Mathematical
Inequalities
& Applications
Volume 26, Number 4 (2023), 1021–1038 doi:10.7153/mia-2023-26-63

NEW REVERSE HÖLDER–TYPE INEQUALITIES AND APPLICATIONS

JORGE A. PAZ MOYADO, YAMILET QUINTANA,
JOSÉ M. RODRIGUEZ AND JOSÉ M. SIGARRETA*

(*Communicated by I. Perić*)

Abstract. In this paper, we establish several Hölder-type inequalities using Jensen-type and Young-type inequalities as key tools. Particularly noteworthy is a reverse Hölder inequality with the Specht's ratio. Furthermore, we obtain a reverse Young-type inequality and we apply these results to the fractional context, both globally and locally.

1. Introduction

Integral inequalities are a fundamental tool in mathematics and have countless applications in various fields [15, 26, 34, 35]. They allow us to establish bounds on integrals and compare the values of different integrals, and are an essential part of many mathematical theories and techniques.

In recent years there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types of fractional derivatives, since these fractional integral inequalities and have numerous applications in the theory of differential equations and applied mathematics, including physics, engineering, and finance. For example, in physics, fractional differential equations are used to model systems that exhibit long-range memory effects, such as viscoelastic materials or anomalous diffusion processes. In finance, fractional differential equations are used to model stock price movements, interest rates, and other financial processes. Some of the inequalities studied are Gronwall, Chebyshev, Hermite-Hadamard-type, Ostrowski-type, Opial-type, Grüss-type, Hardy-type, Petrović-type, Milne-type, Gagliardo-Nirenberg-type, Minkowski-type and Hölder-type inequalities (see, e.g., [3, 4, 5, 10, 11, 13, 16, 25, 32, 33, 36, 37, 39, 40, 41, 42, 43]).

In particular, there are many generalizations of Hölder inequality, see e.g., the papers [3, 4, 5, 19, 23, 38] and the books [34, 35], and their references. See also the preliminary results in Sections 3 and 4 of this paper.

Mathematics subject classification (2020): 26A33, 26A51, 26D15.

Keywords and phrases: Hölder inequality, Hölder-type inequalities, Riemann-Liouville integral operators, conformable fractional integral operators.

[*] Corresponding author.

Motivated by the recent article [9] in which the authors obtain Jensen-type inequalities for convex and m-convex functions, and apply these inequalities to generalized Riemann-Liouville-type integral operators, in the present work we provide several Hölder-type inequalities. Also, we apply them to the generalized Riemann-Liouville-type integral operators defined in [6], which include most of known Riemann-Liouville-type fractional integrals, and to the generalized local fractional integral operators defined in [2, 7, 21, 22], which include most of known fractional conformable integral operators.

The outline of the paper is as follows. Section 2 contains some background. In Sections 3 and 4 we prove the Hölder-type inequalities. The main tools in Section 3 and 4 are a Jensen-type inequality and a Young-type inequality, respectively. Finally, in Sections 5 and 6 we apply our inequalities to the generalized Riemann-Liouville-type integral operators, and to the operators associated to the generalized local fractional derivative, respectively.

2. Basic facts

One of the classical integral inequalities frequently studied is Jensen's inequality, which relates the value of a convex function of an integral to the integral of the convex function. It was proved in 1906 [29], and it can be stated as follows:

Let μ be a probability measure on any measurable space X. If $f : X \to (a,b)$ is μ-integrable and φ is a convex function on (a,b), then

$$\varphi\left(\int_X f\,d\mu\right) \leqslant \int_X \varphi\circ f\,d\mu\,.$$

Our purpose is to prove Hölder-type inequalities. The classical Hölder inequality states that if μ is a measure on any measurable space X, $p,q>1$ are real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, $f\in L^p(X,\mu)$ and $g\in L^q(X,\mu)$, then $fg\in L^1(X,\mu)$ and

$$\int_X |fg|\,d\mu \leqslant \|f\|_p\,\|g\|_q\,. \tag{1}$$

As well known, if $p=q=2$, (1) becomes Cauchy-Schwarz inequality.

Two different and popular forms of proving Hölder inequality are, respectively, Young and Jensen inequalities.

The following Jensen-type inequality for convex functions was established in [9, Theorem 8]:

PROPOSITION 1. *Let μ be a probability measure on any measurable space X and $a\leqslant b$ real constants. If $f : X\to[a,b]$ is a measurable function and φ is a convex function on $[a,b]$, then f and $\varphi\circ f$ are μ-integrable functions and*

$$\varphi\Big(a+b-\int_X f\,d\mu\Big) \leqslant \varphi(a)+\varphi(b)-\int_X \varphi\circ f\,d\mu.$$

3. Two Hölder-type inequalities

In [3, Theorem 2.2] appears the following stability version of Hölder's inequality, incorporating an extra term that measures the deviation from equality.

THEOREM 2. *Let $1<p<\infty$ and let $q=p/(p-1)$ be its conjugate exponent. If $f\in L^p$, $g\in L^q$, $\|f\|_p\|g\|_q>0$, and $1<p\leqslant 2$, then*

$$\|f\|_p\|g\|_q\left(1-\frac{1}{p}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{q/2}}{\|g\|_q^{q/2}}\right\|_2^2\right)_+\leqslant\|fg\|_1$$
$$\leqslant\|f\|_p\|g\|_q\left(1-\frac{1}{q}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{q/2}}{\|g\|_q^{q/2}}\right\|_2^2\right),$$

while if $2\leqslant p<\infty$, the terms $1/p$ and $1/q$ exchange their positions in the preceding inequalities.

In the same direction, we are going to use Proposition 1 in order to obtain two reverse Hölder-type inequalities. In the next result we use the usual convention in measure theory $0\cdot\infty=0/0=0$.

THEOREM 3. *Let μ be a measure on any measurable space X, let $p,q>1$ be real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, $f\in L^p(X,\mu)$ and $g\in L^q(X,\mu)$.*

(1) If $|f|^{1-p}g\in L^\infty(X,\mu)$, then $\|f\|_p\|g\|_q\leqslant\left\||f|^{1-p}g\right\|_\infty\|f\|_p^p$ and the following Hölder-type inequality holds:

$$\|fg\|_1\geqslant\left\||f|^{1-p}g\right\|_\infty\|f\|_p^p-\left(\left\||f|^{1-p}g\right\|_\infty^q\|f\|_p^{pq}-\|f\|_p^q\|g\|_q^q\right)^{1/q}. \tag{2}$$

(2) If $f|g|^{1-q}\in L^\infty(X,\mu)$, then $\|f\|_p\|g\|_q\leqslant\left\|f|g|^{1-q}\right\|_\infty\|g\|_q^q$ and the following Hölder-type inequality holds:

$$\|fg\|_1\geqslant\left\|f|g|^{1-q}\right\|_\infty\|g\|_q^q-\left(\left\|f|g|^{1-q}\right\|_\infty^p\|g\|_q^{pq}-\|f\|_p^p\|g\|_q^p\right)^{1/p}. \tag{3}$$

Proof. Let us prove the first item. Since $|f|^{1-p}g\in L^\infty(X,\mu)$, we have $g(x)=0$ for μ-a.e. x with $f(x)=0$.

We can assume that $\|f\|_p>0$, since otherwise $f=0$ μ-a.e. and the inequality is, in fact, an equality.

As usual, we denote by χ_E the characteristic function of a set E.

