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ON THE JACOBI–DUNKL COEFFICIENTS OF LIPSCHITZ AND DINI–LIPSCHITZ FUNCTIONS ON THE CIRCLE

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Abstract. In this paper, we consider $\mathscr{E}$ the set of infinitely differentiable 2π-periodic functions on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We use the distributions in $\mathscr{E}$, as a tool to prove the continuity of the Jacobi-Dunkl operator. We obtain a generalization of the classical Titchmarsh theorem for the Jacobi-Dunkl coefficients of a set of functions satisfying Lipschitz conditions, with the use of the generalized Jacobi-Dunkl translation operator defined by Vinogradov. In addition, we introduce the discrete Jacobi-Dunkl Dini-Lipschitz class and we obtain an analogue of Younis' theorem in this occurrence.

1. Introduction

Let $\{c_k\}_{k\in\mathbb{Z}}$ be a sequence of complex numbers such that

$$\sum_{k\in\mathbb{Z}} |c_k| < \infty. \tag{1}$$

Then

$$f(x) := \sum_{k\in\mathbb{Z}} c_k e^{ikx},$$

is a continuous 2π-periodic function and c_k, $k \in \mathbb{Z}$ are the Fourier coefficients of f. It is well known that many problems for partial differential equations are reduced to a power series expansion of the desired solution in terms of special functions or orthogonal polynomials (such as Laguerre, Hermite, Jacobi, Jacobi-Dunkl, etc., polynomials). In particular, this is associated with the separation of variables as applied to problems in mathematical physics (see [22, 25]).

One of classical problems in harmonic analysis and approximation theory consists in finding necessary and sufficient conditions on the Fourier coefficients $c_k, k \in \mathbb{Z}$ of a function to belong to a generalized Lipschitz class.

In 1937, E.C. Titchmarsh [26, Theorem 85] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate

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growth of the norm of their Fourier transform, he proved that if $f \in L^2(\mathbb{R})$ with $0 < \delta < 1$, then the following statement

$$\left(\int_{\mathbb{R}} |f(t+h)-f(t)|^2 dt\right)^{1/2} = O(h^{\delta}) \quad \text{as } h \to 0,$$

is equivalent to

$$\int_{|\lambda|\geqslant N} |\widehat{f}(\lambda)|^2 d\lambda = O(N^{-2\delta}) \quad \text{as } N \to \infty,$$

where $\widehat{f}$ stands for the Fourier transform of f.

Later, Younis generalized this theorem by replacing $O(h^{\delta})$ by

$$O\left(\frac{h^{\delta}}{(\log \frac{1}{h})^{\gamma}}\right), \quad 0 < \delta < 1, \ \gamma > 0.$$

In 1967, R. P. Boas [4] found necessary and sufficient conditions on the Fourier coefficients c_k, $k \in \mathbb{Z}$, satisfying the condition (1), to ensure that f belong to a generalized Lipschitz class. More precisely, in the case $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}^+$ (that is for cosine series with non-negative coefficients), he showed that $f \in \mathrm{Lip}(\delta)$, $0 < \delta < 1$, if and only if

$$\sum_{k=n}^{\infty} c_k = O(n^{-\delta}),$$

or, equivalently,

$$\sum_{k=1}^{n} kc_k = O(n^{1-\delta}).$$

After the publication of these articles, this theory has been widely studied by several authors. It is extended to functions of several variables on $\mathbb{R}^n$ and on the torus group $\mathbb{T}^n$ was studied by Younis [28, 29], and has also been generalized to general compact Lie groups [28]. Recently, it has also been extended to the case of compact Groups [8]. Titchmarsh's theorem [26] was also extended by Bray [5] to higher dimensional Euclidean spaces in a more general setting using multipliers by modifying the technique given in the seminal paper of Platonov [19] in the case of rank one noncompact symmetric spaces. For an overview of extensions of this theorem in different settings we refer to [1, 8, 9, 10, 11, 12, 14, 16, 17, 19, 24, 27].

To our knowledge, these theorems for the discrete Jacobi-Dunkl transform have not derived yet. In our current research, we are concerned with the Jacobi-Dunkl expansions on $I = [-\pi, \pi]$. By using some elements and results related to the discrete harmonic analysis associated with Jacobi-Dunkl transform introduced in [7], we try to explore the validity of these results in case of functions of the wider Lipschitz class in the weighted spaces $\mathbb{L}_2^{(\alpha,\beta)}$. For this purpose, we use the generalized Jacobi-Dunkl translation operator which was defined by Vinogradov in [21].

