_{Q^{N+1}_{k(x)}} f(y)dy$. Hence from Lebesgue's differentiation Theorem we have, for all most every $x\in X$, that

$$f(x) = \sum_{R\in\mathscr{D}^0} \left\langle f, \frac{\chi(R)}{\mu(R)^{\frac{1}{2}}} \right\rangle \frac{\chi(R)(x)}{\mu(R)^{\frac{1}{2}}} + \sum_{h\in\mathscr{H}_0} \langle f,h\rangle h(x). \tag{1}$$

From (1) and Proposition 2 we have the following result.

PROPOSITION 3. *Let $\mathscr{D}$ be a dyadic family on the space of homogeneous type (X,d,μ). Let $\mathscr{B} = \left\{ \frac{\chi_R}{\mu(R)^{\frac{1}{2}}}, R \in \mathscr{D}^0 \right\} \cup \mathscr{H}_0$, where $\mathscr{H}_0 = \left\{ h : Q(h) \in \mathscr{D}^j, j \geqslant 0 \right\}$. Then for all $f \in L^p(X,\mu)$ with $1 < p < \infty$ we get*

$$\|f\|_p \sim \left(\left\| \sum_{R \in \mathscr{D}^0} \left\langle f, \frac{\chi_R}{\mu(R)^{1/2}} \right\rangle \frac{\chi_R}{\mu(R)^{1/2}} \right\|_p^p + \left\| \left(\sum_{h \in \mathscr{H}_0} |\langle f,h \rangle|^2 |h|^2 \right)^{1/2} \right\|_p^p \right)^{1/p}$$

3. Analytical properties of Herz spaces in space of homogeneous type

In this section we introduce the Herz spaces on measure metric spaces. Our start point is the following simple result which is obtained directly from the above definition of Herz spaces in $\mathbb{R}^n$ given in the Introduction.

LEMMA 2. *Let $f \in \mathscr{K}_{p,q}(\mathbb{R}^n)$ with $1 < p,q < \infty$, then*

$$\|f\|_{\mathscr{K}_{p,q}} \sim \sum_{i=1}^{2^n} \|f_i\|_{\mathscr{K}_{p,q}}$$

where $f_i = f\chi_{\mathfrak{C}_i}$ and $\mathfrak{C}_i$ denote each usual quadrant in $\mathbb{R}^n$.

From this result we introduce our Herz spaces in the abstract context of spaces of homogeneous type.

DEFINITION 5. Let (X,d,μ) be a space of homogeneous type, $1 < p,q < \infty$ and M the number of quadrants associated to the dyadic family $\mathscr{D}$ on X. For $i = 1,\ldots,M$, let Q_0^i be given a dyadic cube in the quadrant $\mathfrak{C}_i$. Set $(Q_n^i)_{n\in\mathbb{N}}$ to denote the sequences of *non trivial ancestors* of Q_0^i, that is $Q_{n-1}^i \subset Q_n^i$ strictly for each $n \in \mathbb{N}$, with $Q_n^i \in \mathscr{D}$. If C_l^i denote the crowns $Q_l^i \setminus Q_{l-1}^i$ for $l \in \mathbb{N}$, we define the *Herz space on* (X,d,μ), $\mathscr{K}_{p,q} = \mathscr{K}_{p,q}(X,\mu)$, as the space of all μ-measurable functions f such that

$$\|f\|_{\mathscr{K}_{p,q}} := \sum_{i=1}^{M} (\|f\chi_{Q_0^i}\|_p^q + \sum_{l\in\mathbb{N}} \|f\chi_{C_l^i}\|_p^q)^{\frac{1}{q}} \tag{2}$$

is finite.

Notice that from the following result, which is an immediate consequence of the properties of the L^p norms and l_q norms, we have that $\left(\mathscr{K}_{p,q}, \|.\|_{\mathscr{K}_{p,q}}\right)$ is a normed space.

PROPOSITION 4. *Let (X,d,μ) be a space of homogeneous type, $\mathfrak{C}_i$ a quadrant, $1 < p,q < \infty$ and $\hat{\mathscr{K}}_{p,q}$ the set of all functions μ-measurables, f, such that $f(x) = 0$ for $x \in X \setminus \mathfrak{C}_i$ and $\|f\|^* < \infty$ where*

$$\|f\|^* = (\|f\chi_{Q_0^i}\|_p^q + \sum_{l\in\mathbb{N}} \|f\chi_{C_l^i}\|_p^q)^{\frac{1}{q}}.$$

Then $(\hat{\mathscr{K}}_{p,q}, \|f\|^*)$ *is a normed space.*

In the sequel in this work, given a space of homogeneous type (X,d,μ) with a dyadic family $\mathscr{D}$, from the above proposition and the Definition 5 we will consider without loss of generality that X has a unique quadrant.

On the other hand, Definition 5 is independent of the initial dyadic cube Q_0. In fact we have the following result which shows that if we consider two different initial dyadic cubes then the Herz spaces associated to them are the same space.