If we apply Proposition 1 with the convex function $\varphi(t)=t^q$ on $[0,b]$ to the function and the probability measure

$$\frac{|fg|}{w}\chi_{\{f\neq 0\}},\qquad w\,d\mu,\qquad \text{with } w=\frac{|f|^p}{\|f\|_p^p},$$

respectively, and with

$$a=0, \qquad b=\big\||f|^{1-p}g\big\|_\infty \|f\|_p^p \geqslant \frac{|fg|}{|f|^p/\|f\|_p^p}\,\chi_{\{f\neq 0\}} = \frac{|fg|}{w}\,\chi_{\{f\neq 0\}}.$$

Since $w\,d\mu$ is a probability measure,

$$b=\int_X b\,w\,d\mu \geqslant \int_X \frac{|fg|}{w}\,\chi_{\{f\neq 0\}}\,w\,d\mu = \int_X |fg|\,d\mu, \tag{4}$$

and we obtain

$$\left(b-\int_X \frac{|fg|}{w}\,\chi_{\{f\neq 0\}}\,w\,d\mu\right)^q \leqslant b^q - \int_X \frac{|fg|^q}{w^q}\,\chi_{\{f\neq 0\}}\,w\,d\mu. \tag{5}$$

Since $\frac{1}{p}+\frac{1}{q}=1$, we deduce $p-pq=-q$ and

$$\begin{aligned}\frac{|fg|^q}{w^q}\,\chi_{\{f\neq 0\}}\,w = w^{1-q}|f|^q\,|g|^q\,\chi_{\{f\neq 0\}} &= \frac{|f|^{p-pq}}{\|f\|_p^{p-pq}}\,|f|^q\,|g|^q\,\chi_{\{f\neq 0\}}\\ &= \|f\|_p^q\,|g|^q\,\chi_{\{f\neq 0\}} = \|f\|_p^q\,|g|^q\end{aligned}$$

μ-a.e., where the last equality holds because $g(x)=0$ for μ-a.e. x with $f(x)=0$.

Hence, (5) becomes

$$\left(b-\int_X |fg|\,d\mu\right)^q \leqslant b^q - \int_X \|f\|_p^q\,|g|^q\,d\mu = b^q - \|f\|_p^q\,\|g\|_q^q. \tag{6}$$

Since (4) implies $b \geqslant \int_X |fg|\,d\mu$, we have

$$b^q - \|f\|_p^q\,\|g\|_q^q \geqslant \left(b-\int_X |fg|\,d\mu\right)^q \geqslant 0,$$

and so, $\|f\|_p\,\|g\|_q \leqslant b = \big\||f|^{1-p}g\big\|_\infty \|f\|_p^p$. Hence, (6) implies

$$b-\left(b^q-\|f\|_p^q\,\|g\|_q^q\right)^{1/q} \leqslant \int_X |fg|\,d\mu,$$

$$\big\||f|^{1-p}g\big\|_\infty \|f\|_p^p - \left(\big\||f|^{1-p}g\big\|_\infty^q \|f\|_p^{pq} - \|f\|_p^q\,\|g\|_q^q\right)^{1/q} \leqslant \int_X |fg|\,d\mu.$$

If we change the roles of f,p and g,q, the previous argument gives the second item. $\square$

4. A reverse Hölder inequality

A classical extension of Hölder inequality states that if μ is a measure on any measurable space X, $p_1,\dots,p_n>1$ are real numbers such that $\frac{1}{p_1}+\ldots+\frac{1}{p_n}=1$, and $f_k \in L^{p_k}(X,\mu)$ for $1 \leqslant k \leqslant n$, then $f_1\cdots f_n \in L^1(X,\mu)$ and

$$\|f_1\cdots f_n\|_1 \leqslant \|f_1\|_{p_1}\cdots\|f_n\|_{p_n}. \tag{7}$$

Since Hölder inequality is a very important result in Analysis, there are several versions of reverse of Hölder inequality of the following type:

$$\|f_1\|_{p_1}\cdots\|f_n\|_{p_n}\leqslant A\,\|f_1\cdots f_n\|_1\,, \tag{8}$$

with different hypothesis. For instance, in [35, p. 146] and [8, Theorem 3] appear inequalities as (8) with $n=2$. Also, (8) is proved in [35, p. 141] for any n when the functions $f_1,\dots,f_n$ are bounded and greater than a positive constant. We are going to prove (8) with weaker hypotheses.

To make the proof easier to read, first of all, we state several technical lemmas. Let us start with an elementary fact.

LEMMA 4. *If $f\in C^1[a,b]$ and $f'=g_1g_2$ with $g_1,g_2\in C[a,b]$, g_1 positive and g_2 decreasing on $[a,b]$, then f attains its minimum value on $[a,b]$ on the set $\{a,b\}$.*

LEMMA 5. *If $0<a<1$, $\alpha_k,\beta_k,\lambda_k>0$ for $1\leqslant k\leqslant n$, then the function*

$$F(x)=\prod_{k=1}^{n}x_k^{\alpha_k}$$

attains its minimum value on the set

$$E=\Big\{x\in\mathbb{R}^n:\sum_{k=1}^{n}\lambda_k x_k^{\beta_k}=1,\ a x_k^{\beta_k}\leqslant x_i^{\beta_i}\ \textit{for}\ 1\leqslant i,k\leqslant n,\\ x_k>0\ \textit{for}\ 1\leqslant k\leqslant n\Big\}$$

at the boundary ∂E (the boundary E is understood to be a subset of the hypersurface $\sum_{k=1}^{n}\lambda_k x_k^{\beta_k}=1$ in $\mathbb{R}^n$).

Proof. Since F is a continuous function on the compact set E, it attains its minimum value on E.

Note that it suffices to show that the minimum value of the function

$$f(x)=\Big(1-\sum_{k=1}^{n-1}\lambda_k x_k^{\beta_k}\Big)^{\alpha_n/\beta_n}\prod_{k=1}^{n-1}x_k^{\alpha_k}$$

on the set

$$G=\Big\{x\in\mathbb{R}^{n-1}:a x_k^{\beta_k}\leqslant\lambda_n^{-1}\Big(1-\sum_{i=1}^{n-1}\lambda_i x_i^{\beta_i}\Big)\leqslant a^{-1}x_k^{\beta_k},\ x_k>0\ \text{for}\ 1\leqslant k\leqslant n-1\\ a x_k^{\beta_k}\leqslant x_i^{\beta_i}\ \text{for}\ 1\leqslant i,k\leqslant n-1\Big\}$$

is attained at ∂G. We have for $1 \leqslant j \leqslant n-1$

$$\begin{aligned}\frac{\partial f}{\partial x_j} &= -\lambda_j \beta_j x_j^{\beta_j - 1} \frac{\alpha_n}{\beta_n} \Big(1 - \sum_{k=1}^{n-1} \lambda_k x_k^{\beta_k}\Big)^{\alpha_n/\beta_n - 1} \prod_{k=1}^{n-1} x_k^{\alpha_k} \\ &\quad + \Big(1 - \sum_{k=1}^{n-1} \lambda_k x_k^{\beta_k}\Big)^{\alpha_n/\beta_n} \frac{\alpha_j}{x_j} \prod_{k=1}^{n-1} x_k^{\alpha_k} \\ &= \Big(1 - \sum_{k=1}^{n-1} \lambda_k x_k^{\beta_k}\Big)^{\alpha_n/\beta_n - 1} \frac{1}{x_j} \prod_{k=1}^{n-1} x_k^{\alpha_k} \\ &\quad \cdot \Big(-\lambda_j \beta_j x_j^{\beta_j} \frac{\alpha_n}{\beta_n} + \alpha_j \Big(1 - \sum_{k=1}^{n-1} \lambda_k x_k^{\beta_k}\Big)\Big) \\ &= \Big(1 - \sum_{k=1}^{n-1} \lambda_k x_k^{\beta_k}\Big)^{\alpha_n/\beta_n - 1} \frac{1}{x_j} \prod_{k=1}^{n-1} x_k^{\alpha_k} \\ &\quad \cdot \Big(\alpha_j \Big(1 - \sum_{k \neq j} \lambda_k x_k^{\beta_k}\Big) - \Big(\lambda_j \beta_j \frac{\alpha_n}{\beta_n} + \lambda_j \alpha_j\Big) x_j^{\beta_j}\Big).\end{aligned}$$