We conclude this introduction by giving the organization of this paper.

In the next Section, we state some basic notions and results from the discrete harmonic analysis associated with the Jacobi-Dunkl transform that will be needed throughout this paper.

In Section 3, we consider $\mathscr{E}$ the set of all infinitely differentiable 2π-periodic functions on the circle $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, we also define $\mathscr{E}'$ the set of even distributions on $\mathbb{T}$ (that is, continuous linear functionals on $\mathscr{E}$) and we prove that the Jacobi-Dunkl differential operator $\Lambda_{\alpha,\beta}$ is a continuous linear operator on the space $\mathscr{E}$.

In Section 4, we study among other things the validity of Titchmarsh's theorem in the case of functions of Lipschitz class in the space $\mathbb{L}_2^{(\alpha,\beta)}$, while in Section 5, we extend this theorem to Younis's theorem in the case of functions of Dini-Lipschitz class.

2. Preliminaries

In this Section, we will recall some properties of Jacobi and Jacobi-Dunkl polynomials, we present the information we need about the discrete harmonic analysis on the image under the Jacobi-Dunkl transform. For this purpose, we refer the reader to [2, 3, 6, 7, 15, 20, 21].

Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are the sets of non-negative integers, integers and real numbers respectively, f_e and f_o are the even and odd parts of a function f, i.e.,

$$f_e(t)=\frac{f(t)+f(-t)}{2} \quad \text{and} \quad f_o(t)=\frac{f(t)-f(-t)}{2}, \ t\in I.$$

We shall always assume that α and β are arbitrary real numbers with

$$\alpha\geqslant\beta\geqslant-\frac{1}{2},\ \alpha\neq-\frac{1}{2}, \quad \text{and set} \quad \rho:=\alpha+\beta+1.$$

We shall consider functions $f(t)$ on $I:=[-\pi,\pi]$. It is convenient to extend them to 2π-periodic functions on $\mathbb{R}$ or, equivalently, regard each $f(t)$ as function on the circle $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$. Unless otherwise stated, I stands for the closed interval $[-\pi,\pi]$ and I_0 stands for the open interval $(-\pi,\pi)$.

The Jacobi polynomials $\varphi_n^{(\alpha,\beta)}$ are defined by

$$\varphi_n^{(\alpha,\beta)}(t):=R_n^{(\alpha,\beta)}(\cos(t)), \tag{2}$$

for all $n\in\mathbb{N}$ and $t\in[0,\pi]$, with $x\mapsto R_n^{(\alpha,\beta)}(x)$ is the normalized Jacobi polynomial of degree n such that $R_n^{(\alpha,\beta)}(1)=1$, and are defined as (for more details see [23]).

$$R_n^{(\alpha,\beta)}(x)=\frac{\Gamma(\alpha+1)}{\Gamma(n+\rho)}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n+\rho+k)}{\Gamma(\alpha+1+k)}\left(\frac{x-1}{2}\right)^k. \tag{3}$$

Note that for all $n\in\mathbb{N}$ and $t\in[0,\pi]$, we have

$$|\varphi_n^{(\alpha,\beta)}(t)|\leqslant 1 \quad \text{and} \quad \varphi_n^{(\alpha,\beta)}(-t)=\varphi_n^{(\alpha,\beta)}(t). \tag{4}$$

The Jacobi operator $\mathscr{B} = \mathscr{B}_{\alpha,\beta}$ defined on $\mathscr{C}^2(I_0)$ is given by

$$\mathscr{B}f := \frac{1}{A}(Af')' = f'' + \frac{A'}{A}f',$$

where $A = A_{\alpha,\beta}$ is the weight function given in the relation

$$A(\theta) := (1-\cos\theta)^{\alpha}(1+\cos\theta)^{\beta}|\sin\theta|,\ \alpha \geqslant \beta \geqslant -\frac{1}{2},\ \alpha \neq -\frac{1}{2}. \tag{5}$$

