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ON THE UNIFORM CONVERGENCE AND INTEGRABILITY OF SPECIAL TRIGONOMETRIC INTEGRALS

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Abstract. Necessary and sufficient conditions for the uniform convergence of trigonometric Fourier integrals are well-established when admissible monotone or general monotone functions are considered. In this paper, we generalize these main results by giving such conditions for the uniform convergence of sine and cosine integrals $\int_0^\infty f_1(x)\sin(ux^p)\,dx$ and $\int_0^\infty f_2(x)\cos(ux^p)\,dx$ in case of admissible general monotone functions f_1 and f_2. Moreover, we give necessary and sufficient conditions for the L^q-integrability with the power weights of these integrals when non-negative functions f_1 and f_2 belong to the class $\overline{GM}_{p\theta}$.

1. Preliminaries on uniform convergence

Let $f_1, f_2 : \mathbb{R}_+ \to \mathbb{C}$ be measurable functions in Lebesgue's sense and $p > 0$, where $\mathbb{R}_+ = (0,\infty)$. We consider the uniform convergence of the sine and the cosine integrals

$$F_1(p,u) := F_1(f_1(x);p,u) = \int_0^\infty f_1(x)\sin(ux^p)\,dx, \tag{1.1}$$

$$F_2(p,u) := F_2(f_2(x);p,u) = \int_0^\infty f_2(x)\cos(ux^p)\,dx \tag{1.2}$$

in $u \in \overline{\mathbb{R}}_+ = [0,\infty)$, where we mean the uniform convergence of

$$S_1(p,b,u) := S_1(f_1(x);p,b,u) = \int_0^b f_1(x)\sin(ux^p)\,dx, \quad b \to \infty,$$

$$S_2(p,b,u) := S_2(f_1(x);p,b,u) = \int_0^b f_2(x)\cos(ux^p)\,dx, \quad b \to \infty,$$

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respectively.

We recall that the uniform convergence of the sine and the cosine integrals

$$F_1(1,u)=\int_0^\infty f_1(x)\sin(ux)\,dx, \tag{1.3}$$

$$F_2(1,u)=\int_0^\infty f_2(x)\cos(ux)\,dx \tag{1.4}$$

in $u\in\overline{\mathbb{R}}_+$ has been studied by many authors, see e.g. [2, 3, 9, 16].

Throughout the paper, we deal with functions f_1 and f_2 defined on $\overline{\mathbb{R}}_+$, of bounded variation on $\mathbb{R}_+$, vanishing at infinity, and such that $x^p f_1(x)\in L^1([0,1])$ for $p>0$ and $f_2(x)\in L^1([0,1])$. We will denote the above presumptions in the following way: $f_1\in\Phi_p$ $(p>0)$ and $f_2\in\Psi$, respectively. It is clear that $\Psi\subset\Phi_{p_1}\subset\Phi_{p_2}$ for $0<p_1<p_2$. Then local integrability of $f_1(x)\sin(ux^p)$ and $f_2(x)\cos(ux^p)$ are ensured.

It is clear that for $p>0$ and $f_1\in\Phi_p$ or $f_2\in\Psi$ we have

$$F_1(f_1(x);p,u)=\int_0^\infty \frac{1}{p}x^{1/p-1}f_1(x^{1/p})\sin(ux)\,dx=F_1\left(\tfrac{1}{p}x^{1/p-1}f_1(x^{1/p});1,u\right) \tag{1.5}$$

or

$$F_2(f_2(x);p,u)=F_2\left(\tfrac{1}{p}x^{1/p-1}f_2(x^{1/p});1,u\right), \tag{1.6}$$

respectively.

DEFINITION 1.1. ([3, 12]) Suppose that $f:\mathbb{R}_+\to\mathbb{C}$ is a function locally of bounded variation on $\mathbb{R}_+$. We say that f is *general monotone* with majorant β, or shortly, $f\in GM(\beta)$, if there exist positive constants C, A and a function $\beta:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\int_x^{2x}|df(t)|\leqslant C\beta(x)$$

for all $x>A$.

If we consider majorant ${}_0\beta(x)=|f(x)|$, then class $GM({}_0\beta)$ contains M, the class of non-negative, monotone non-increasing functions (see [14, 15]).

In case of

$${}_1\beta(x)=\frac{1}{x}\int_{x/c}^{cx}|f(t)|\,dt,\quad (x\in\mathbb{R}_+)$$

functions from $GM({}_1\beta)=MVBVF$ are called *mean value bounded variation functions*, where $c>1$ (see [13]).

There are several intermediate classes that have been considered in various publications. Further if we choose

$$ {}_2\beta(x) = \frac{1}{x}\sup_{a\geqslant B(x)}\int_a^{2a}|f(t)|\,dt, \quad (x\in\mathbb{R}_+) $$

class $GM({}_2\beta) = SBVF_2$ consists of *supremum bounded variation functions of second type*, where B is a positive non-decreasing function such that $B(x)\to\infty$ as $x\to\infty$. Note that $M \subsetneq GM({}_0\beta) \subsetneq GM({}_1\beta) \subsetneq GM({}_2\beta)$ (see [9]).

We recall the main results regarding the uniform convergence of (1.3) and (1.4) from [3].

THEOREM 1.1. ([3, Theorem 3]) *Suppose that a function $f_1 : \mathbb{R}_+ \to \mathbb{C}$ belongs to Φ_1.*

(i) *If $f_1 \in GM(\beta)$ and $x\beta(x)\to 0$ as $x\to\infty$, or equivalently,*

$$ x\int_x^{2x}|df_1(t)| \to 0 \quad \text{as } x\to\infty, \tag{1.7} $$

then the integral (1.3) converges uniformly in $u\in\overline{\mathbb{R}}_+$ and

$$ \|F_1(1)-S_1(1,a)\|_\infty \leqslant C_s \max_{x\geqslant a/2}\left(x\int_x^{2x}|df_1(t)|\right). \tag{1.8} $$

(ii) *Let a non-negative function f_1 satisfy*

$$ f_1(x) \leqslant \frac{C}{x}\max_{a\geqslant x/c}\int_a^{2a} f_1(t)\,dt \quad \text{for some } c>1. \tag{1.9} $$

Then the uniform convergence of the integral (1.3) implies

$$ xf_1(x)\to 0 \quad \text{as } x\to\infty. \tag{1.10} $$

THEOREM 1.2. ([3, Theorem 2]) *Suppose that a function $f_2 : \mathbb{R}_+ \to \mathbb{C}$ belongs to Ψ.*

(i) *If $f_2 \geqslant 0$ or $f_2 \in GM(\beta)$, where $x\beta(x)\to 0$ as $x\to\infty$, or equivalently,*

$$ x\int_x^{2x}|df_2(t)| \to 0 \quad \text{as } x\to\infty, \tag{1.11} $$

and

$$ \int_0^\infty f_2(x)\,dx \ \text{converges}, \tag{1.12} $$

then the integral (1.4) *converges uniformly in* $u \in \overline{\mathbb{R}}_+$ *and*

$$\|F_2(1) - S_2(1,a)\|_\infty \leqslant \max_{x>a} \left| \int_a^x f_2(t)\,dt \right| + C_c \max_{x \geqslant a/2} \left(x \int_x^{2x} |df_2(t)| \right). \tag{1.13}$$

(ii) *If* $f_2 \geqslant 0$ *or* $f_2 \in GM(\beta)$, *where* $x\beta(x) \to 0$ *as* $x \to \infty$ *and the integral* (1.4) *converges uniformly in* $u \in \overline{\mathbb{R}}_+$, *then* (1.12) *holds.*

Using the above theorems, one can obtain an assumption for the uniform convergence of the integrals (1.3) and (1.4) for certain classes of general monotone functions, e.g. $GM(_2\beta)$.