Since the last factor of $\partial f / \partial x_j$ is a decreasing function on x_j and the other factors are positive, Lemma 4 implies that the minimum value of f as a function of the variable x_j on any interval I contained in the domain of f is attained at the boundary of I, for each $1 \leqslant j \leqslant n$.

This implies that the minimum value of the function f on G is attained at ∂G, and the conclusion of the lemma holds. □

Young inequality

$$xy \leqslant \frac{1}{p} x^p + \frac{1}{q} y^q$$

for $x, y \geqslant 0$ and $1/p + 1/q = 1$, is a very important result in Analysis, since it is a key tool in the proof of Hölder inequality. Its reverse inequality was given in [45] with Specht's ratio as follows:

$$S\Big(\frac{x^p}{y^q}\Big) xy \geqslant \frac{1}{p} x^p + \frac{1}{q} y^q \tag{9}$$

where the Specht's ratio [44] is defined on $\mathbb{R}^+$ as

$$S(a) = \frac{a^{\frac{1}{a-1}}}{e \log a^{\frac{1}{a-1}}}.$$

There are also several versions of the additive-type refined Young inequality (and its reverse), see [3], even for n real numbers (see [4]).

We are going to prove a version of (9) for n real numbers, also with Specht's ratio.

PROPOSITION 6. *If* $0 < a < 1$, $p_1, \ldots, p_n > 1$ *and* $x_1, \ldots, x_n \geqslant 0$ *are real numbers such that* $\frac{1}{p_1} + \ldots + \frac{1}{p_n} = 1$ *and* $a x_k^{p_k} \leqslant x_i^{p_i}$ *for* $1 \leqslant i, k \leqslant n$, *then there exists a positive* *constant A, which just depends on $a, p_1, \dots, p_n$, such that*

$$\frac{1}{p_1} x_1^{p_1} + \cdots + \frac{1}{p_n} x_n^{p_n} \leqslant A x_1 \cdots x_n. \tag{10}$$

In fact, if $\mathscr{P}_n$ denote the group of permutations of $\{1, \dots, n\}$, then the best value of A is the maximum on the finite set

$$\begin{aligned} A &= \max_{1 \leqslant m < n,\, \sigma \in \mathscr{P}_n} \Big(a + (1-a) \sum_{k=1}^{m} \frac{1}{p_{\sigma(k)}} \Big) a^{-1+\sum_{k=1}^{m} 1/p_{\sigma(k)}} \\ &\leqslant e^{-1} a^{\frac{-1}{1-a}} \frac{1-a}{-\log a} = S(a). \end{aligned}$$

Proof. If $x = 0$, then the inequality trivially holds. If $x \neq 0$, then the hypothesis $a x_k^{p_k} \leqslant x_i^{p_i}$ for $1 \leqslant i, k \leqslant n$ gives $x_k > 0$ for $1 \leqslant k \leqslant n$. Define

$$\begin{aligned} E_1 = \Big\{ x \in \mathbb{R}^n : \sum_{k=1}^{n} \frac{1}{p_k} x_k^{p_k} = 1,\ a x_k^{p_k} \leqslant x_i^{p_i} \ \text{ for } \ 1 \leqslant i, k \leqslant n, \\ x_k > 0 \ \text{ for } \ 1 \leqslant k \leqslant n \Big\} \end{aligned}$$

and $f_1(x) = x_1 \cdots x_n$. Since f_1 is a positive continuous function on the compact set E_1, there exists

$$\Gamma = \min_{x \in E_1} f_1(x) > 0.$$

Note that Γ just depends on $a, p_1, \dots, p_n$.

Define $t = \sum_{k=1}^{n} \frac{1}{p_k} x_k^{p_k} > 0$, then $\sum_{k=1}^{n} \frac{1}{p_k} (x_k / t^{1/p_k})^{p_k} = 1$ and

$$\Gamma \leqslant \prod_{k=1}^{n} \frac{x_k}{t^{1/p_k}} = t^{-\sum_{k=1}^{n} \frac{1}{p_k}} \prod_{k=1}^{n} x_k = t^{-1} \prod_{k=1}^{n} x_k.$$

Hence,

$$\Gamma \sum_{k=1}^{n} \frac{1}{p_k} x_k^{p_k} = t\,\Gamma \leqslant \prod_{k=1}^{n} x_k$$

and so, (10) holds with $A = 1/\Gamma$.

Let us compute Γ now. By Lemma 5, we know that Γ is attained at a point in ∂E_1; thus, there exist $1 \leqslant i_1, j_1 \leqslant n$ with $i_1 \neq j_1$ and $x_{i_1}^{p_{i_1}} = a x_{j_1}^{p_{j_1}}$ for that point. Hence,

$$\Gamma = \min_{x \in E_2} f_2(x)$$

with

$$f_2(x) = a^{1/p_{i_1}} x_{j_1}^{1+p_{j_1}/p_{i_1}} \prod_{k \neq i_1, j_1} x_k$$

and

$$E_2 = \Big\{ x = (x_1, \dots, x_{i_1-1}, x_{i_1+1}, \dots, x_n) \in \mathbb{R}^{n-1} : \\ \sum_{k \neq i_1, j_1} \frac{1}{p_k} x_k^{p_k} + \Big(\frac{1}{p_{j_1}} + \frac{a}{p_{i_1}} \Big) x_{j_1}^{p_{j_1}} = 1, \\ a x_k^{p_k} \leqslant x_i^{p_i} \text{ for } i, k \neq i_1,\ x_k > 0 \text{ for } k \neq i_1 \Big\}.$$

By Lemma 5, we know that Γ is attained at a point in ∂E_2; thus, there exist $1 \leqslant i_2, j_2 \leqslant n$ with $i_2, j_2 \neq i_1$, $i_2 \neq j_2$ and $x_{i_2}^{p_{i_2}} = a x_{j_2}^{p_{j_2}}$ for that point. Hence, we have two cases:

If $j_2 \neq j_1$, then

$$\Gamma = \min_{x \in E_{3,1}} f_{3,1}(x)$$

with

$$f_{3,1}(x) = a^{1/p_{i_1} + 1/p_{i_2}} x_{j_1}^{1 + p_{j_1}/p_{i_1}} x_{j_2}^{1 + p_{j_2}/p_{i_2}} \prod_{k \neq i_1, j_1, i_2, j_2} x_k$$

and

$$E_{3,1} = \Big\{ x \in \mathbb{R}^{n-2} : \sum_{k \neq i_1, j_1, i_2, j_2} \frac{1}{p_k} x_k^{p_k} + \Big(\frac{1}{p_{j_1}} + \frac{a}{p_{i_1}} \Big) x_{j_1}^{p_{j_1}} + \Big(\frac{1}{p_{j_2}} + \frac{a}{p_{i_2}} \Big) x_{j_2}^{p_{j_2}} = 1, \\ a x_k^{p_k} \leqslant x_i^{p_i} \text{ for } i, k \neq i_1, i_2,\ x_k > 0 \text{ for } k \neq i_1, i_2 \Big\}.$$

If $j_2 = j_1$, then $x_{i_2}^{p_{i_2}} = a x_{j_1}^{p_{j_1}} = x_{i_1}^{p_{i_1}}$,

$$\Gamma = \min_{x \in E_{3,2}} f_{3,2}(x)$$

with

$$f_{3,2}(x) = a^{1/p_{i_1} + 1/p_{i_2}} x_{j_1}^{1 + p_{j_1}/p_{i_1} + p_{j_1}/p_{i_2}} \prod_{k \neq i_1, j_1, i_2} x_k$$

and

$$E_{3,2} = \Big\{ x \in \mathbb{R}^{n-2} : \sum_{k \neq i_1, j_1, i_2} \frac{1}{p_k} x_k^{p_k} + \Big(\frac{1}{p_{j_1}} + \frac{a}{p_{i_1}} + \frac{a}{p_{i_2}} \Big) x_{j_1}^{p_{j_1}} = 1, \\ a x_k^{p_k} \leqslant x_i^{p_i} \text{ for } i, k \neq i_1, i_2,\ x_k > 0 \text{ for } k \neq i_1, i_2 \Big\}.$$

Applying this argument iteratively, we obtain

$$\Gamma = \min_{x \in E_0'} x_1 \cdots x_n$$

with

$$E_0' = \Big\{ x \in E_1 : a^{e_{k,i}} x_k^{p_k} = x_i^{p_i} \text{ with } e_{k,i} \in \mathbb{Z} \text{ for } 1 \leqslant i, k \leqslant n \Big\}.$$

Since $0<a<1$ and $ax_k^{p_k}\leqslant x_i^{p_i}\leqslant a^{-1}x_k^{p_k}$ for $1\leqslant i,k\leqslant n$ for every $x\in E_1$,

$$\Gamma=\min_{x\in E_0} x_1\cdots x_n$$

with

$$E_0=\Big\{x\in\mathbb{R}^n:\ \sum_{k=1}^{n}\frac{1}{p_k}x_k^{p_k}=1,\ a^{e_{k,i}}x_k^{p_k}=x_i^{p_i}\ \text{ with } e_{k,i}\in\{-1,0,1\}$$
$$\text{for }\ 1\leqslant i,k\leqslant n \text{ with some } e_{k,i}\neq 0\Big\}$$

(recall that $ax_{j_1}^{p_{j_1}}=x_{i_1}^{p_{i_1}}$ and so, $e_{j_1,i_1}\neq 0$). Then $E_0=\cup_{m=1}^{n-1}E_0^m$, where

$$E_0^m=\Big\{x\in E_0:\ \exists\sigma\in\mathscr{P}_n \text{ with } ax_{\sigma(k)}^{p_{\sigma(k)}}=x_{\sigma(i)}^{p_{\sigma(i)}}\ \text{ for }\ 1\leqslant k\leqslant m<i\leqslant n\Big\}.$$

If $x\in E_0^m$ and $\sigma\in\mathscr{P}_n$ satisfies $ax_{\sigma(k)}^{p_{\sigma(k)}}=x_{\sigma(i)}^{p_{\sigma(i)}}$ for $1\leqslant k\leqslant m<i\leqslant n$, define $t=x_{\sigma(1)}^{p_{\sigma(1)}}$. We have

$$\begin{aligned}1&=\sum_{k=1}^{n}\frac{1}{p_{\sigma(k)}}x_{\sigma(k)}^{p_{\sigma(k)}}=\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}t+\sum_{k=m+1}^{n}\frac{1}{p_{\sigma(k)}}at\\&=t\Big(\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}+a\sum_{k=m+1}^{n}\frac{1}{p_{\sigma(k)}}\Big).\end{aligned}$$

If $1\leqslant i\leqslant m$, then

$$x_{\sigma(i)}=\Big(\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}+a\sum_{k=m+1}^{n}\frac{1}{p_{\sigma(k)}}\Big)^{-1/p_{\sigma(i)}}.$$

If $m<i\leqslant n$, then

$$x_{\sigma(i)}=a^{1/p_{\sigma(i)}}\Big(\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}+a\sum_{k=m+1}^{n}\frac{1}{p_{\sigma(k)}}\Big)^{-1/p_{\sigma(i)}}.$$

Hence,

$$\begin{aligned}\prod_{i=1}^{n}x_i&=\prod_{i=1}^{m}\Big(\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}+a\sum_{k=m+1}^{n}\frac{1}{p_{\sigma(k)}}\Big)^{-1/p_{\sigma(i)}}\\&\qquad\cdot\prod_{i=m+1}^{n}a^{1/p_{\sigma(i)}}\Big(\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}+a\sum_{k=m+1}^{n}\frac{1}{p_{\sigma(k)}}\Big)^{-1/p_{\sigma(i)}}\\&=\Big(\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}+a\sum_{k=m+1}^{n}\frac{1}{p_{\sigma(k)}}\Big)^{-\sum_{i=1}^{n}1/p_{\sigma(i)}}a^{\sum_{i=m+1}^{n}1/p_{\sigma(i)}}\\&=\Big(\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}+a\sum_{k=m+1}^{n}\frac{1}{p_{\sigma(k)}}\Big)^{-1}a^{\sum_{k=m+1}^{n}1/p_{\sigma(k)}}\\&=\Big(a+(1-a)\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}\Big)^{-1}a^{1-\sum_{k=1}^{m}1/p_{\sigma(k)}}\end{aligned}$$

and so,

$$\Gamma = \min_{1\leqslant m<n,\,\sigma\in\mathscr{P}_n} \Big(a+(1-a)\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}\Big)^{-1} a^{1-\sum_{k=1}^{m}1/p_{\sigma(k)}}.$$

Let us find a lower bound for Γ, which is very good when n grows.

Consider the function $u:[0,1]\to\mathbb{R}$ given by

$$u(s)=\big(a+(1-a)s\big)^{-1}a^{1-s}.$$

Thus,

$$\begin{aligned} u'(s) &= -(1-a)\big(a+(1-a)s\big)^{-2}a^{1-s}+\big(a+(1-a)s\big)^{-1}a^{1-s}(-\log a)\\ &= \big(a+(1-a)s\big)^{-2}a^{1-s}\big(-(1-a)-\big(a+(1-a)s\big)\log a\big). \end{aligned}$$

The function $v(s)=-(1-a)-\big(a+(1-a)s\big)\log a$ is increasing, $v(0)=-1+a-a\log a$ and $v(1)=-1+a-\log a$.

If $w_1(a)=-1+a-a\log a$, then $w_1'(a)=-\log a>0$ and $w_1(a)<w_1(1)=0$ for every $0<a<1$.

If $w_2(a)=-1+a-\log a$, then $w_2'(a)=1-1/a<0$ and $w_2(a)>w_2(1)=0$ for every $0<a<1$.