For $1 \leqslant p < \infty$, we consider the Banach space $\mathbb{L}_p^{(\alpha,\beta)}$ of all measurable functions $f(t)$ on I with finite norm

$$\|f\|_p := \left(\int_{-\pi}^{\pi} |f(t)|^p A(t)dt\right)^{1/p}.$$

For $p = \infty$, we define the Banach space $\mathbb{L}_\infty^{(\alpha,\beta)} = \mathscr{C}(I)$ to be the set of all continuous functions $f(t)$ on I endowed with the norm

$$\|f\|_\infty = \max_{t\in I}|f(t)|.$$

For all $n \in \mathbb{N}$, $\varphi_n^{(\alpha,\beta)}$ is the unique even $\mathscr{C}^\infty$-solution in $(0,\pi)$ of the differential equation

$$\mathscr{B}f(t) = -\lambda_n^2 f(t), \quad f(0) = 1, \quad f'(0) = 0,$$

where

$$\lambda_n = \lambda_n^{(\alpha,\beta)} := \operatorname{sgn}(n)\sqrt{|n|(|n|+\rho)}, \quad n \in \mathbb{Z}.$$

The Jacobi function $\varphi_n^{(\alpha,\beta)}$, $n \in \mathbb{N}$ satisfies the following inequalities.

LEMMA 1. *The following inequalities are valid for Jacobi functions* $\varphi_n^{(\alpha,\beta)}$:

a) For $t \in [0,\pi/2]$, *we have*

$$1 - \varphi_{|n|}^{(\alpha,\beta)}(t) \leqslant k_1\lambda_n^2 t^2, \quad \forall n \in \mathbb{Z}. \tag{6}$$

b) For $t \in [0,1]$ *and* $t|n| \leqslant 1$, *we have*

$$1 - \varphi_{|n|}^{(\alpha,\beta)}(t) \geqslant k_2\lambda_n^2 t^2, \quad \forall n \in \mathbb{Z}. \tag{7}$$

Proof. See [18, Proposition 3.5 and Lemma 3.1]. □

LEMMA 2. *The following inequality is true*

$$1 - \varphi_{|n|}^{(\alpha,\beta)}(t) \geqslant k_3, \tag{8}$$

for $t|n| \geqslant 1$, *where* k_3 *is a certain constant.*

Proof. See [18, Proposition 3.3]. □

The Jacobi-Dunkl operator $\Lambda = \Lambda_{\alpha,\beta}$ is defined on I by

$$\Lambda f := \frac{1}{A}(Af)' = f' + \frac{A'}{A} f_o, \tag{9}$$

with

$$\frac{A'(t)}{A(t)} = \left(\alpha + \frac{1}{2}\right)\cot\frac{t}{2} - \left(\beta + \frac{1}{2}\right)\tan\frac{t}{2}, \quad t \in I_0 \setminus \{0\}. \tag{10}$$

Note that if f is even, then $\Lambda f = f'$, if f is odd, then $\Lambda f = (Af)'/A$ and if f is an even $\mathscr{C}^\infty$-function, then we have

$$\Lambda^2 f = \mathscr{B} f.$$

From [7], for all $n \in \mathbb{Z}$, the differential-difference equation

$$\begin{cases} \Lambda f(t) = i\lambda_n f(t), & n \in \mathbb{Z}, \\ \quad f(0) = 1, \end{cases} \tag{11}$$

admits a unique $\mathscr{C}^\infty$-solution $\psi_n^{(\alpha,\beta)}(t)$ on I. It is related to the Jacobi polynomial and to its derivative by

$$\psi_n^{(\alpha,\beta)}(t) := \begin{cases} \varphi_{|n|}^{(\alpha,\beta)}(t) - \dfrac{i}{\lambda_n}\dfrac{d}{dt}\varphi_{|n|}^{(\alpha,\beta)}(t) & \text{if} \quad n \in \mathbb{Z}^*, \\ 1 & \text{if} \quad n = 0. \end{cases}$$

We note that, for all $n \in \mathbb{Z}$ and $t \in I$, we have

$$\psi_{-n}^{(\alpha,\beta)}(t) = \psi_n^{(\alpha,\beta)}(-t) = \overline{\psi_n^{(\alpha,\beta)}(t)} \quad \text{and} \quad |\psi_n^{(\alpha,\beta)}(t)| \leqslant 1. \tag{12}$$