THEOREM 1.3. ([9, Theorem 2.6]) *Assume* $f_1 : \mathbb{R}_+ \to \mathbb{C}$ *belongs to the class* $GM(_2\beta) \cap \Phi_1$.

(i) *If* (1.10) *holds, then the integral* (1.3) *converges uniformly in* $u \in \overline{\mathbb{R}}_+$.

(ii) *Conversely, if* $f_1 \geqslant 0$ *and* (1.3) *converges uniformly in* u, *then* (1.10) *is satisfied.*

THEOREM 1.4. ([10, Theorem 1.3]) *Assume* $f_2 : \mathbb{R}_+ \to \mathbb{C}$ *belongs to the class* $GM(_2\beta) \cap \Psi$.

(i) *If* (1.12) *and*

$$xf_2(x) \to 0 \quad \text{as } x \to \infty \tag{1.14}$$

hold, then the integral (1.4) *converges uniformly in* $u \in \overline{\mathbb{R}}_+$.

(ii) *Conversely, if* $f_2 \geqslant 0$ *and* (1.4) *converges uniformly in* u, *then* (1.12) *and* (1.14) *are satisfied.*

In [2], it is proved that the non-negativity assumption can be removed from Theorem 1.3 in case of the class $GM(_1\beta)$.

THEOREM 1.5. ([2, Theorem 3.3]) *Assume* $f_1 : \mathbb{R}_+ \to \mathbb{R}$ *belongs to the class* $GM(_1\beta) \cap \Phi_1$.*Then, a necessary and sufficient condition for the sine integral* (1.3) *to converge uniformly in* $u \in \overline{\mathbb{R}}_+$ *is* (1.10).

The proof of [2, Theorem 3.3] is omitted because combining the proofs of [2, Theorem 3.1] and [6, Theorem 3.1] gives the required result. By analyzing the proofs of these theorems, and noting

$$\cos \xi t \geqslant \frac{1}{\sqrt{2}} \quad \text{for} \quad \xi = \xi(x) = \frac{\pi}{16x},\ t \in [x, 4x]$$

regarding [2, Theorem 3.1] and

$$\cos \xi t \geqslant \frac{1}{\sqrt{2}} \quad \text{for} \quad \xi = \xi(x) = \frac{\pi}{4\lambda x},\ t \in \left[\frac{x}{\lambda}, \lambda x\right]$$

regarding [6, Theorem 3.1], we can deduce the following:

THEOREM 1.6. *Assume $f_2 : \mathbb{R}_+ \to \mathbb{R}$ belongs to class $GM({}_1\beta)\cap\Psi$. Then, the necessary and sufficient conditions for the cosine integral* (1.4) *to converge uniformly in $u \in \overline{\mathbb{R}}_+$ are* (1.12) *and* (1.14).

We remark that the investigation of the uniform convergence of trigonometric integrals (1.3)–(1.4) is motivated by and often studied together with the uniform convergence of trigonometric series. For more background, see for example papers [3, 4, 6, 18, 19]. The investigation of (1.1)–(1.2) is motivated by the recent papers of M. R. Gabdullin [7], S. Kęska [11], and K. A. Oganesyan [17], where the uniform convergence of series $\sum_{k=1}^{\infty} c_k \sin k^{\alpha}x$ and $\sum_{k=1}^{\infty} c_k \cos k^{\alpha}x$ has been considered (in [11], only the special case $\sum_{k=1}^{\infty} c_k \sin\sqrt{k}x$ is considered).

2. Results on uniform convergence

We extend Theorems 1.1 and 1.2 to the setting of the integrals (1.1) and (1.2).

THEOREM 2.1. *Suppose that a function $f_1 : \mathbb{R}_+ \to \mathbb{C}$ belongs to Φ_p with $p>0$. If $f_1 \in GM(\beta)$ and $x\beta(x)\to 0$ as $x\to\infty$, or equivalently* (1.7) *holds, then the integral* (1.1) *converges uniformly in $u\in\overline{\mathbb{R}}_+$ and*

$$\|F_1(p)-S_1(p,a)\|_\infty \leqslant C \max_{x\geqslant a/2^{1+1/p}} \left(x\int_x^{2x} |df_1(t)| \right). \tag{2.1}$$

THEOREM 2.2. *Suppose that a function $f_2 : \mathbb{R}_+ \to \mathbb{C}$ belongs to Ψ, and $p>0$. If $f_2\in GM(\beta)$,* (1.12) *and $x\beta(x)\to 0$ as $x\to\infty$, or equivalently* (1.11) *hold, then the integral* (1.2) *converges uniformly in $u\in\overline{\mathbb{R}}_+$ and*

$$\|F_2(p)-S_2(p,a)\|_\infty \leqslant \max_{x>a}\left|\int_a^x f_2(t)\,dt\right| + C \max_{x\geqslant a/2^{1+1/p}} \left(x\int_x^{2x} |df_2(t)| \right). \tag{2.2}$$

Now, we formulate criteria for the uniform convergence of the integrals (1.1) and (1.2) for general monotone functions from $GM({}_2\beta)$.

THEOREM 2.3. *Assume $f_1\in GM({}_2\beta)\cap\Phi_p$ with $p>0$.*

(i) *If $f_1 : \mathbb{R}_+\to\mathbb{C}$ and* (1.10) *holds, then integral* (1.1) *converges uniformly in $u\in\overline{\mathbb{R}}_+$.*

(ii) *Let $f_1 : \mathbb{R}_+\to\mathbb{R}$ and suppose that $I_s(x) := \int_x^{2x} |f_1(t)|\,dt$ is bounded at infinity. Then the uniform convergence of the integral* (1.1) *implies* (1.10).

THEOREM 2.4. *Assume $f_2\in GM({}_2\beta)\cap\Psi$ and $p>0$.*

(i) *If $f_2 : \mathbb{R}_+\to\mathbb{C}$,* (1.12) *and* (1.14) *hold, then integral* (1.2) *converges uniformly in $u\in\overline{\mathbb{R}}_+$.*

(ii) *Let* $f_2 : \mathbb{R}_+ \to \mathbb{R}$ *and suppose that* $I_c(x) := \int_x^{2x} |f_2(t)|\,dt$ *is bounded at infinity. Then the uniform convergence of the integral* (1.2) *implies* (1.12) *and* (1.14).