Therefore, $v(0)=w_1(a)<0$ and $v(1)=w_2(a)>0$, and so, $u'(s_0)=0$ if and only if

$$-(1-a)-\big(a+(1-a)s_0\big)\log a=0 \qquad \Longleftrightarrow \qquad a+(1-a)s_0=\frac{1-a}{-\log a}$$
$$\Longleftrightarrow \qquad s_0=\frac{1-a+a\log a}{-(1-a)\log a}.$$

Since $u'<0$ on $(0,s_0)$ and $u'>0$ on $(s_0,1)$, we have $u(s)\geqslant u(s_0)$ for every $s\in(0,1)$. We have

$$a+(1-a)s_0=a+(1-a)\frac{1-a+a\log a}{-(1-a)\log a}=\frac{1-a}{-\log a}$$

and

$$1-s_0=1+\frac{1-a+a\log a}{(1-a)\log a}=\frac{1-a+\log a}{(1-a)\log a}$$
$$a^{1-s_0}=e^{(1-s_0)\log a}=e^{1+\frac{\log a}{1-a}}=e\,a^{\frac{1}{1-a}}.$$

Hence,

$$\Gamma=\min_{1\leqslant m<n,\,\sigma\in\mathscr{P}_n} u\Big(\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}\Big)\geqslant u(s_0)=\frac{-\log a}{1-a}\,e\,a^{\frac{1}{1-a}}. \quad \square$$

THEOREM 7. *Let μ be a measure on any measurable space X, $0<a<1$, $p_1,\dots,p_n>1$ be real numbers such that $\frac{1}{p_1}+\dots+\frac{1}{p_n}=1$, and $f_k:X\to\mathbb{C}$ measurable functions with $f_1\cdots f_n\in L^1(X,\mu)$ and $a|f_k|^{p_k}\leqslant|f_i|^{p_i}$ μ-a.e. for $1\leqslant i,k\leqslant n$. Then $f_k\in L^{p_k}(X,\mu)$ for $1\leqslant k\leqslant n$ and*

$$\|f_1\|_{p_1}\cdots\|f_n\|_{p_n}\leqslant A\,\|f_1\cdots f_n\|_1, \tag{11}$$

where

$$A = \max_{1 \leqslant m < n,\, \sigma \in \mathscr{P}_n} \Big(a + (1-a) \sum_{k=1}^{m} \frac{1}{p_{\sigma(k)}} \Big) a^{-1+\sum_{k=1}^{m} 1/p_{\sigma(k)}}$$
$$\leqslant e^{-1} a^{\frac{-1}{1-a}} \frac{1-a}{-\log a} = S(a).$$

REMARK 8. Note that the inequality

$$\|f_1\|_{p_1} \cdots \|f_n\|_{p_n} \leqslant S(a)\, \|f_1 \cdots f_n\|_1$$

holds with a constant which is the known Specht's ratio (for two variables). Hence, this constant just depends on a; in particular, it does not depend on $n, p_1, \dots, p_n, f_1, \dots, f_n$, and this is an important fact in the theory of $L^{p(\cdot)}$ spaces with variable exponent (see e.g. [14, 17, 18, 20, 27]).

Proof. Proposition 6 gives

$$A\, |f_1(x) \cdots f_n(x)| \geqslant \frac{1}{p_1} |f_1(x)|^{p_1} + \cdots + \frac{1}{p_n} |f_n(x)|^{p_n}$$

for μ-a.e. $x \in X$. If we integrate this inequality with respect to μ, then we obtain

$$A\, \|f_1 \cdots f_n\|_1 \geqslant \frac{1}{p_1} \|f_1\|_{p_1}^{p_1} + \cdots + \frac{1}{p_n} \|f_n\|_{p_n}^{p_n}. \tag{12}$$

Since $f_1 \cdots f_n \in L^1(X,\mu)$, (12) implies $f_k \in L^{p_k}(X,\mu)$ for $1 \leqslant k \leqslant n$.

If $\|f_k\|_{p_k} = 0$ for some $1 \leqslant k \leqslant n$, then the inequality is direct.

If $\|f_k\|_{p_k} > 0$ for every $1 \leqslant k \leqslant n$, applying (12) to the functions $f_k / \|f_k\|_{p_k}$, we obtain

$$A \left\| \frac{f_1}{\|f_1\|_{p_1}} \cdots \frac{f_n}{\|f_n\|_{p_n}} \right\|_1 \geqslant \frac{1}{p_1} \left\| \frac{f_1}{\|f_1\|_{p_1}} \right\|_{p_1}^{p_1} + \cdots + \frac{1}{p_n} \left\| \frac{f_n}{\|f_n\|_{p_n}} \right\|_{p_n}^{p_n}$$
$$= \frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1,$$

and so,

$$A\, \|f_1 \cdots f_n\|_1 \geqslant \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}. \qquad \square$$

5. Generalized Riemann-Liouville-type integral operators

One of the first operators that can be called fractional is the Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$, with $Re(\alpha) > 0$, defined as follows (see [24]).

DEFINITION 9. Let $a < b$ and $f \in L^1((a,b);\mathbb{R})$. The *right and left side Riemann-Liouville fractional integrals of order* α, with $Re(\alpha) > 0$, are defined, respectively, by

$${}^{RL}J_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \tag{13}$$

and

$$ {}^{RL}J^{\alpha}_{b^-}f(t)=\frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)\,ds, \tag{14}$$

with $t\in(a,b)$.

When $\alpha\in(0,1)$, their corresponding *Riemann-Liouville fractional derivatives* are given by

$$\begin{aligned}\left({}^{RL}D^{\alpha}_{a^+}f\right)(t)&=\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{a^+}f(t)\right)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t\frac{f(s)}{(t-s)^{\alpha}}\,ds,\\ \left({}^{RL}D^{\alpha}_{b^-}f\right)(t)&=-\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{b^-}f(t)\right)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b\frac{f(s)}{(s-t)^{\alpha}}\,ds.\end{aligned}$$

Now, we give the definition of a general fractional integral in [6] (see also [12]).

DEFINITION 10. Let $a<b$ and $\alpha\in\mathbb{R}^+$. Let $g:[a,b]\to\mathbb{R}$ be a positive function on $(a,b]$ with continuous positive derivative on (a,b), and $G:[0,g(b)-g(a)]\times(0,\infty)\to\mathbb{R}$ a continuous function which is positive on $(0,g(b)-g(a)]\times(0,\infty)$. Let us define the function $T:[a,b]\times[a,b]\times(0,\infty)\to\mathbb{R}$ by

$$T(t,s,\alpha)=\frac{G(|g(t)-g(s)|,\alpha)}{g'(s)}.$$

The *right and left integral operators*, denoted respectively by J^{α}_{T,a^+} and J^{α}_{T,b^-}, are defined for each measurable function f on $[a,b]$ as

$$J^{\alpha}_{T,a^+}f(t)=\int_a^t\frac{f(s)}{T(t,s,\alpha)}\,ds, \tag{15}$$

$$J^{\alpha}_{T,b^-}f(t)=\int_t^b\frac{f(s)}{T(t,s,\alpha)}\,ds, \tag{16}$$

with $t\in[a,b]$.

We say that $f\in L^1_T[a,b]$ if $J^{\alpha}_{T,a^+}|f|(t), J^{\alpha}_{T,b^-}|f|(t)<\infty$ for every $t\in[a,b]$.