For all $n, p \in \mathbb{Z}$, we have the orthogonality formula given by (see [7])

$$\int_{-\pi}^{\pi} \psi_n^{(\alpha,\beta)}(t)\overline{\psi_p^{(\alpha,\beta)}(t)}A(t)dt = (w_n^{(\alpha,\beta)})^{-1}\delta_{n,p}, \tag{13}$$

where

$$w_n^{(\alpha,\beta)} = \left(\int_{-\pi}^{\pi} |\psi_n^{(\alpha,\beta)}(t)|^2 A(t)dt\right)^{-1} : \quad w_0^{(\alpha,\beta)} = \frac{\Gamma(\rho+1)}{2^{2\rho}\Gamma(\alpha+1)\Gamma(\beta+1)},$$

and

$$w_n^{(\alpha,\beta)} = \frac{(2|n|+\rho)\Gamma(\alpha+|n|+1)\Gamma(\rho+|n|)}{2^{2\rho+1}(\Gamma(\alpha+1))^2\Gamma(|n|+1)\Gamma(\beta+|n|+1)}, \quad \forall n \in \mathbb{Z}^*.$$

By using the relation (see [7])

$$\frac{d}{dt}\varphi_{|n|}^{(\alpha,\beta)}(t) = -\frac{\lambda_n^2}{4(\alpha+1)}\sin(2t)\,\varphi_{|n|-1}^{(\alpha+1,\beta+1)}(t),$$

the function $\psi_n^{(\alpha,\beta)}$ can be written in the form

$$\psi_n^{(\alpha,\beta)}(t) = \varphi_{|n|}^{(\alpha,\beta)}(t) + i\frac{\lambda_n}{4(\alpha+1)}\sin(2t)\varphi_{|n|-1}^{(\alpha+1,\beta+1)}(t). \tag{14}$$

The discrete Jacobi-Dunkl transform (or the Jacobi-Dunkl coefficients) of a function f in $\mathbb{L}_1^{(\alpha,\beta)}$ is defined by (see [7])

$$c_n(f) := \int_{-\pi}^{\pi} f(t)\overline{\psi_n^{(\alpha,\beta)}(t)}A(t)dt, \quad \forall n \in \mathbb{Z}. \tag{15}$$

Now, we consider the Jacobi-Dunkl expansion of f given by

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n(f)\psi_n^{(\alpha,\beta)}(t)w_n^{(\alpha,\beta)}, \quad \forall t \in I. \tag{16}$$

THEOREM 1. (Parseval formula) *If $f \in \mathbb{L}_2^{(\alpha,\beta)}$, then we have*

$$\|f\|_2 = \left(\sum_{n=-\infty}^{+\infty} |c_n(f)|^2 w_n^{(\alpha,\beta)}\right)^{1/2}. \tag{17}$$

Proof. See [7, Theorem 3.4]. □

In the following, we need to recall some results cited by Vinogradov in [21], where he introduced the generalized Jacobi-Dunkl translation operator [21, Lemma 1]. First, we will introduce some notations that we require. We denote by

$$x_+^\lambda := \begin{cases} x^\lambda & \text{if } x > 0,\ \lambda \in \mathbb{R}, \\ 0 & \text{if } x \leqslant 0, \end{cases}$$

$$x_+ := x_+^1.$$

$$a_{\alpha,\beta} := \int_0^1 r^{2\beta+1}(1-r^2)^{\alpha-\beta-1}dr = \frac{\Gamma(\beta+1)\Gamma(\alpha-\beta)}{2\Gamma(\alpha+1)}, \ \alpha > \beta > -1.$$

$$b_\beta := \int_0^\pi (\sin\theta)^{2\beta}d\theta = \frac{\sqrt{\pi}\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+1)}, \ \beta > -\frac{1}{2}.$$

$$c_{\alpha,\beta} := a_{\alpha,\beta}b_\beta = \frac{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}{2\Gamma(\alpha+1)}, \ \alpha > \beta > -\frac{1}{2}.$$

$$G_{\alpha,\beta} := \begin{cases} \mathbb{R}\setminus\{2n\pi\}_{n\in\mathbb{Z}} & \text{if } \alpha > \beta \geqslant -\frac{1}{2}, \\ \mathbb{R}\setminus\{n\pi\}_{n\in\mathbb{Z}} & \text{if } \alpha = \beta > -\frac{1}{2}, \\ \emptyset & \text{if } \alpha = \beta = -\frac{1}{2}. \end{cases}$$