Using Lemma 3.2 we can show, similarly as in [2, Remark 4.2], that the hypothesis of $I_s(x)$ and $I_c(x)$ being bounded is not needed if we assume that f_1 or f_2 are non-negative, respectively.

Similarly, as in [2, Theorem 3.3] we can formulate criteria for the uniform convergence of the integrals (1.1) and (1.2) for the class $GM({}_1\beta)$.

THEOREM 2.5. *Assume* $f_1 : \mathbb{R}_+ \to \mathbb{R}$ *belongs to the class* $GM({}_1\beta) \cap \Phi_p$ *with* $p > 0$. *Then, a necessary and sufficient condition for the sine integral* (1.1) *to converge uniformly in* $u \in \overline{\mathbb{R}}_+$ *is* (1.10).

THEOREM 2.6. *Assume* $f_2 : \mathbb{R}_+ \to \mathbb{R}$ *belongs to the class* $GM({}_1\beta) \cap \Psi$ *and* $p > 0$. *Then, the necessary and sufficient conditions for the cosine integral* (1.2) *to converge uniformly in* $u \in \overline{\mathbb{R}}_+$ *are* (1.12) *and* (1.14).

We show some examples of general monotone functions which can be considered in the main theorems to favor reading. In every case, except for the last one, $p > 0$ is arbitrary.

EXAMPLE 2.1. Let $f_\alpha(x) = (1+x)^{-\alpha}$ for an $\alpha > 0$. Then f_α is decreasing, $f_\alpha \in M \cap \Psi$, and by Theorems 2.3 and 2.4 (or, Theorems 2.5 and 2.6), both $\int_0^\infty f_\alpha(x)\sin(ux^p)\,dx$ and $\int_0^\infty f_\alpha(x)\cos(ux^p)\,dx$ are uniformly convergent in u for $\alpha > 1$ but not for $0 < \alpha \leqslant 1$.

EXAMPLE 2.2. Let $g_\alpha(x) = (1+x)^{-\alpha}\sin x$ for an $\alpha \in (0,1) \cup (2,\infty)$. Then g_α is not decreasing, however, $g_\alpha \in GM({}_2\beta) \cap \Psi$. Indeed, there exist positive constants C, C' and A such that

$$\int_x^{2x} |dg_\alpha(t)| \leqslant \int_x^{2x} \left|\frac{\cos t}{(1+t)^\alpha}\right| dt + \int_x^{2x} \left|\frac{\alpha \sin t}{(1+t)^{\alpha+1}}\right| dt \leqslant \frac{(\alpha+1)x}{(1+x)^\alpha}$$

$$\leqslant C \frac{x^{1-\frac{2-\alpha}{1-\alpha}}\left(1+2x^{\frac{2-\alpha}{1-\alpha}}\right)^\alpha}{(1+x)^\alpha} \int_{x^{\frac{2-\alpha}{1-\alpha}}}^{2\cdot x^{\frac{2-\alpha}{1-\alpha}}} \left|\frac{\sin t}{(1+t)^\alpha}\right| dt$$

$$\leqslant \frac{C'}{x} \int_{x^{\frac{2-\alpha}{1-\alpha}}}^{2\cdot x^{\frac{2-\alpha}{1-\alpha}}} \left|\frac{\sin t}{(1+t)^\alpha}\right| dt \leqslant \frac{C'}{x} \sup_{a \geqslant x^{\frac{2-\alpha}{1-\alpha}}} \int_a^{2a} \left|\frac{\sin t}{(1+t)^\alpha}\right| dt$$

for $x > A$. Due to Theorems 2.3 and 2.4, both $\int_0^\infty g_\alpha(x)\sin(ux^p)\,dx$ and $\int_0^\infty g_\alpha(x)\cos(ux^p)\,dx$ are uniformly convergent in u for $\alpha > 2$ but not for $0 < \alpha < 1$.

EXAMPLE 2.3. Let $h_\alpha(x) = e^{-\alpha x}$ for an $\alpha > 0$. Then $h_\alpha \in M \cap \Psi$, and both $\int_0^\infty h_\alpha(x)\sin(ux^p)\,dx$ and $\int_0^\infty h_\alpha(x)\cos(ux^p)\,dx$ are uniformly convergent.

EXAMPLE 2.4. For $i(x) = ((1+x)\ln(2+x))^{-1}$, $i \in M \cap \Psi$, and Theorem 2.3 (or, 2.5) implies the uniform convergence of $\int_0^\infty i(x)\sin(ux^p)\,dx$ in u while $\int_0^\infty i(x)\cos(ux^p)\,dx$ is not uniformly convergent in u since (1.12) is not satisfied for i.

EXAMPLE 2.5. Consider $j(x) = x^{-2}$ and $p > 1$. Then $j \in M \cap \Phi_p$. Integrals $\int_0^\infty j(x)\cos(ux^p)\,dx$, $\int_0^\infty j(x)\cos(ux)\,dx$ and $\int_0^\infty j(x)\sin(ux)\,dx$ do not converge, since $j \notin \Psi$ and $j \notin \Phi_1$, respectively. However, $\int_0^\infty j(x)\sin(ux^p)\,dx$ converges uniformly in u due to Theorem 2.3 (or, 2.5). Moreover,

$$\begin{aligned}
J_1(p,u) &= \int_0^\infty \frac{1}{x^2}\sin(ux^p)\,dx = \frac{1}{p}\int_0^\infty \frac{px^{p-1}}{x^{p+1}}\sin(ux^p)\,dx \\
&= \frac{1}{p}\int_0^\infty \frac{1}{x^{1+1/p}}\sin(ux)\,dx = \frac{1}{p}\Gamma\left(-\frac{1}{p}\right)\sin\frac{-\pi}{2p}\,u^{1/p} \\
&= \Gamma\left(1-\frac{1}{p}\right)\sin\frac{\pi}{2p}\,u^{1/p}
\end{aligned}$$

since

$$\Gamma(z) = \frac{u^z}{\sin\frac{\pi z}{2}}\int_0^\infty x^{z-1}\sin(ux)\,dx, \qquad u \in \mathbb{R}_+,\ 0 < \mathrm{Re}(z) < 1,$$

(see [8, p. 893]), that can be extended for $-1 < \mathrm{Re}(z) < 0$ using integration by parts and

$$\Gamma(z) = \frac{u^z}{\cos\frac{\pi z}{2}}\int_0^\infty x^{z-1}\cos(ux)\,dx, \qquad u \in \mathbb{R}_+,\ 0 < \mathrm{Re}(z) < 1,$$

(also see [8, p. 893]).

3. Proofs of Theorems 2.1–2.6

First, we show an important property of class $GM(\beta)$.