Theorems 7 and 3 have, respectively, the following direct consequences for generalized Riemann-Liouville-type integral operators.

PROPOSITION 11. *Let* $0<a<1$, $p_1,\dots,p_n>1$ *be real numbers such that* $\frac{1}{p_1}+\dots+\frac{1}{p_n}=1$, $c<d$ *real constants and* $d\mu(s)=ds/T(d,s,\alpha)$ *on* $[c,d]$. *If* $f_k:[c,d]\to\mathbb{C}$ *are measurable functions with* $f_1\cdots f_n\in L^1(\mu,[c,d])$ *and* $a|f_k|^{p_k}\leqslant|f_i|^{p_i}$ μ*-a.e. for* $1\leqslant i,k\leqslant n$, *then* $f_k\in L^{p_k}(\mu,[c,d])$ *for* $1\leqslant k\leqslant n$ *and*

$$\left(\int_c^d\frac{|f_1(s)|^{p_1}}{T(d,s,\alpha)}\,ds\right)^{1/p_1}\cdots\left(\int_c^d\frac{|f_n(s)|^{p_n}}{T(d,s,\alpha)}\,ds\right)^{1/p_n}\leqslant A\int_c^d\frac{|f_1(s)\cdots f_n(s)|}{T(d,s,\alpha)}\,ds, \tag{17}$$

where

$$\begin{aligned} A &= \max_{1\leqslant m<n,\, \sigma\in\mathscr{P}_n} \Big(a+(1-a)\sum_{k=1}^{m}\frac{1}{p_{\sigma(k)}}\Big)a^{-1+\sum_{k=1}^{m}1/p_{\sigma(k)}} \\ &\leqslant e^{-1}a^{\frac{-1}{1-a}}\,\frac{1-a}{-\log a}. \end{aligned}$$

PROPOSITION 12. *Let* $c<d$ *be real constants,* $d\mu(s)=ds/T(d,s,\alpha)$ *on* $[c,d]$*, let* $p,q>1$ *be real numbers such that* $\frac{1}{p}+\frac{1}{q}=1$*,* $f\in L^p(\mu,[c,d])$ *and* $g\in L^q(\mu,[c,d])$*.*

(1) *If* $|f|^{1-p}g\in L^\infty[c,d]$*, then*

$$\left(\int_c^d\frac{|f(s)|^p}{T\big(d,s,\alpha\big)}\,ds\right)^{1/p}\left(\int_c^d\frac{|g(s)|^q}{T\big(d,s,\alpha\big)}\,ds\right)^{1/q}\leqslant \big\||f|^{1-p}g\big\|_\infty\int_c^d\frac{|f(s)|^p}{T\big(d,s,\alpha\big)}\,ds$$

and the following Hölder-type inequality holds:

$$\begin{aligned} \int_c^d\frac{|fg|}{T(d,s,\alpha)}\,ds\geqslant\ & \big\||f|^{1-p}g\big\|_\infty\int_c^d\frac{|f(s)|^p}{T\big(d,s,\alpha\big)}\,ds \\ & -\left(\big\||f|^{1-p}g\big\|_\infty^q\left(\int_c^d\frac{|f(s)|^p}{T\big(d,s,\alpha\big)}\,ds\right)^q\right. \\ & \left.-\left(\int_c^d\frac{|f(s)|^p}{T\big(d,s,\alpha\big)}\,ds\right)^{q/p}\left(\int_c^d\frac{|g(s)|^q}{T\big(d,s,\alpha\big)}\,ds\right)\right)^{1/q}. \end{aligned}$$

(2) *If* $f|g|^{1-q}\in L^\infty[c,d]$*, then*

$$\left(\int_c^d\frac{|f(s)|^p}{T\big(d,s,\alpha\big)}\,ds\right)^{1/p}\left(\int_c^d\frac{|g(s)|^q}{T\big(d,s,\alpha\big)}\,ds\right)^{1/q}\leqslant \big\|f|g|^{1-q}\big\|_\infty\int_c^d\frac{|g(s)|^q}{T\big(d,s,\alpha\big)}\,ds$$

and the following Hölder-type inequality holds:

$$\begin{aligned} \int_c^d\frac{|fg|}{T(d,s,\alpha)}\,ds\geqslant\ & \big\|f|g|^{1-q}\big\|_\infty\int_c^d\frac{|g(s)|^q}{T\big(d,s,\alpha\big)}\,ds \\ & -\left(\big\|f|g|^{1-q}\big\|_\infty^p\left(\int_c^d\frac{|g(s)|^q}{T\big(d,s,\alpha\big)}\,ds\right)^p\right. \\ & \left.-\left(\int_c^d\frac{|f(s)|^p}{T\big(d,s,\alpha\big)}\,ds\right)\left(\int_c^d\frac{|g(s)|^q}{T\big(d,s,\alpha\big)}\,ds\right)^{p/q}\right)^{1/p}. \end{aligned}$$

6. Generalized local fractional derivative

Let us recall the definition of generalized local fractional derivative in [2,7,21,22]. Given $s \in \mathbb{R}$, we denote by $\lceil s \rceil$ the *upper integer part* of s, i.e., the smallest integer greater than or equal to s.

DEFINITION 13. Given an interval $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and a positive continuous function $F(t,\alpha)$ on $I \times (0,\infty)$, the *derivative* $G_F^\alpha f$ of f of order α at the point $t \in I$ is defined by

$$G_F^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} f\big(t - khF(t,\alpha)\big). \tag{18}$$

If $a = \inf\{t \in I\}$ (respectively, $b = \sup\{t \in I\}$), then $G_F^\alpha f(a)$ (respectively, $G_F^\alpha f(b)$) is defined with $h \to 0^-$ (respectively, $h \to 0^+$) instead of $h \to 0$ in the limit.

If $F(t,\alpha) = 1$ when $\alpha \in \mathbb{N}$, then we obtain a conformable local fractional derivative of any order. See [1,28,30] for more information on conformable fractional derivatives. If $F(t,\alpha)$ depends on t when $\alpha \in \mathbb{N}$, then we get a non-conformable local fractional derivative of any order.

DEFINITION 14. Let I be an interval $I \subseteq (0,\infty)$, $f : I \to \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. The *conformable derivative* $G^\alpha f$ of f of order α at the point $t \in I$ is defined by

$$G^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} f\big(t - kht^{\lceil \alpha \rceil - \alpha}\big). \tag{19}$$

Note that $F(t,\alpha) = t^{\lceil \alpha \rceil - \alpha} = 1$ for every $\alpha \in \mathbb{N}$. We know from the classical calculus that if f is a function defined in a neighborhood of the point t, and there exists the n-th derivative $D^n f(t)$, then

$$D^n f(t) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t - kh).$$

Therefore, if $\alpha = n \in \mathbb{N}$ and f is smooth enough, then Definition 14 coincides with the classical definition of the n-th derivative. The same holds for any choice of F with $F(t,\alpha) = 1$ for $t \in I$ and $\alpha \in \mathbb{N}$.

Let I be an interval $I \subseteq \mathbb{R}$, $a,t \in I$ and $\alpha \in \mathbb{R}$. The integral operator $J_{F,a}^\alpha$ is defined for every locally integrable function f on I as

$$J_{F,a}^\alpha(f)(t) = \int_a^t \frac{f(s)}{F(s,\alpha)}\, ds.$$

The following results in [2, 7, 21, 22, 31] contain some basic properties of this integral operator.