PROPOSITION 5. *Let* (X,d,μ) *be a space of homogeneous type with a dyadic family* $\mathscr{D}$*. Let* Q_0 *and* Q_0^* *be two given different dyadic cubes, then there exist two positives constant* C_1 *and* C_2 *such that*

$$C_1\|f\|_{\mathscr{K}_{p,q,Q_0}(X)} \leqslant \|f\|_{\mathscr{K}_{p,q,Q_0^*}(X)} \leqslant C_2\|f\|_{\mathscr{K}_{p,q,Q_0}(X)},$$

where $\mathscr{K}_{p,q,Q}(X)$ *denote* $\|.\|_{\mathscr{K}_{p,q}(X)}$ *defined from of initial dyadic cube* Q.

Proof. Given Q_0 and Q_0^*, we consider the sequences of their non trivial ancestors Q_n and Q_n^* respectively with $n \in \mathbb{N}$. Let C_n and C_n^* be the crowns associated to Q_0 and Q_0^* respectively. That is, $C_n = Q_{n+1} \setminus Q_n$ and $C_n^* = Q_{n+1}^* \setminus Q_n^*$ with $n \in \mathbb{N}$.

Let Q' be the first common ancestor dyadic cube of Q_0 and Q_0^*. Let i and j be in $\mathbb{N}$ such that $Q_i^* = Q'$ and $Q_j = Q'$. We note that $C_{j+k} = C_{i+k}^*$ for all $k \in \mathbb{N}$. Hence $C_l = C_{l+(i-j)}^* \forall l \geqslant (j-1)$.

Also $Q' = Q_0 \cup C_1 \cup \ldots \cup C_j$ and $Q' = Q_0^* \cup C_1^* \cup \ldots \cup C_i^*$.

Thus,

$$(\|f\chi_{Q_0}\|_p^q + \sum_{l=1}^{j} \|f\chi_{C_l}\|_p^q) \leqslant \|f\chi_{Q'}\|_p^q \leqslant (j+1)^{q-1}(\|f\chi_{Q_0}\|_p^q + \sum_{l=1}^{j} \|f\chi_{C_l}\|_p^q).$$

and

$$(\|f\chi_{Q_0^*}\|_p^q + \sum_{l=1}^{i} \|f\chi_{C_l^*}\|_p^q) \leqslant \|f\chi_{Q'}\|_p^q \leqslant (i+1)^{q-1}(\|f\chi_{Q_0^*}\|_p^q + \sum_{l=1}^{i} \|f\chi_{C_l^*}\|_p^q).$$

Therefore

$$\begin{aligned}
\|f\|_{\mathscr{K}_{p,q,Q_0}(X)}^q &= \|f\chi_{Q_0}\|_p^q + \sum_{l=1}^{j} \|f\chi_{C_l}\|_p^q + \sum_{l=j+1}^{\infty} \|f\chi_{C_l}\|_p^q \\
&\leqslant \|f\chi_{Q'}\|_p^q + \sum_{l=j+1}^{\infty} \|f\chi_{C_l}\|_p^q \\
&\leqslant (i+1)^{q-1}(\|f\chi_{Q_0^*}\|_p^q + \sum_{l=1}^{i} \|f\chi_{C_l^*}\|_p^q) + \sum_{l=j+1}^{\infty} \|f\chi_{C_{l+(i-j)}^*}\|_p^q \\
&\leqslant [(i+1)^{q-1} + 1]\|f\|_{\mathscr{K}_{p,q,Q_0^*}(X)}^q,
\end{aligned}$$

and

$$\begin{aligned}\|f\|^q_{\mathscr{K}_{p,q,Q_0}(X)} &= \|f\chi_{Q_0}\|^q_p + \sum_{l=1}^{j} \|f\chi_{C_l}\|^q_p + \sum_{l=j+1}^{\infty} \|f\chi_{C_l}\|^q_p \\ &\geqslant (j+1)^{-q+1}\|f\chi_{Q'}\|^q_p + \sum_{l=j+1}^{\infty} \|f\chi_{C_l}\|^q_p \\ &\geqslant (j+1)^{-q+1}(\|f\chi_{Q_0^*}\|^q_p + \sum_{l=1}^{i} \|f\chi_{C_l^*}\|^q_p) + \sum_{l=j+1}^{\infty} \|f\chi_{C^*_{l+(i-j)}}\|^q_p \\ &\geqslant (j+1)^{-q+1}\|f\|^q_{\mathscr{K}_{p,q,Q_0^*}(X)}. \quad \square\end{aligned}$$

As in the classical Euclidean setting, in this abstract context of spaces of homogeneous type we have that the Herz space is a Banach space. Although the proof follows standard arguments we give an outline for completeness. For this, first we state the basic results of convergence in Herz spaces in homogeneous type. Their proofs follow the basic Lebesgue Monotone and Dominated convergence Theorem.