LEMMA 3.1. *If $f(x) \in GM(\beta)$ with some $\beta(x)$, where $x\beta(x) \to 0$ $(x \to \infty)$, then $g(x) = x^{1/p-1}f(x^{1/p}) \in GM(\beta')$ for any $p > 0$ with $\beta'(x) = \frac{1}{x}\max_{t \geqslant x^{1/p}/2}(t\beta(t))$ and $x\beta'(x) \to 0$ $(x \to \infty)$.*

Proof. Calculations give us

$$\begin{aligned}
\int_x^{2x} |dg(t)| &= \int_x^{2x} \left| d\left(t^{1/p-1} f(t^{1/p})\right)\right| \\
&\leqslant \left|\frac{1}{p}-1\right| \int_x^{2x} t^{1/p-2}\left|f(t^{1/p})\right| dt + \int_x^{2x} t^{1/p-1}\left|df(t^{1/p})\right| \\
&\leqslant \left|\frac{1}{p}-1\right| \frac{1}{x}\int_x^{2x} t^{1/p-1}\left|f(t^{1/p})\right| dt + C_0\, x^{1/p-1}\int_x^{2x} \left|df(t^{1/p})\right| \\
&\leqslant |1-p| \frac{1}{x}\int_{x^{1/p}}^{(2x)^{1/p}} |f(s)|\, ds + C_0 x^{1/p-1} \int_{x^{1/p}}^{(2x)^{1/p}} |df(s)|
\end{aligned}$$

where $C_0 = \max\{1, 2^{1/p-1}\}$. Since for any $f(x) \in GM(\beta)$, we have

$$\begin{aligned}
|f(x)| &\leqslant \int_x^{\infty} |df(t)| \leqslant \frac{1}{\ln 2}\int_{x/2}^{\infty} \frac{1}{t^2}\left(t\int_t^{2t} |df(s)|\right) \\
&\leqslant \frac{1}{\ln 2}\,\frac{1}{x}\max_{t\geqslant x/2}\left(t\int_t^{2t} |df(s)|\right) \leqslant \frac{C}{\ln 2}\,\frac{1}{x}\max_{t\geqslant x/2}\left(t\beta(t)\right),
\end{aligned}$$

(see [3]), we can obtain

$$\begin{aligned}
\int_x^{2x} |dg(t)| &\leqslant \frac{C|1-p|}{\ln 2}\,\frac{1}{x}\int_{x^{1/p}}^{(2x)^{1/p}} \frac{1}{s}\max_{t\geqslant s/2}\left(t\beta(t)\right) ds + C_0 \frac{1}{x}\sum_{k=0}^{n_0-1} x^{1/p}\int_{2^k x^{1/p}}^{2^{k+1}x^{1/p}} |df(s)| \\
&\leqslant \frac{C|1-p|}{\ln 2}\,\frac{1}{x}\left(2^{1/p}-1\right)\max_{t\geqslant x^{1/p}/2}\left(t\beta(t)\right) + C_0\frac{1}{x}\sum_{k=0}^{n_0-1}\frac{2^k x^{1/p}}{2^k}\beta\left(2^k x^{1/p}\right) \\
&\leqslant \left(\frac{C|1-p|}{\ln 2}\left(2^{1/p}-1\right) + C_0 n_0\right)\frac{1}{x}\max_{t\geqslant x^{1/p}/2}\left(t\beta(t)\right) \qquad (3.1)
\end{aligned}$$

where n_0 is the integer so that $2^{n_0-1} \leqslant 2^{1/p} \leqslant 2^{n_0}$. This means $g(x) \in GM(\beta')$ with

$$\beta'(x) = \frac{1}{x}\max_{t\geqslant x^{1/p}/2}\left(t\beta(t)\right).$$

Therefore, $x\beta'(x) \to 0\ (x\to\infty)$. $\square$

Proof of Theorem 2.1. Suppose $f_1 \in GM(\beta)\cap\Phi_p$. Then $g_1(x) = \frac{1}{p}x^{1/p-1}f_1(x^{1/p}) \in GM(\beta')\cap\Phi_1$. Using (1.5) and Lemma 3.1 we get from Theorem 1.1 the uniform convergence of (1.1). Moreover, we can obtain (2.1) from (1.8) keeping in mind (3.1):

$$
\begin{aligned}
&\|F_1(f_1(x);p)-S_1(f_1(x);p,a)\|_\infty \\
&\quad=\|F_1(g_1(x);1)-S_1(g_1(x);1,a^p)\|_\infty \leqslant C_s \max_{x\geqslant a^p/2}\left(x\int_x^{2x}|dg_1(t)|\right)\\
&\quad\leqslant C_s \max_{x\geqslant a^p/2}\left(\left(\frac{C|1-p|}{p\ln 2}(2^{1/p}-1)+\frac{C_0n_0}{p}\right)\max_{t\geqslant x^{1/p}/2}\left(t\int_t^{2t}|df_1(s)|\right)\right)\\
&\quad= C'\max_{t\geqslant a/2^{1+1/p}}\left(t\int_t^{2t}|df_1(s)|\right),
\end{aligned}
\tag{3.2}
$$

where $C'=C_s\left(\frac{C|1-p|}{p\ln 2}(2^{1/p}-1)+\frac{C_0n_0}{p}\right)$.

This completes the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2. Suppose $f_2\in GM(\beta)\cap\Psi$. Since $g_2(x)=\frac{1}{p}x^{1/p-1}f_2(x^{1/p})$ $\in GM(\beta')\cap\Psi$, using (1.6) and Lemma 3.1, we get from Theorem 1.2 the uniform convergence of (1.2). Moreover, we can obtain (2.2) from (1.13) using (3.1) and (3.2):

$$
\begin{aligned}
&\|F_2(f_2(x);p)-S_2(f_2(x);p,a)\|_\infty\\
&\quad=\|F_2(g_2(x);1)-S_2(g_2(x);1,a^p)\|_\infty\\
&\quad\leqslant \max_{x>a^p}\left|\int_{a^p}^{x}g_2(t)\,dt\right|+C_c\max_{x\geqslant a^p/2}\left(x\int_x^{2x}|dg_2(t)|\right)\\
&\quad\leqslant \max_{t>a}\left|\int_a^t f_2(s)\,ds\right|+C''\max_{t\geqslant a/2^{1+1/p}}\left(t\int_t^{2t}|df_2(s)|\right),
\end{aligned}
$$

where $C''=C_c\left(\frac{C|1-p|}{p\ln 2}(2^{1/p}-1)+\frac{C_0n_0}{p}\right)$.

The proof of Theorem 2.2 is complete. $\square$

LEMMA 3.2. *If* $f(x)\in GM({}_2\beta)$, *then* $g(x)=x^{1/p-1}f(x^{1/p})\in GM({}_2\beta)$, *for any* $p>0$.