PROPOSITION 15. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$, $0 < \alpha \leqslant 1$ and f a differentiable function on I such that f' is a locally integrable function on I. Then, we have for all $t \in I$*

$$J_{F,a}^{\alpha}\big(G_F^{\alpha}(f)\big)(t) = f(t) - f(a).$$

PROPOSITION 16. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$ and $\alpha \in (0,1]$.*

$$G_F^{\alpha}\big(J_{F,a}^{\alpha}(f)\big)(t) = f(t),$$

for every continuous function f on I and $a, t \in I$.

Theorems 7 and 3 have, respectively, the following direct consequences for the integral operator $J_{F,c}^{\alpha}$.

PROPOSITION 17. *Let $0 < a < 1$, $p_1, \dots, p_n > 1$ be real numbers such that $\frac{1}{p_1} + \dots + \frac{1}{p_n} = 1$, $c < d$ real constants and $d\mu(s) = ds/F(s,\alpha)$ on $[c,d]$. If $f_k : [c,d] \to \mathbb{C}$ are measurable functions with $f_1 \cdots f_n \in L^1(\mu, [c,d])$ and $a|f_k|^{p_k} \leqslant |f_i|^{p_i}$ μ-a.e. for $1 \leqslant i,k \leqslant n$, then $f_k \in L^{p_k}(\mu, [c,d])$ for $1 \leqslant k \leqslant n$ and*

$$\left(\int_c^d \frac{|f_1(s)|^{p_1}}{F(s,\alpha)}\, ds\right)^{1/p_1} \cdots \left(\int_c^d \frac{|f_n(s)|^{p_n}}{F(s,\alpha)}\, ds\right)^{1/p_n} \leqslant A \int_c^d \frac{|f_1(s) \cdots f_n(s)|}{F(s,\alpha)}\, ds, \tag{20}$$

i.e.,

$$\big(J_{F,c}^{\alpha}(|f_1|^{p_1})(d)\big)^{1/p_1} \cdots \big(J_{F,c}^{\alpha}(|f_n|^{p_n})(d)\big)^{1/p_n} \leqslant A\, J_{F,c}^{\alpha}(|f_1 \cdots f_n|)(d), \tag{21}$$

where

$$\begin{aligned} A &= \max_{1 \leqslant m < n,\, \sigma \in \mathscr{P}_n} \left(a + (1-a) \sum_{k=1}^{m} \frac{1}{p_{\sigma(k)}}\right) a^{-1+\sum_{k=1}^{m} 1/p_{\sigma(k)}} \\ &\leqslant e^{-1} a^{\frac{-1}{1-a}} \frac{1-a}{-\log a}. \end{aligned}$$

PROPOSITION 18. *Let $c < d$ be real constants, $d\mu(s) = ds/F(s,\alpha)$ on $[c,d]$, let $p, q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\mu, [c,d])$ and $g \in L^q(\mu, [c,d])$.*

(1) *If $|f|^{1-p} g \in L^{\infty}[c,d]$, then*

$$\big(J_{F,c}^{\alpha}(|f|^p)(d)\big)^{1/p} \big(J_{F,c}^{\alpha}(|g|^q)(d)\big)^{1/q} \leqslant \big\||f|^{1-p} g\big\|_{\infty} J_{F,c}^{\alpha}(|f|^p)(d)$$

and the following Hölder-type inequality holds:

$$\begin{aligned} &J_{F,c}^{\alpha}(|fg|)(d) \geqslant \big\||f|^{1-p} g\big\|_{\infty} J_{F,c}^{\alpha}(|f|^p)(d) \\ &- \Big(\big\||f|^{1-p} g\big\|_{\infty}^{q} \big(J_{F,c}^{\alpha}(|f|^p)(d)\big)^{q} - \big(J_{F,c}^{\alpha}(|f|^p)(d)\big)^{q/p} \big(J_{F,c}^{\alpha}(|g|^q)(d)\big)\Big)^{1/q}. \end{aligned}$$

(2) *If $f|g|^{1-q} \in L^{\infty}[c,d]$, then*

$$\big(J_{F,c}^{\alpha}(|f|^p)(d)\big)^{1/p} \big(J_{F,c}^{\alpha}(|g|^q)(d)\big)^{1/q} \leqslant \big\|f|g|^{1-q}\big\|_{\infty} J_{F,c}^{\alpha}(|f|^p)(d)$$

and the following Hölder-type inequality holds:

$$
\begin{aligned}
&J^{\alpha}_{F,c}(|fg|)(d) \geqslant \left\|f|g|^{1-q}\right\|_{\infty} J^{\alpha}_{F,c}(|g|^{q})(d) \\
&- \left(\left\|f|g|^{1-q}\right\|_{\infty}^{p} \left(J^{\alpha}_{F,c}(|g|^{q})(d)\right)^{p} - \left(J^{\alpha}_{F,c}(|f|^{p})(d)\right) \left(J^{\alpha}_{F,c}(|g|^{q})(d)\right)^{p/q} \right)^{1/p}.
\end{aligned}
$$

Acknowledgements. We would like to thank the referee for his/her comments which have improved the presentation of the paper.

The research of Yamilet Quintana, José M. Rodríguez and José M. Sigarreta is supported by a grant from Agencia Estatal de Investigación (PID2019-106433GB-I00 /MCIN/AEI/10.13039/501100011033), Spain.

The research of Yamilet Quintana and José M. Rodríguez is supported by the Madrid Government (Comunidad de Madrid-Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M23), and in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

The research of Yamilet Quintana is partially supported by the grant CEX2019-000904-S funded by MCIN/AEI/10.13039/501100011033.

REFERENCES

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. **279** (2015) 57–66, https://doi.org/10.1016/j.cam.2014.10.016.

[2] R. Abreu-Blaya, A. Fleitas, J. E. Nápoles Valdés, R. Reyes, J. M. Rodríguez, J. M. Sigarreta, *On the conformable fractional logistic models*, Math. Methods Appl. Sci. **43:7** (2020) 4156–4167.

[3] J. M. Aldaz, *A stability version of Hölder's inequality*, J. Math. Anal. Appl. **343** (2008) 842–852.

[4] J. M. Aldaz, *Comparison of differences between arithmetic and geometric means*, Tamkang J. Math. **42** (2011) 445–451.

[5] J. M. Aldaz, *Strengthened Cauchy-Schwarz and Hölder Inequalities*, J. Ineq. Pure Appl. Math. **10:4** (2009) 116.

[6] P. Bosch, H. J. Carmenate, J. M. Rodríguez, J. M. Sigarreta, *Generalized inequalities involving fractional operators of the Riemann-Liouville type*, AIMS Math. **7:1** (2022) 1470–1485, https://doi.org/10.3934/math.2022087.

[7] P. Bosch, J. F. Gómez-Aguilar, J. M. Rodríguez, J. M. Sigarreta, *Analysis of dengue fever outbreak by generalized fractional derivative*, Fractals **28:8** (2020) Art. 2040038.

[8] P. Bosch, E. D. Molina, J. M. Rodríguez, J. M. Sigarreta, *Generalized inequalities involving Inequalities on the Generalized ABC Index*, Mathematics **9:10** (2021) 1151.

[9] P. Bosch, Y. Quintana, J. M. Rodríguez, J. M. Sigarreta, *Jensen-type inequalities for m-convex functions*, Open Math. **20** (2022) 11–13, https://doi.org/10.1515/math-2022-0061.