LEMMA 3. *Let (X,d,μ) be a space of homogeneous type and $1 < p,q < \infty$.*

(a) If $(f_n)_{n\in\mathbb{N}} \subset \mathscr{K}_{p,q}(X)$ and for almost every $x \in X$ we have that $0 \leqslant f_n(x) \leqslant f_{n+1}(x)$, forall $n \in \mathbb{N}$. If $G(x) = \lim_{x\to\infty} f_n(x)$ then we have

$$\|G\|_{\mathscr{K}_{p,q}(X)} = \lim_{n\to\infty} \|f_n\|_{\mathscr{K}_{p,q}(X)}.$$

(b) If $(f_n)_{n\in\mathbb{N}} \subset \mathscr{K}_{p,q}(X)$ and for almost every $x \in X$, we have that $|f_n(x)| \leqslant H(x)$ $\forall n \in \mathbb{N}$,where $H \in \mathscr{K}_{p,q}(X)$,then if there exist for almost every $x \in X$ the limit $F(x) = \lim_{n\to\infty} f_n(x)$, we get that

$$\|F\|_{\mathscr{K}_{p,q}(X)} = \lim_{n\to\infty} \|f_n\|_{\mathscr{K}_{p,q}(X)}.$$

From de above result we get the completeness of Herz spaces in spaces of homogeneous type.

THEOREM 6. *Let (X,d,μ) be a space of homogeneous type. For $1 < p,q < \infty$, the Herz space $\mathscr{K}_{p,q}(X)$ is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{K}_{p,q}(X)$. We take a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ with $k \in \mathbb{N}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\|_{\mathscr{K}_{p,q}(X)} \leqslant \frac{1}{2^k}. \tag{3}$$

With this subsequence we consider the following two series, for $x \in X$.

(I) $f_{n_1}(x)+\sum_{k=1}^{\infty}(f_{n_{k+1}}(x)-f_{n_k}(x))$.

(II) $|f_{n_1}(x)|+\sum_{k=1}^{\infty}|f_{n_{k+1}}(x)-f_{n_k}(x)|$.

From the above Lemma, part (a), we get that the serie (II) is convergent at almost every point $x\in X$. Moreover, with $G(x)=|f_{n_1}(x)|+\sum_{k=1}^{\infty}|(f_{n_{k+1}}(x)-f_{n_k}(x))|$ from (3) we get that $G\in\mathscr{K}_{p,q}(X)$. On the other hand, from the convergence of serie (II), we get that the serie (I) also is convergent at almost every point $x\in X$.

We write $F(x)$ to denote the serie (I) for those points $x\in X$ where the serie is convergent. That is

$$F(x)=f_{n_1}(x)+\sum_{k=1}^{\infty}(f_{n_{k+1}}(x)-f_{n_k}(x)).$$

Thus, with $F_k(x)$ the k-th partial sum of (I) we have that $|F_k(x)|\leqslant G(x)$.

So, from the above lemma part (b) we get that

$$\|F\|_{\mathscr{K}_{p,q}(X)}=\lim_{k\to\infty}\|F_k\|_{\mathscr{K}_{p,q}(X)}\leqslant\|G\|_{\mathscr{K}_{p,q}(X)}<\infty.$$

Therefore $F\in\mathscr{K}_{p,q}(X)$.

On the other hand,

$$\begin{aligned}F_{k-1}(x)&=f_{n_1}(x)+\sum_{i=1}^{k-1}(f_{n_{i+1}}(x)-f_{n_i}(x))\\&=f_{n_k}(x).\end{aligned}$$

Thus $f_{n_k}(x)\to F(x)$ where $k\to\infty$, at almost every point $x\in X$.

The theorem will be proved if we show that $f_n\to F$ in $\mathscr{K}_{p,q}(X)$. For this, it is enough to prove that $f_{n_k}\to F$ in $\mathscr{K}_{p,q}(X)$ when $k\to\infty$ and use that the sequence (f_n) is a Cauchy sequence in $\mathscr{K}_{p,q}(X)$.

Notice that for $k\in\mathbb{N}$ we have that

$$\|F-f_{n_k}\|^q_{\mathscr{K}_{p,q}(X)}=\|\left(F-f_{n_k}\right)\chi_{Q_0}\|^q_p+\sum_{l=1}^{\infty}\|\left(F-f_{n_k}\right)\chi_{C_l}\|^q_p. \tag{4}$$

For a μ-measurable $A\subset X$ we get that

$$|\left(F(x)-f_{n_k}(x)\right)\chi_A(x)|^p\leqslant 2^p|G(x)\chi_A(x)|^p.$$

Since $G\in\mathscr{K}_{p,q}(X)$ then $|G(x)\chi_A(x)|^p\in L^1(X,\mu)$.

Thus, from Lebesgue Dominated Convergence Theorem in $L^1(X,\mu)$ we have that

$$\begin{aligned}\lim_{k\to\infty}\int_X|F(x)-f_{n_k}(x)|^p\chi_A(x)d\mu(x)&=\int_X\lim_{k\to\infty}|F(x)-f_{n_k}(x)|^p\chi_A(x)d\mu(x)\\&=0.\end{aligned}$$

In particular, with $A=Q_0$ or $A=C_l$ where $l\in\mathbb{N}$, and Q_0 and C_l are the initial dyadic cube and the crowns respectively in the norm-$\mathscr{K}_{p,q}(X)$ we have that

$$\|(F-f_{n_k})\chi_{C_l}\|^q_p\to 0\quad\text{if}\quad k\to\infty,$$

with $l = 0, 1, \ldots$ where for $l = 0$ we write $C_l = Q_0$.