Proof. By Lemma 3.1, we have that $g(x)\in GM(\beta')$ with

$$
\begin{aligned}
\beta'(x)&=\frac{1}{x}\max_{t\geqslant x^{1/p}/2} t\,{}_2\beta(t)=\frac{1}{x}\max_{t\geqslant x^{1/p}/2}\sup_{a\geqslant B(t)}\int_a^{2a}|f(s)|\,ds\\
&\leqslant\frac{1}{x}\sup_{a\geqslant B(x^{1/p}/2)}\int_a^{2a}|f(s)|\,ds=\frac{p}{x}\sup_{a\geqslant B(x^{1/p}/2)}\int_{a^p}^{(2a)^p}t^{1/p-1}\left|f(t^{1/p})\right|dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{p}{x} \sup_{a \geqslant B(x^{1/p}/2)} \sum_{k=0}^{n_0-1} \int_{2^k a^p}^{2^{k+1} a^p} |g(t)|\, dt \leqslant \frac{p n_0}{x} \sup_{a \geqslant B(x^{1/p}/2)} \sup_{b \geqslant a^p} \int_b^{2b} |g(t)|\, dt \\
&\leqslant \frac{p n_0}{x} \sup_{b \geqslant (B(x^{1/p}/2))^p} \int_b^{2b} |g(t)|\, dt
\end{aligned}
$$

where n_0 is the integer so that $2^{n_0-1} \leqslant 2^p \leqslant 2^{n_0}$. This means $g(x) \in GM({}_2\beta)$. $\square$

Proof of Theorem 2.3. Part (i). Using Theorem 2.1 we obtain the uniform convergence of (1.1).

Part (ii). Suppose that $f_1 : \mathbb{R}_+ \to \mathbb{R}$, $f_1 \in GM({}_2\beta)$, $I_s(x)$ is bounded at infinity, $p > 0$ and (1.1) converges uniformly in u. We start with setting

$$
\xi = \xi(x) = \frac{\pi}{2(4x)^p}, \quad x \in \mathbb{R}_+.
$$

Then for $x \leqslant t \leqslant 4x$, we have

$$
\frac{\pi}{2^{2p+1}} \leqslant \xi(x) t^p \leqslant \frac{\pi}{2},
$$

therefore

$$
\sin(\xi(x) t^p) \geqslant \sin \frac{\pi}{2^{2p+1}}.
$$

Using this inequality, we can show, analogously as in the proof of [2, Theorem 3.1], that $I_s(x)^3 \to 0$ as $x \to \infty$. This completes the proof of part (ii) and that of Theorem 2.3. $\square$

Proof of Theorem 2.4. Part (i). Using Theorem 2.2 we obtain the uniform convergence of (1.2).

Part (ii). Suppose that $f_2 : \mathbb{R}_+ \to \mathbb{R}$, $f_2 \in GM({}_2\beta)$, $I_c(x)$ is bounded at infinity, $p > 0$ and (1.2) converges uniformly in u. Taking $u = 0$, we immediately get (1.12) since $\cos(ux^p) \equiv 1$. Moreover, for

$$
\xi = \xi(x) = \frac{\pi}{4(4x)^p}, \quad x \in \mathbb{R}_+.
$$

and $x \leqslant t \leqslant 4x$, we have

$$
\frac{\pi}{2^{2p+2}} \leqslant \xi(x) t^p \leqslant \frac{\pi}{4}.
$$

Therefore

$$
\cos(\xi(x) t^p) \geqslant \frac{1}{\sqrt{2}}.
$$

Using this inequality, we can show, analogously as in the proof of [2, Theorem 3.1], that $I_c(x)^3 \to 0$ as $x \to \infty$. The proof of part (ii) and that of Theorem 2.4. is complete. $\square$

LEMMA 3.3. *If* $f(x) \in GM({}_1\beta)$*, then* $g(x) = x^{1/p-1} f(x^{1/p}) \in GM({}_1\beta)$*, for any* $p > 0$.

Proof. The proof is similar to that of Lemma 3.2. □

Proof of Theorem 2.5. Lemma 3.3 and Theorem 1.5 yield the required result. □

Proof of Theorem 2.6. Lemma 3.3 and Theorem 1.6 yield the required result. □

4. Weighted integrability

We consider the weighted L^q-convergence, $q \geqslant 1$, of the sine and cosine integrals (1.1)–(1.2) with power weights $x^{-\gamma}$ for certain $\gamma \in \mathbb{R}$.

We recall that the weighted L^q-convergence of sine and cosine integrals (1.3)–(1.4) were considered by various authors, see e.g. [1, 3, 13, 20].

DEFINITION 4.1. ([3]) Suppose that $f : \mathbb{R}_+ \to \mathbb{C}$ is a function locally of bounded variation on $\mathbb{R}_+$. We say that $f \in \overline{GM}_\theta$,with $\theta \in (0,1]$, if there exist positive constants C, A and $c > 1$ such that

$$\int_x^\infty |df(t)| \leqslant C x^{\theta-1} \int_{x/c}^\infty \frac{|f(t)|}{t^\theta}\, dt < \infty$$

for all $x > A$.

It is known that $M \subsetneq \overline{GM}_1 \subseteq \overline{GM}_{\theta_2} \subseteq \overline{GM}_{\theta_1}$ for $0 < \theta_1 \leqslant \theta_2 \leqslant 1$, where M denotes the class of non-negative, monotone non-increasing functions (see [5] and [14, 15]).

THEOREM 4.1. ([3]) *Let* $f_1 : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ *belong to* $\overline{GM}_\theta \cap \Phi_1$*, where* $\theta \in (0,1]$*. If* $q \geqslant 1$ *and* $1 - \theta q < \gamma < 1 + q$*, then*

$$\frac{|F_1(f_1(x);1,u)|^q}{u^\gamma} \in L(\mathbb{R}_+) \Leftrightarrow \int_0^\infty x^{\gamma+q-2} f_1^q(x)\, dx < \infty.$$

THEOREM 4.2. ([3]) *Let* $f_2 : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ *belong to class* $\overline{GM}_\theta \cap \Psi$*, where* $\theta \in (0,1]$*. If* $q \geqslant 1$ *and* $1 - \theta q < \gamma < 1$*, then*

$$\frac{|F_2(f_2(x);1,u)|^q}{u^\gamma} \in L(\mathbb{R}_+) \Leftrightarrow \int_0^\infty x^{\gamma+q-2} f_2^q(x)\, dx < \infty.$$

5. Results on weighted integrability

We extend Theorems 4.1 and 4.2 to the integrals (1.1) and (1.2).