[10] P. Bosch, A. Portilla, J. M. Rodríguez, J. M. Sigarreta, *On a generalization of the Opial inequality*, submitted.

[11] P. Bosch, J. M. Rodríguez, J. M. Sigarreta, *On new Milne-type inequalities and applications*, J. Inequal. Appl. **2023** (2023) 3, https://doi.org/10.1186/s13660-022-02910-0.

[12] P. Bosch, J. M. Rodríguez, J. M. Sigarreta, *Oscillation results for a nonlinear fractional differential equation*, AIMS Math. **8** (5) (2023) 12486–12505.

[13] M. P. Cruz, R. Abreu Blaya, P. Bosch, J. M. Rodríguez, J. M. Sigarreta, *On Ostrowski type inequalities for generalized integral operators*, J. Math. **2022** (2022) Article ID 2247246, https://doi.org/10.1155/2022/2247246.

[14] D. CRUZ-URIBE, A. FIORENZE, C. J. NEUGEBAUER, *The maximal operator on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003) 223–238.

[15] Z. CVETKOVSKI, *Inequalities: Theorems, Techniques and Selected Problems*, Springer-Verlag, Berlin, 2012.

[16] Z. DAHMANI, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integral*, Ann. Funct. Anal. **1** (2010) 51–8.

[17] L. DIENING, *Riesz Potential and Sobolev Embeddings of generalized Lebesgue and Sobolev Spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr. **263:1** (2004) 31–43.

[18] L. DIENING, M. RUŽIČKA, *Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Ang. Math. **563** (2003) 197–220.

[19] S. S. DRAGOMIR, C. J. GOH, *A counterpart of Hölder's inequality*, Mitt. Math. Ges. Hamburg **16** (1997) 99–106.

[20] X. FAN, D. ZHAO, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001) 424–446.

[21] A. FLEITAS, J. F. GÓMEZ-AGUILAR, J. E. NÁPOLES VALDÉS, J. M. RODRÍGUEZ, J. M. SIGARRETA, *Analysis of the local Drude model involving the generalized fractional derivative*, Optik **193** (2019) 163008.

[22] A. FLEITAS, J. E. NÁPOLES VALDÉS, J. M. RODRÍGUEZ, J. M. SIGARRETA, *Note on the generalized conformable derivative*, Rev. Uni. Mat. Argen. **62:2** (2021) 443–457.

[23] X. GAO, M. GAO, X. SHANG, *A refinement of Hölder's inequality and applications*, JIPAM. J. Inequal. Pure Appl. Math. **8:2** (2007) Article 44, 9 pp.

[24] R. GORENFLO, F. MAINARDI, *Fractals and Fractional Calculus in Continuum Mechanics*, 1st ed., Vienna, Springer, 1997.

[25] J. HAN, P. OTHMAN MOHAMMED, H. ZENG, *Generalized fractional integral inequalities of Hermite-Hadamard-type for a convex function*, Open Math. **18** (2020) 794–806.

[26] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[27] P. HARJULEHTO, P. HÄSTÖ, *A capacity approach to Poincaré inequalities and Sobolev imbedding in variable exponent Sobolev spaces*, Rev. Mat. Complut. **17:1** (2004) 129–146.

[28] F. JARAD, E. UGURLU, T. ABDELJAWAD, D. BALEANU, *On a new class of fractional operators*, Adv. Differ. Equ. **2017** (2017) 247, `https://doi.org/10.1186/s13662-017-1306-z`.

[29] J. L. W. V. JENSEN, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math. **30:1** (1906) 175–193.

[30] U. N. KATUGAMPOLA, *A new fractional derivative with classical properties*, arXiv:1410.6535.

[31] R. KHALIL, M. AL HORANI, A. YOUSEF, M. SABABHEH, *A new definition of fractional derivative*, J. Comput. Appl. Math. **264** (2014) 65–70.

[32] P. KÓRUS, J. E. NÁPOLES VALDÉS, *q-Hermite-Hadamard inequalities for functions with convex or h-convex q-derivative*, Math. Ineq. Appl. **25** (2) (2022) 601–610.

[33] P. KÓRUS, J. E. NÁPOLES VALDÉS, J. M. RODRÍGUEZ, J. M. SIGARRETA ALMIRA, *Petrović-type inequality via fractional calculus*, submitted.

[34] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[35] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic, 1993.

[36] S. MUBEEN, S. HABIB, M. N. NAEEM, *The Minkowski inequality involving generalized k-fractional conformable integral*, J. Inequal. Appl. **2019** (2019) 81.

[37] K. S. NISAR, F. QI, G. RAHMAN, S. MUBEEN, M. ARSHAD, *Some inequalities involving the extended gamma function and the Kummer confluent hypergeometric K-function, J. Inequal. Appl.* **2018** (2018) 135.

[38] J. PEČARIĆ, V. ŠIMIĆ, *A note on the Hölder inequality*, JIPAM. J. Inequal. Pure Appl. Math. **7:5** (2006), Article 176, 3 pp.

[39] G. RAHMAN, T. ABDELJAWAD, F. JARAD, A. KHAN, K. SOOPPY NISAR, *Certain inequalities via generalized proportional Hadamard fractional integral operators*, Adv. Difference Eq. **2019** (2019) 454.

[40] G. RAHMAN, K. SOOPPY NISAR, B. GHANBARI, T. ABDELJAWAD, *On generalized fractional integral inequalities for the monotone weighted Chebyshev functionals*, Adv. Difference Eq. **2020** (2020) 368.

[41] S. RASHID, M. ASLAM NOOR, K. INAYAT NOOR, Y.-M. CHU, *Ostrowski type inequalities in the sense of generalized K-fractional integral operator for exponentially convex functions*, AIMS Math. **5:3**, 2629–2645.

[42] Y. SAWANO, H. WADADE, *On the Gagliardo-Nirenberg type inequality in the critical Sobolev-Orrey space*, J. Fourier Anal. Appl. **19** (2013) 20–47.

[43] E. SET, M. TOMAR, M. Z. SARIKAYA, *On generalized Grüss type inequalities for k-fractional integrals*, Appl. Math. Comput. **269** (2015) 29–34.

[44] W. SPECHT, *Zer Theorie der elementaren Mittel*, Math. Z. **74** (1960) 91–98.

[45] M. TOMINAGA, *Specht's ratio in the Young inequality*, Sci. Math. Japan **55** (2002) 583–588.

(Received March 11, 2023)

Jorge A. Paz Moyado
Universidad Autónoma de Guerrero
Centro Acapulco
CP 39610, Acapulco de Juárez, Guerrero, Mexico
e-mail: `16376113@uagro.mx`

Yamilet Quintana
Universidad Carlos III de Madrid
Departamento de Matemáticas
Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain
and
Instituto de Ciencias Matematicas (ICMAT)
Campus de Cantoblanco UAM, 28049-Madrid, Spain
e-mail: `yaquinta@math.uc3m.es`

José M. Rodríguez
Universidad Carlos III de Madrid
Departamento de Matemáticas
Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain
e-mail: `jomaro@math.uc3m.es`

José M. Sigarreta
Universidad Autónoma de Guerrero
Centro Acapulco
CP 39610, Acapulco de Juárez, Guerrero, Mexico
e-mail: `jsmathguerrero@gmail.com`