Thus, for $\varepsilon > 0$ and $N \in \mathbb{N}$, for each $l = 0, 1, \ldots$ there exists $K_l \in \mathbb{N}$ such that

$$\|(F - f_{n_k})\chi_{C_l}\|_p^q < \frac{\varepsilon}{2N},$$

for all $k > K_l$.

So, for each $N \in \mathbb{N}$ we get that there exists $K^* = \max\{K_l : l = 0, \ldots, N\}$ such that

$$\sum_{l=0}^{N} \|(F - f_{n_k})\chi_{C_l}\|_p^q < \frac{\varepsilon}{2}. \tag{5}$$

On the other hand, because F and f_{n_k} belong to $\mathscr{K}_{p,q}(X)$ for each $k \in \mathbb{N}$, we have that $\|F - f_{n_k}\|_{\mathscr{K}_{p,q}(X)}^q = \sum_{l=0}^{\infty} \|(F - f_{n_k})\chi_{C_l}\|_p^q < \infty$. So there exists $N^* \in \mathbb{N}$ such that

$$\sum_{l=N^*+1}^{\infty} \|(F - f_{n_k})\chi_{C_l}\|_p^q < \frac{\varepsilon}{2}. \tag{6}$$

Therefore, with $N = N^*$ in (5), from (6) and (4) we have that f_{n_k} converges to F in $\mathscr{K}_{p,q}(X)$. Thus, because $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, we get that f_n converges to F in $\mathscr{K}_{p,q}(X)$. $\square$

Finally, we present the Herz space with dyadic support which is a dense subspace of Herz space.

DEFINITION 6. Let (X, d, μ) be a space of homogeneous type, $\mathscr{D}$ a dyadic family and $1 < p, q < \infty$. We define the *Herz spaces with dyadic support* as

$$\mathscr{K}_{p,q}^{\mathscr{D}}(X) = \{f \in \mathscr{K}_{p,q}(X) : \exists Q \in \tilde{\mathscr{D}} \text{ such that } supp(f) \subseteq Q\}.$$

Notice that as we consider spaces of homogeneous type with only one quadrant, from property (d.5) in Definition 1 we get that $\mathscr{K}_{p,q}^{\mathscr{D}}(X)$ is the subspace of all functions in $\mathscr{K}_{p,q}(X)$ with bounded support. The following result state that the space $\mathscr{K}_{p,q}^{\mathscr{D}}(X)$ is dense in $\mathscr{K}_{p,q}(X)$.

PROPOSITION 7. *Let (X, d, μ) be a space of homogeneous type, $\mathscr{D}$ a dyadic family and $1 < p, q < \infty$. If $f \in \mathscr{K}_{p,q}(X)$ and $\varepsilon > 0$ then there exists a function $f_\varepsilon \in \mathscr{K}_{p,q}^{\mathscr{D}}(X)$ such that $\|f - f_\varepsilon\|_{\mathscr{K}_{p,q}(X)} < \varepsilon$.*

Proof. Let $f \in \mathscr{K}_{p,q}(X)$, Q_0 and C_l the initial dydic cube and the crowns in the definition of $\mathscr{K}_{p,q}(X)$-norm. Thus,

$$\|f\|_{\mathscr{K}_{p,q}(X)}^q = \|f\chi_{Q_0}\|_p^q + \sum_{l \in \mathbb{N}} \|f\chi_{C_l}\|_p^q < \infty.$$

We write $g_0 = f\chi_{Q_0}$ and $g_l = f\chi_{C_l}$ with $l \in \mathbb{N}$. So, $\sum\limits_{l \in \mathbb{N} \cup \{0\}} \|g_l\|_p^q < \infty$ and then

$$\sum_{l=N}^{\infty} \|g_l\|_p^q \to 0 \text{ if } N \to \infty.$$

Therefore, given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\sum\limits_{l=N}^{\infty} \|g_l\|_p^q < \varepsilon$ for $N \geqslant N_0$.

We consider the function $f_\varepsilon = \sum_{l=0}^{N_0} g_l \in \mathscr{K}_{p,q}^{\mathscr{D}}(X)$. Then $\|f_\varepsilon\|_{\mathscr{K}_{p,q}(X)} \leqslant \|f\|_{\mathscr{K}_{p,q}(X)} < \infty$. Therefore we get that

$$\begin{aligned}\|f - f_\varepsilon\|_{\mathscr{K}_{p,q}(X)}^q &= \|(f - f_\varepsilon)\chi_{Q_0}\|_p^q + \sum_{l \in \mathbb{N}} \|(f - f_\varepsilon)\chi_{C_l}\|_p^q \\ &= \sum_{l=N_0+1}^{\infty} \|(f - f_\varepsilon)\chi_{C_l}\|_p^q < \varepsilon. \quad \square\end{aligned}$$

4. Geometrical properties and Democracy inequality for Haar system in Herz spaces

In this section we study Democracy inequality for the truncated Haar system in the Herz spaces in spaces of homogeneous type. As we mentioned in the Introduction the Democracy inequality for the Haar system in the Euclidean setting detect the Lebesgue spaces. In particular for the Herz spaces in [8] the authors proved Theorem 1.