THEOREM 5.1. *Let* $p>0$ *and a function* $f_1:\mathbb{R}_+\to\overline{\mathbb{R}}_+$ *belong to* $\overline{GM}_{p\theta}\cap\Phi_p$, *where* $p\theta\in(0,1]$ *and* $\theta\in(0,1)$. *If* $q\geqslant 1$ *and* $1-\theta q<\gamma<1+q$, *then*

$$\frac{|F_1(f_1(x);p,u)|^q}{u^\gamma}\in L(\mathbb{R}_+)\Leftrightarrow\int_0^\infty u^{p\gamma+q-p-1}f_1^q(u)\,du<\infty. \tag{5.1}$$

THEOREM 5.2. *Let* $p>0$ *and a function* $f_2:\mathbb{R}_+\to\overline{\mathbb{R}}_+$ *belong to* $\overline{GM}_{p\theta}\cap\Psi$, *where* $p\theta\in(0,1]$ *and* $\theta\in(0,1)$. *If* $q\geqslant 1$ *and* $1-\theta q<\gamma<1$, *then*

$$\frac{|F_2(f_2(x);p,u)|^q}{u^\gamma}\in L(\mathbb{R}_+)\Leftrightarrow\int_0^\infty u^{p\gamma+q-p-1}f_2^q(u)\,du<\infty. \tag{5.2}$$

It is clear that Theorems 5.1 and 5.2 do not work when $\theta=1$. As Lemma 6.2 shows, in order to prove (5.1) and (5.2), the assumptions of these theorems must be changed. Similarly as in the proofs of Theorems 5.1 and 5.2, we can show the following theorem:

THEOREM 5.3. *Suppose* $0<p\leqslant 1$.

(i) *Let a function* $f_1:\mathbb{R}_+\to\overline{\mathbb{R}}_+$ *be such that* $x^{1/p-1}f_1(x^{1/p})\in\overline{GM}_1\cap\Phi_p$. *If* $q\geqslant 1$ *and* $1-q<\gamma<1+q$, *then* (5.1) *holds.*

(ii) *Let a function* $f_2:\mathbb{R}_+\to\overline{\mathbb{R}}_+$ *be such that* $x^{1/p-1}f_2(x^{1/p})\in\overline{GM}_1\cap\Psi$. *If* $q\geqslant 1$ *and* $1-q<\gamma<1$, *then* (5.2) *holds.*

We show some examples of general monotone functions for which the main integrability theorems can be applied. We suppose that $p>0$, $q\geqslant 1$.

EXAMPLE 5.1. Let $f_\alpha(x)=e^{-\alpha x}$ for an $\alpha>0$. It is clear that $f_\alpha\in M\cap\Psi$. Applying Theorems 5.1–5.3, we get

$$\frac{|F_1(f_\alpha(x);p,u)|^q}{u^\gamma}\in L(\mathbb{R}_+)$$

for $1-\frac{q}{p}<\gamma<1+q$, $p>1$ or $1-q<\gamma<1+q$, $p\leqslant 1$ and

$$\frac{|F_2(f_\alpha(x);p,u)|^q}{u^\gamma}\in L(\mathbb{R}_+)$$

for $1-\frac{q}{p}<\gamma<1$, $p>1$ or $1-q<\gamma<1$, $p\leqslant 1$.

EXAMPLE 5.2. Let

$$g_\alpha(x)=\begin{cases}0, & \text{if } x\in[0,1)\\ x^{-\alpha}, & \text{if } x\in[2^{2k},2^{2k+1}),\ \ k=0,1,2,\dots\end{cases}$$

for $\alpha>0$. One can see that for any $x\in[2^{2r},2^{2r+2})$, $r\in\mathbb{Z}_0^+$,

$$\int_x^\infty |dg_\alpha(t)|\leqslant C_1\sum_{k=r}^\infty\left(2^{2k}\right)^{-\alpha}=C_1\left(2^{2r}\right)^{-\alpha}\frac{1}{1-2^{-\alpha}}$$

$$\leqslant\frac{C_1\alpha}{(1-2^{-\alpha})(2^{-2\alpha}-2^{-3\alpha})}\int_x^\infty\frac{|g_\alpha(t)|}{t}\,dt,$$

which means $g_\alpha\in\overline{GM}_1\cap\Psi$. By Theorems 5.1–5.3,

$$\frac{|F_1(g_\alpha(x);p,u)|^q}{u^\gamma}\in L(\mathbb{R}_+)\Leftrightarrow\int_0^\infty x^{p\gamma+q-p-1}x^{\alpha q}\,dx<\infty\Leftrightarrow\gamma<1-\frac{q}{p}+\frac{\alpha q}{p}$$

for $1-\frac{q}{p}<\gamma<1+q$, $p>1$ or $1-q<\gamma<1+q$, $p\leqslant1$ and

$$\frac{|F_2(g_\alpha(x);p,u)|^q}{u^\gamma}\in L(\mathbb{R}_+)\Leftrightarrow\gamma<1-\frac{q}{p}+\frac{\alpha q}{p}$$

for $1-\frac{q}{p}<\gamma<1$, $p>1$ or $1-q<\gamma<1$, $p\leqslant1$.

6. Proofs of Theorems 5.1–5.2

We show the following property of the classes $\overline{GM}_{p\theta}$.

LEMMA 6.1. *Let $p>0$ and $f(x)\in\overline{GM}_{p\theta}$, where $p\theta\in(0,1]$ and $\theta\in(0,1)$. Then $g(x)=x^{1/p-1}f(x^{1/p})\in\overline{GM}_\theta$.*

Proof. Elementary calculations give us

$$\int_x^\infty|dg(t)|=\int_x^\infty\left|d\left(t^{1/p-1}f(t^{1/p})\right)\right|$$

$$\leqslant\left|\frac{1}{p}-1\right|\int_x^\infty t^{1/p-2}\left|f(t^{1/p})\right|dt+\int_x^\infty t^{1/p-1}\left|df(t^{1/p})\right|$$

$$=\left|\frac{1}{p}-1\right|\int_x^\infty\frac{t^{1/p-1}\left|f(t^{1/p})\right|}{t^{1-\theta+\theta}}\,dt+\int_x^\infty t^{1/p-1}\left|df(t^{1/p})\right|$$

$$\leqslant\left|\frac{1}{p}-1\right|x^{\theta-1}\int_x^\infty\frac{t^{1/p-1}\left|f(t^{1/p})\right|}{t^\theta}\,dt+I. \tag{6.1}$$

Taking $C_1 = \max\left\{1, 2^{1/p-1}\right\}$, we have

$$
\begin{aligned}
I &= \sum_{k=0}^{\infty} \int_{2^k x}^{2^{k+1}x} t^{1/p-1} \left| df(t^{1/p}) \right| \\
&\leqslant C_1 \sum_{k=0}^{\infty} \left(2^k x\right)^{1/p-1} \int_{2^k x}^{2^{k+1}x} \left| df(t^{1/p}) \right| \\
&= C_1 \sum_{k=0}^{\infty} \left(2^k x\right)^{1/p-1} \int_{\left(2^k x\right)^{1/p}}^{\left(2^{k+1}x\right)^{1/p}} |df(z)| \\
&\leqslant C_1 C \sum_{k=0}^{\infty} \left(2^k x\right)^{1/p-1} \left(\left(2^k x\right)^{1/p}\right)^{p\theta-1} \int_{\left(2^k x\right)^{1/p}/c}^{\infty} \frac{|f(z)|}{z^{p\theta}}\, dz \\
&\leqslant C_1 C \sum_{k=0}^{\infty} \left(2^k x\right)^{\theta-1} \int_{\left(2^k x\right)^{1/p}/c}^{\infty} \frac{|f(z)|}{z^{p\theta}}\, dz \\
&\leqslant C_1 C x^{\theta-1} \int_{x^{1/p}/c}^{\infty} \frac{|f(z)|}{z^{p\theta}}\, dz \sum_{k=0}^{\infty} \left(2^{\theta-1}\right)^k \\
&= \frac{C_1 C}{p(1-2^{\theta-1})} x^{\theta-1} \int_{x/c^p}^{\infty} \frac{t^{1/p-1}\left|f(t^{1/p})\right|}{t^{\theta}}\, dt.
\end{aligned} \tag{6.2}
$$

Using (6.1) and (6.2), we get

$$
\int_{x}^{\infty} |dg(t)| \leqslant C' x^{\theta-1} \int_{x/c'}^{\infty} \frac{|g(t)|}{t^{\theta}}\, dt,
$$

where $C' = \left|\frac{1}{p} - 1\right| + \frac{C_1 C}{p(1-2^{\theta-1})}$ and $c' = c^p$. Thus $g(x) \in \overline{GM}_{\theta}$. $\square$

LEMMA 6.2. *If* $g(x) = x^{1/p-1} f(x^{1/p}) \in \overline{GM}_1$, *where* $p \in (0,1]$, *then* $f(x) \in \overline{GM}_p$.

Proof. Let $g(x) = x^{1/p-1} f(x^{1/p}) \in \overline{GM}_1$, where $p \in (0,1]$. Then

$$
\begin{aligned}
\int_{x}^{\infty} |df(t)| &= \int_{x^p}^{\infty} |df(u^{1/p})| = \int_{x^p}^{\infty} \left| d\left(p u^{1-1/p} \tfrac{1}{p} u^{1/p-1} f(u^{1/p})\right) \right| \\
&\leqslant \left(\frac{1}{p} - 1\right) \int_{x^p}^{\infty} \frac{\left|f(u^{1/p})\right|}{u}\, du + p \int_{x^p}^{\infty} u^{1-1/p} \left| d\left(\tfrac{1}{p} u^{1/p-1} f(u^{1/p})\right) \right|
\end{aligned}
$$

$$
\begin{aligned}
&= (1-p)\int_x^\infty \frac{|f(z)|}{z}\,dz + px^{p-1}\int_{x^p}^\infty \left|d\left(\tfrac{1}{p}u^{1/p-1}f(u^{1/p})\right)\right| \\
&\leqslant (1-p)x^{p-1}\int_x^\infty \frac{|f(z)|}{z^p}\,dz + Cpx^{p-1}\int_{x^p/c}^\infty \frac{\frac{1}{p}u^{1/p-1}\left|f(u^{1/p})\right|}{u}\,du \\
&= (1-p)x^{p-1}\int_x^\infty \frac{|f(z)|}{z^p}\,dz + Cpx^{p-1}\int_{x/c^{1/p}}^\infty \frac{|f(z)|}{z^p}\,dz \\
&\leqslant C''x^{p-1}\int_{x/c''}^\infty \frac{|f(z)|}{z^p}\,dz,
\end{aligned}
$$

where $C'' = (1-p)+Cp$ and $c'' = c^{1/p}$. Thus $f(x) \in \overline{GM}_p$. $\square$

Proof of Theorem 5.1. Let $p>0$ and a function $f_1 \in \overline{GM}_{p\theta} \cap \Phi_p$, where $p\theta \in (0,1]$ and $\theta \in (0,1)$. By (1.5),

$$
\frac{|F_1(f_1(x);p,u)|^q}{u^\gamma} \in L(\mathbb{R}_+) \Leftrightarrow \frac{\left|F_1\left(\frac{1}{p}x^{1/p-1}f_1(x^{1/p});1,u\right)\right|^q}{u^\gamma} \in L(\mathbb{R}_+).
$$

Using Lemma 6.1 we get that $x^{1/p-1}f_1(x^{1/p}) \in \overline{GM}_\theta$. Thus by Theorem 4.1 with $q \geqslant 1$ and $1-\theta q < \gamma < 1+q$

$$
\begin{aligned}
\frac{\left|F_1\left(\frac{1}{p}x^{1/p-1}f_1(x^{1/p});1,u\right)\right|^q}{u^\gamma} &\Leftrightarrow \int_0^\infty x^{\gamma+q-2}\left(\frac{1}{p}x^{1/p-1}f_1(x^{1/p})\right)^q dx < \infty \\
&\Leftrightarrow \int_0^\infty x^{p\gamma+q-p-1}f_1^q(x)\,dx < \infty.
\end{aligned}
$$

This ends our proof. $\square$

Proof of Theorem 5.2. Let $p>0$ and a function $f_2 \in \overline{GM}_{p\theta} \cap \Psi$, where $p\theta \in (0,1]$ and $\theta \in (0,1)$. Using (1.6), Lemma 6.1 and Theorem 4.2 with $q \geqslant 1$ and $1-\theta q < \gamma < 1$ we obtain

$$
\begin{aligned}
\frac{|F_2(f_2(x);p,u)|^q}{u^\gamma} \in L(\mathbb{R}_+) &\Leftrightarrow \frac{\left|F_2\left(\frac{1}{p}x^{1/p-1}f_2(x^{1/p});1,u\right)\right|^q}{u^\gamma} \in L(\mathbb{R}_+) \\
&\Leftrightarrow \int_0^\infty x^{\gamma+q-2}\left(\frac{1}{p}x^{1/p-1}f_2(x^{1/p})\right)^q dx < \infty \\
&\Leftrightarrow \int_0^\infty x^{p\gamma+q-p-1}f_2^q(x)\,dx < \infty
\end{aligned}
$$

and our proof is completed. $\square$

7. Conclusions

In Theorems 2.1 and 2.2, we gave sufficient conditions for the uniform convergence of the integrals (1.1) and (1.2) with certain general monotone functions f_1 and f_2 while in Theorems 2.3 and 2.4, we gave necessary and sufficient conditions for the uniform convergence regarding $GM(_2\beta)$ functions. In addition in Theorems 2.5 and 2.6, we saw that the previously given conditions are necessary and sufficient in case of not necessarily non-negative $GM(_1\beta)$ functions. From these theorems, we conclude the following.