The following results show that the situation can be different when we replace the Euclidean setting by another space of homogeneous type.

PROPOSITION 8. *Let (X, d, μ) be the space of homogeneous type with $X = \{2^j : j \in \mathbb{N} \bigcup \{0\}\}$, d the usual Euclidean metric and the measure μ given by $\mu(E) = \sum_{x \in E} x$ for each subset E in X. With the dyadic family $\mathscr{D} = X \cap D$, where D is the usual dyadic family in $\mathbb{R}$, the truncated Haar system associated is democratic in $\mathscr{K}_{p,q}(X)$ for all $1 < p, q < \infty$.*

Proof. We will denote with Q the dyadic cubes in $\mathscr{D}$ and with I the dyadic intervals in $\mathbb{R}$. So, the dyadic cubes in this setting are

- For $j < -1$, $\mathscr{D}^j = \{\{1, \ldots, 2^{-j-1}\}, \{2^{-j}\}, \{2^{-j+1}\}, \ldots\}$.
- For $j \geqslant -1$, $\mathscr{D}^j = \mathscr{D}^{-1} = \{\{1\}, \{2\}, \{4\}, \ldots \{2^\ell\}, \ldots\}$.

Also we have that $\mathscr{H}_0 = \{h \in \mathscr{H} : Q(h) \in \tilde{\mathscr{D}}^j, j \geqslant 0\} = \emptyset$. Therefore the truncated Haar system is $\beta = \{\frac{\chi_R}{\mu(R)^{1/2}}, R \in \mathscr{D}^0\}$.

Let $F \subset \beta$ be a finite subset of β and we consider the function $g_F = \sum_{\varphi \in F} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}}}$.

In the $\mathscr{K}_{p,q}$-norm we shall consider the initial dyadic cube $Q_0 = \{1\}$ and the crowns $C_l = \{2^l\}$, with $l \in \mathbb{N}$. Notice that for $\varphi \in F$ there exists a dyadic cube $R \in \mathscr{D}^0$ such that $R = supp(\varphi)$. Hence we have that

$$\begin{aligned}\|\varphi\|_{\mathscr{K}_{p,q}} &= \left(\left\| \frac{\chi_R}{\mu(R)^{1/2}} \chi_{Q_0} \right\|_p^q + \sum_{l=1}^{\infty} \left\| \frac{\chi_R}{\mu(R)^{1/2}} \chi_{C_l} \right\|_p^q \right)^{1/q} \\ &= \mu(R)^{-1/2+1/p}.\end{aligned}$$

Thus

$$\begin{aligned}\|g_F\|_{\mathscr{K}_{p,q}} &= \left(\left\| \left(\sum_{\varphi\in F} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}}} \right) \chi_{Q_0} \right\|_p^q + \sum_{l=1}^{\infty} \left\| \left(\sum_{\varphi\in F} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}}} \right) \chi_{C_l} \right\|_p^q \right)^{1/q} \\ &= \left(\sum_{\substack{R\in\mathscr{D}^0 \\ \varphi=\frac{\chi_R}{\mu(R)^{1/2}}\in F}} \left\| \frac{\chi_R}{\mu(R)^{1/p}} \right\|_p^q \right)^{1/q} \\ &= |F|^{1/q}.\end{aligned}$$

Then the truncated Haar system is democratic in $\mathscr{K}_{p,q}(X)$ for all $1 < p,q < \infty$. $\square$

PROPOSITION 9. *Let $\mathscr{D}$ be the dyadic family and (X,d,μ) be the space of homogeneous type in the above proposition. Then for $p \neq q$, $||.||_{\mathscr{K}_{p,q}(X)}$ and $||.||_r$ are not equivalent norms, $\forall r > 1$.*

Proof. We proceed by contradiction. We suppose that $1 < p,q,r < \infty$ and $||.||_{\mathscr{K}_{p,q}(X)}$ and $||.||_r$ are equivalent where $p \neq q$.

It is not difficult to see that $\|\sum_{\varphi\in F} \frac{\varphi}{\|\varphi\|_r}\|_r \sim |F|^{\frac{1}{r}}$ for each finite subset F of the truncated Haar system $\beta = \{\frac{\chi_R}{\mu(R)^{1/2}}, R \in \mathscr{D}^0\}$.

Also, from the Proposition 8 we have that $\|\sum_{\varphi\in F} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}(X)}}\|_{\mathscr{K}_{p,q}(X)} \sim |F|^{\frac{1}{q}}$.

Therefore from the equivalence between norms $||.||_{\mathscr{K}_{p,q}(X)}$ and $||.||_r$ we have that $|F|^{1/q} \sim |F|^{1/r}$ for every finite subset F and so $r = q$.

On the other hand, we consider the set $E = \{2^l\}$ with $l \in \mathbb{N} \cup \{0\}$. Thus $\|\chi_E\|_r = 2^{l/r}$ and $\|\chi_E\|_{\mathscr{K}_{p,q}(X)} = 2^{l/p}$. So again from the equivalence of $||.||_{\mathscr{K}_{p,q}(X)}$ and $||.||_r$ we have that $2^{l/r} \sim 2^{l/p}$ and hence $r = p$. That is $p = q$. $\square$

On the other hand, the truncated Haar system have the same property of democracy that the Haar system in Lebesgue spaces. In [2] the authors proved the following result.

THEOREM 10. *Let (X,d,μ) be a space of homogeneous type and let $\mathscr{H}$ the Haar system associated to the dyadic family $\mathscr{D}$. Then $\mathscr{H}$, when normalized to L^p, is a democratic basis for $L^p(X,d,\mu)$ with $1 < p < \infty$, and even more,*

$$\left\| \left(\sum_{h\in F} \frac{|h|^2}{\|h\|_p^2} \right)^{1/2} \right\|_p \sim \left\| \sum_{h\in F} \frac{h}{\|h\|_p} \right\|_p \sim |F|^{1/p}$$

for each finite subset $F \subset \mathscr{H}$.

As a consequence of this Theorem, we obtain that the truncated Haar basis is democratic in $L^p(X,d,\mu)$. More precisely, we have the following result

COROLLARY 1. *Let (X,d,μ) be a space of homogeneous type and let $\mathscr{H}$ the Haar system associated to the dyadic family $\mathscr{D} = \bigcup_{j\in\mathbb{Z}} \mathscr{D}^j$. Then the truncated Haar system β given by $\beta = \left\{\frac{\chi_R}{\mu(R)^{1/2}} : R \in \mathscr{D}^0\right\} \cup \mathscr{H}_0$, where $\mathscr{H}_0 = \{h \in \mathscr{H} : Q(h) \in \widetilde{\mathscr{D}}^j,$ $j \geqslant 0\}$ is democratic in $L^p(X,\mu)$, $1 < p < \infty$, when is normalized to L^p. Furthermore, for every finite subset $F \subset \beta$ and $1 < p < \infty$ we have*

$$\left\|\sum_{\varphi\in F} \frac{\varphi}{\|\varphi\|_p}\right\|_p \sim |F|^{1/p}.$$

Proof. Let $p \in (1,\infty)$ and $F \subset \beta$ finite.

We consider the function $g_F = \sum_{\varphi\in F} \dfrac{\varphi}{\|\varphi\|_{L^p(X,\mu)}}$, which is a function in $L^p(X,\mu)$.

So from the characterization given in Proposition 3, we have that

$$\|g_F\|_p \sim \left(\left\|\sum_{R\in\mathscr{D}^0}\left\langle g_F, \frac{\chi_R}{\mu(R)^{1/2}}\right\rangle \frac{\chi_R}{\mu(R)^{1/2}}\right\|_p^p + \left\|\left(\sum_{h\in\mathscr{H}_0} |\langle g_F, h\rangle|^2 |h|^2\right)^{1/2}\right\|_p^p\right)^{1/p} \tag{7}$$

Note that if $\varphi \in F$, then there is either $Q \in \mathscr{D}^0$ such that $\varphi = \frac{\chi_Q}{\mu(Q)1/2}$ or there exists $h \in \mathscr{H}_0$ such that $\varphi = h$.

If $\varphi = \frac{\chi_Q}{\mu(Q)^{1/2}}$ for some $Q \in \mathscr{D}^0$, then

$$\langle\varphi, h\rangle = \int_X \varphi(x)h(x)d\mu(x) = 0 \tag{8}$$

for all $h \in \mathscr{H}_0$, since $\int_X h d\mu = 0$.

A similar argument shows that if $\varphi = h$ for some $h \in \mathscr{H}_0$

$$\langle\varphi, \chi_R\rangle = \int_X \varphi(x)\chi_R(x)d\mu(x) = 0 \tag{9}$$

for all $R \in \mathscr{D}^0$.

Therefore from (7) to (9)

$$\|g_F\|_p \sim \left(\left\|\sum_{R\in\mathscr{D}^0}\left\langle g_{F_0}, \frac{\chi_R}{\mu(R)1/2}\right\rangle \frac{\chi_R}{\mu(R)^{1/2}}\right\|_p^p + \left\|\left(\sum_{h\in\mathscr{H}_0} |\langle g_{F_1}, h\rangle|^2 |h|^2\right)^{1/2}\right\|_p^p\right)^{1/p},$$

where $F_0 = \{\varphi \in F : \varphi = \frac{\chi_Q}{\mu(Q)^{1/2}}, Q \in \mathscr{D}^0\}$ and $F_1 = \{\varphi \in F : \varphi = h, h \in \mathscr{H}_0\}$.