THEOREM 7.1. *Suppose that a function* $f:\mathbb{R}_+\to\mathbb{C}$ *belongs to* Ψ. *If* $f\in GM(\beta)$, *the integral* $\int_0^\infty f(t)\,dt$ *is convergent and* $x\beta(x)\to 0$ *as* $x\to\infty$, *or equivalently*

$$x\int_x^{2x}|df(t)|\to 0 \quad as \;\; x\to\infty,$$

then the integral

$$F(p,u):=F(f(x);p,u)=\int_0^\infty f(x)e^{iux^p}\,dx$$

converges uniformly in $u\in\overline{\mathbb{R}}_+$ *and*

$$\|F(p)-S(p,a)\|_\infty\leqslant\max_{x>a}\left|\int_a^x f(t)\,dt\right|+C\max_{x\geqslant a/2^{1+1/p}}\left(x\int_x^{2x}|df(t)|\right)$$

where

$$S(p,a,u):=S(f(x);p,a,u)=\int_0^a f(x)e^{iux^p}\,dx.$$

THEOREM 7.2. *Assume* $f\in GM(_2\beta)\cap\Psi$ *and* $p>0$.

(i) *If* $f:\mathbb{R}_+\to\mathbb{C}$, (7.1) *and* (7.2) *hold, then the integral* $F(p,u)$ *converges uniformly in* $u\in\overline{\mathbb{R}}_+$.

(ii) *Let* $f:\mathbb{R}_+\to\mathbb{R}$ *and suppose that* $I(x):=\int_x^{2x}|f(t)|\,dt$ *is bounded at infinity. Then the uniform convergence of the integral* $F(p,u)$ *implies*

$$xf(x)\to 0 \quad as \;\; x\to\infty, \tag{7.1}$$

and that

$$\int_0^\infty f(x)\,dx \;\; converges. \tag{7.2}$$

THEOREM 7.3. *Assume $f:\mathbb{R}_+\to\mathbb{R}$ belongs to the class $GM({}_1\beta)\cap\Psi$ and $p>0$. Then, the necessary and sufficient conditions for the integral $F(p,u)$ to converge uniformly in $u\in\overline{\mathbb{R}}_+$ are* (7.1) *and* (7.2)*.*

In Theorems 5.1 and 5.2, we presented sufficient and necessary conditions for the weighted L^q-convergence of the integrals (1.1) and (1.2) with certain non-negative functions f_1 and f_2 belonging to the class $\overline{GM}_{p\theta}$, where $p\theta\in(0,1]$ and $\theta\in(0,1)$. In Theorem 5.3, we gave a necessary and sufficient condition for the weighted L^q-convergence of the integrals (1.1) and (1.2) when $p\in(0,1]$ and $x^{1/p-1}f_1(x^{1/p})\in\overline{GM}_1$ or $x^{1/p-1}f_2(x^{1/p})\in\overline{GM}_1$, respectively. From these theorems, we conclude the following.

THEOREM 7.4. *Let $p>0$ and a function $f:\mathbb{R}_+\to\overline{\mathbb{R}}_+$ belong to $\overline{GM}_{p\theta}\cap\Psi$, where $p\theta\in(0,1]$ and $\theta\in(0,1)$. If $q\geqslant 1$ and $1-\theta q<\gamma<1$, then*

$$\frac{|F(f(x);p,u)|^q}{u^\gamma}\in L(\mathbb{R}_+)\Leftrightarrow\int_0^\infty u^{p\gamma+q-p-1}f^q(u)\,du<\infty. \tag{7.3}$$

THEOREM 7.5. *Let $0<p\leqslant 1$ and a function $f:\mathbb{R}_+\to\overline{\mathbb{R}}_+$ be such that $x^{1/p-1}f(x^{1/p})\in\overline{GM}_1\cap\Psi$. If $q\geqslant 1$ and $1-q<\gamma<1$, then* (7.3) *holds.*

REFERENCES

[1] J. BENEDETTO AND H. HEINIG, *Weighted Fourier Inequalities: New Proofs and Generalizations*, J. Fourier Anal. Appl., **9** (2003), 1–37.

[2] A. DEBERNARDI, *Uniform convergence of sine transforms of general monotone functions*, Math. Nachr., **290** (2017), 2815–2825.

[3] M. DYACHENKO, E. LIFLYAND AND S. TIKHONOV, *Uniform convergence and integrability of Fourier integrals*, J. Math. Anal. Appl., **372** (2010), 328–338.

[4] M. DYACHENKO AND S. TIKHONOV, *Convergence of trigonometric series with general monotone coefficients*, C. R. Acad. Sci. Paris, Ser. I, **345** (2007), 123–126.

[5] M. DYACHENKO AND S. TIKHONOV, *Integrability and continuity of functions represented by trigonometric series: coefficient criteria*, Studia Math., **193**, 3 (2009), 285–306.

[6] L. FENG, V. TOTIK AND S. P. ZHOU, *Trigonometric series with a generalized monotonicity condition*, Acta Math. Sin. (Engl. Ser.), **30** (2014), 1289–1296.

[7] M. R. GABDULLIN, *Trigonometric series with noninteger harmonics*, J. Math. Anal. Appl., **508**, 1 (2022), 125792.

[8] A. JEFFREY AND D. ZWILLINGER, *Table of Integrals, Series, and Products*, Seventh Edition, Academic Press, 2007.

[9] P. KÓRUS, *On the uniform convergence of special sine integrals*, Acta Math. Hungar., **133** (2011), 82–91.

[10] P. KÓRUS, *Uniform convergence of double trigonometric integrals*, Colloq. Math., **154** (2018), 107–119.

[11] S. KĘSKA, *On the Uniform Convergence of Sine Series with Square Root*, J. Funct. Spaces, **2019** (2019), Article ID 1342189.

[12] E. LIFLYAND AND S. TIKHONOV, *A concept of general monotonicity and applications*, Math. Nachr., **284**, 8–9 (2011), 1083–1098.

[13] E. LIFLYAND AND S. TIKHONOV, *Extended solution of Boas' conjecture on Fourier transforms*, C. R. Acad. Sci. Paris, Ser. I, **346** (2008), 1137–1142.

[14] E. LIFLYAND AND S. TIKHONOV, *The Fourier transforms of general monotone functions*, In: Analysis and Mathematical Physics, Trends in Mathematics, Birkhäuser, Basel, 2009, pp. 373–395.

[15] E. LIFLYAND AND S. TIKHONOV, *Two-sided weighted Fourier inequalities*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **11**, 2 (2012), 341–362.

[16] F. MÓRICZ, *On the uniform convergence of sine integrals*, J. Math. Anal. Appl., **354** (2009), 213–219.

[17] K. A. OGANESYAN, *Uniform convergence criterion for non-harmonic sine series*, Mat. Sb., **212**, 1 (2021), 78–118; Sb. Math., **212**, 1 (2021), 70–110.

[18] B. SZAL, *A new class of numerical sequences and its applications to uniform convergence of sine series*, Math. Nachr., **284**, 14–15 (2011), 1985–2002.

[19] S. TIKHONOV, *Trigonometric series with general monotone coefficients*, J. Math. Anal. Appl., **326** (2007), 721–735.

[20] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, second edition, Clarendon Press, Oxford, 1948.

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