Notice that if $\varphi \in F_0$ we have $\|\varphi\|_p = \mu(Q)^{1/p-1/2}$. In the sequel, we write $X_{F_0} = \bigcup R$, with $R \in \mathscr{D}^0$, $R = supp(\varphi)$ and $\varphi \in F_0$. Then using (9) and that the cubes in $\mathscr{D}^0$ are disjoint, we have that

$$\begin{aligned}
&\left\| \sum_{R\in\mathscr{D}^0} \left\langle g_{F_0}, \frac{\chi_R}{\mu(R)^{1/2}} \right\rangle \frac{\chi_R}{\mu(R)^{1/2}} \right\|_p^p \\
&= \left\| \sum_{R\in\mathscr{D}^0} \left\langle \sum_{\varphi\in F_0} \frac{\varphi}{\|\varphi\|_p}, \frac{\chi_R}{\mu(R)^{1/2}} \right\rangle \frac{\chi_R}{\mu(R)^{1/2}} \right\|_p^p \\
&= \int_X \left| \sum_{R\in\mathscr{D}^0} \left\langle \sum_{\varphi\in F_0} \frac{\varphi}{\|\varphi\|_p}, \frac{\chi_R}{\mu(R)^{1/2}} \right\rangle \frac{\chi_R(x)}{\mu(R)^{1/2}} \right|^p d\mu(x) \\
&= \int_{X_{F_0}} \left| \left\langle \frac{\varphi}{\|\varphi\|_p}, \frac{\chi_R}{\mu(R)^{1/2}} \right\rangle \frac{\chi_R(x)}{\mu(R)^{1/2}} \right|^p d\mu(x) \\
&= \sum_{\varphi\in F_0} \int_{supp(\varphi)} \left(\mu(supp(\varphi))^{1/2-1/p} \frac{\chi_{supp(\varphi)}(x)}{\mu(supp(\varphi))^{1/2}} \right)^p d\mu(x) \\
&= \sum_{\varphi\in F_0} \int_{supp(\varphi)} \left(\mu(supp(\varphi))^{-1/p} \chi_{supp(\varphi)}(x) \right)^p d\mu(x) \\
&= \sum_{\varphi\in F_0} \mu(supp(\varphi))^{-1} \mu(supp(\varphi)) \\
&= |F_0|
\end{aligned}$$

where $F_0 = \left\{ \varphi \in F : \varphi = \frac{\chi_Q}{\mu(Q)^{1/2}}, Q \in \mathscr{D}^0 \right\} \subseteq F$.

On the other hand, from (8) and since $\mathscr{H}_0$ is an orthonormal system, we have that

$$\begin{aligned}
\left\| \left(\sum_{h\in\mathscr{H}_0} |\langle g_{F_1}, h\rangle|^2 |h|^2 \right)^{1/2} \right\|_p^p &= \left\| \left(\sum_{h\in\mathscr{H}_0} \left| \left\langle \sum_{\varphi\in F_1} \frac{\varphi}{\|\varphi\|_p}, h \right\rangle \right|^2 |h|^2 \right)^{1/2} \right\|_p^p \\
&= \left\| \left(\sum_{h\in\mathscr{H}_0} \left| \sum_{\varphi\in F_1} \left\langle \frac{\varphi}{\|\varphi\|_p}, h \right\rangle \right|^2 |h|^2 \right)^{1/2} \right\|_p^p \\
&= \left\| \left(\sum_{h\in\mathscr{H}_0} \left| \sum_{\substack{\tilde{h}\in\mathscr{H}_0: \\ \varphi=\tilde{h}\in F}} \left\langle \frac{\tilde{h}}{\|\tilde{h}\|_p}, h \right\rangle \right|^2 |h|^2 \right)^{1/2} \right\|_p^p \\
&= \left\| \left(\sum_{\substack{h\in\mathscr{H}_0: \\ \varphi=h\in F_1}} \frac{1}{\|h\|_p^2} |h|^2 \right)^{1/2} \right\|_p^p
\end{aligned}$$

$$= \left\| \left(\sum_{h \in F_1} \frac{|h|^2}{\|h\|_p^2} \right)^{1/2} \right\|_p^p$$
$$\sim |F_1|$$

where in the last step we have used Theorem (10). From the latter and from (10) we have from (7) that

$$\|g_F\|_p \sim \left(\left\| \sum_{R \in \mathscr{D}^0} \left\langle g_F, \frac{\chi_R}{\mu(R)^{1/2}} \right\rangle \frac{\chi_R}{\mu(R)^{1/2}} \right\|_p^p + \left\| \left(\sum_{h \in \mathscr{H}_0} |\langle g_F, h \rangle|^2 |h|^2 \right)^{1/2} \right\|_p^p \right)^{1/p}$$
$$\sim (|F_0| + |F_1|)^{1/p}$$
$$= |F|^{1/p}. \quad \square$$

We need some geometric condition on the space (X,d,μ) to obtain an analogous result to Theorem 1 in spaces of homogeneous type.

DEFINITION 7. Let (X,d,μ) be a space of homogeneous type. We say that the dyadic family $\mathscr{D}$ satisfy the *geometric property of concentration* if there exists a cube $Q_* \in \mathscr{D}$ that satisfies that the series $\sum_{j \in \mathbb{N}} |\mathscr{D}^j_{Q_*}|$ diverges, where $\mathscr{D}^j_{Q_*} = \{Q \in \tilde{D}^j : Q \subset Q_*\}$.

Now we are in conditions to state and prove the democracy result for Herz spaces in spaces of homogeneous type.

THEOREM 11. *Let (X,d,μ) be a unbounded space of homogeneous type that admits a dyadic family $\mathscr{D}$ which satisfies the geometric property of concentration and let β the truncated Haar basis associated with the dyadic family $\mathscr{D}$*

$$\beta = \left\{ \frac{\chi_R}{\mu(R)^{1/2}}, R \in \mathscr{D}^0 \right\} \cup \mathscr{H}_0,$$

with $\mathscr{H}_0 = \left\{ h : Q(h) \in \tilde{\mathscr{D}}^j, j \geqslant 0 \right\}$. If β is democratic in $\mathscr{K}_{p,q}(X,\mu)$ with $1 < p,q < \infty$ then $p = q$.

Proof. Let us assume that the basis is democratic. It would suffice to prove that for each positive integer M, there are two subsets F_1 and F_2 of β with $|F_1| = |F_2| = M$ such that

$$\left\| \sum_{\varphi \in F_1} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}}} \right\|_{\mathscr{K}_{p,q}} \sim |F_1|^{1/p} \tag{10}$$

$$\left\| \sum_{\varphi \in F_2} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}}} \right\|_{\mathscr{K}_{p,q}} \sim |F_2|^{1/q} \tag{11}$$

since in such a case we would have $p = q$. Since the dyadic family $\mathscr{D}$ satisfies the geometric property of concentration, this guarantees us that there is at least one cube Q_* in $\mathscr{D}^j$ with $j \geqslant 0$, for which we can find at least M cubes $\{Q_i : i = 1,\dots,M\} \subset \mathscr{D}$ such that $Q_i \subset Q_*$ for all $i = 1,\dots,M$ and for each positive integer M.

Also since Q_* is a dyadic cube we get that $Q_* \subseteq A$, where A is the initial cube or the crowns of the definition of the norm $\mathscr{K}_{p,q}$. Then $\chi_A \chi_{Q_*} = 0$ if $A \cap Q_* = \emptyset$ and $\chi_A \chi_{Q_*} = \chi_{Q_*}$ if $A \cap Q_* \neq \emptyset$.

We consider $\varphi \in \beta$ such that $supp(\varphi) \subset Q_*$. Taking into account the previous observation, $\varphi \chi_A = \varphi$ if $supp(\varphi) \subset A$. Then

$$\|\varphi\|_{\mathscr{K}_{p,q}} = \left(\|\varphi \chi_{Q_0}\|_p^q + \sum_{l=1}^{\infty} \|\varphi \chi_{C_l}\|_p^q \right)^{1/q} = \|\varphi\|_p.$$

We fix $M \in \mathbb{N}$. Let $Q_1, Q_2 \dots Q_M$ be dyadic cubes in $\tilde{\mathscr{D}}$ such that $Q_i \subset Q_*$, $i = 1,2\dots,M$.

For each dyadic cube Q_i, $i = 1,\dots,M$, we take a function $\varphi_i \in \mathscr{H}_0$ such that $supp(\varphi_i) \subseteq Q_i$. Let $F_1 = \{\varphi_i \in \beta, i = 1,2..,M\}$, then

$$\begin{aligned}\left\| \sum_{\varphi \in F_1} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}}} \right\|_{\mathscr{K}_{p,q}} &= \left(\left\| \left(\sum_{\varphi \in F_1} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}}} \right) \chi_{Q_0} \right\|_p^q + \sum_{l=1}^{\infty} \left\| \left(\sum_{\varphi \in F_1} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}}} \right) \chi_{C_l} \right\|_p^q \right)^{1/q} \\ &= \left\| \sum_{\varphi \in F_1} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}}} \right\|_p \\ &\sim |F_1|^{1/p}\end{aligned}$$

where in the last step we use the Corollary (1).

To the choice of the set F_2 we use the hypothesis that the space X is not bounded in the following way. For each $i = 1,2\dots,M$ we take $\hat{\varphi}_i = \chi_{\hat{Q}_i}$ where $\hat{Q}_i \in \mathscr{D}^0$ and $\hat{Q}_i \subset C_i$ and so $\|\hat{\varphi}_i\|_{\mathscr{K}_{p,q}} = \|\hat{\varphi}_i\|_p$. Then for $F_2 = \{\hat{\varphi}_i : i = 1,\dots,M\}$ we have that

$$\begin{aligned}\left\| \sum_{\varphi \in F_2} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}}} \right\|_{\mathscr{K}_{p,q}} &= \left(\left\| \left(\sum_{\varphi \in F_2} \frac{\varphi}{\|\varphi\|\mathscr{K}_{p,q}} \right) \chi_{Q_0} \right\|_p^q + \sum_{l=1}^{\infty} \left\| \left(\sum_{\varphi \in F_2} \frac{\varphi}{\|\varphi\|_{\mathscr{K}_{p,q}}} \right) \chi_{C_l} \right\|_p^q \right)^{1/q} \\ &= \left(\sum_{\varphi \in F_2} \left\| \frac{\varphi}{\|\varphi\|_p} \right\|_p^q \right)^{1/q} \\ &= |F_2|^{1/q}. \quad \square\end{aligned}$$

REMARK 1. In the above theorem the condition of unboundedness for the space of homogeneous type is not restrictive. In fact if X is bounded, $\mathscr{K}_{p,q}$ is the Lebesgue space L^p.

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