For $h,t \in G_{\alpha,\beta}$ and $\theta,\chi \in I$,

$$\sigma_{h,t,\theta}(\chi) := \frac{\cos\frac{\theta}{2}\cos(\chi) - \cos\frac{h}{2}\cos\frac{t}{2}}{\sin\frac{h}{2}\sin\frac{t}{2}}$$

and

$$Q(h,t,\theta,\chi) := 1-\cos^2\frac{h}{2}-\cos^2\frac{t}{2}-\cos^2\frac{\theta}{2}+2\cos\frac{h}{2}\cos\frac{t}{2}\cos\frac{\theta}{2}\cos(\chi).$$

For $\alpha>\beta>-\frac{1}{2}$,

$$W(h,t,\theta) := \frac{|\sin\frac{h}{2}\sin\frac{t}{2}\sin\frac{\theta}{2}|^{-2\alpha}}{2^{\rho+2}c_{\alpha,\beta}}\int_0^{\pi}\left(1-\sigma_{h,t,\theta}+\sigma_{\theta,h,t}+\sigma_{t,\theta,h}\right)(\chi) \times Q_+^{\alpha-\beta-1}(h,t,\theta,\chi)\sin^{2\beta}(\chi)d\chi.$$

For $\alpha>\beta=-\frac{1}{2}$,

$$W(h,t,\theta) := \frac{|\sin\frac{h}{2}\sin\frac{t}{2}\sin\frac{\theta}{2}|^{-2\alpha}}{2^{\alpha+7/2}a_{\alpha,-\frac{1}{2}}}[\left(1-\sigma_{h,t,\theta}+\sigma_{\theta,h,t}+\sigma_{t,\theta,h}\right)(0)Q_+^{\alpha-\frac{1}{2}}(h,t,\theta,0) + \left(1-\sigma_{h,t,\theta}+\sigma_{\theta,h,t}+\sigma_{t,\theta,h}\right)(\pi)Q_+^{\alpha-\frac{1}{2}}(h,t,\theta,\pi)].$$

For $\alpha=\beta>-\frac{1}{2}$,

$$W(h,t,\theta) := \frac{\left(1-\cos^2(h)-\cos^2(t)-\cos^2(\theta)+2\cos(h)\cos(t)\cos(\theta)\right)_+^{\alpha-\frac{1}{2}}}{2b_\alpha|\sin(h)\sin(t)\sin(\theta)|^{2\alpha}} \times\left(1+\frac{\sin(h+t)}{\sin(\theta)}\right)\left(1-\frac{\cos(\theta)-\cos(h)\cos(t)}{\sin(h)\sin(t)}\right).$$

The generalized Jacobi-Dunkl translation operator is defined for $f\in\mathbb{L}_2^{(\alpha,\beta)}$ and $t,h\in I$ by

$$\mathscr{T}^h f(t) := \begin{cases}\int_{-\pi}^{\pi} f(\theta)W(h,t,\theta)A(\theta)d\theta & \text{if } h,t\in G_{\alpha,\beta},\\ f(t+h) & \text{if } h\notin G_{\alpha,\beta} \text{ or } t\notin G_{\alpha,\beta}.\end{cases}$$

It is also shown that for $f\in\mathbb{L}_2^{(\alpha,\beta)}$

$$c_n(\mathscr{T}^h f)=\psi_n^{(\alpha,\beta)}(h)c_n(f), \tag{18}$$

for all $n\in\mathbb{Z}$, $h\in I$, and the product formula

$$\mathscr{T}^h\psi_n^{(\alpha,\beta)}(t)=\psi_n^{(\alpha,\beta)}(h)\psi_n^{(\alpha,\beta)}(t), \tag{19}$$

holds.

THEOREM 2. *If $f\in\mathbb{L}_2^{(\alpha,\beta)}$, then $\mathscr{T}^h f\in\mathbb{L}_2^{(\alpha,\beta)}$ and we have*

$$\|\mathscr{T}^h f\|_2\leqslant\|f\|_2,\quad \forall h\in I. \tag{20}$$

Proof. See [21, Theorem 3]. □

For every $f\in\mathbb{L}_2^{(\alpha,\beta)}$, we define the differences $\Delta_h^m f$ of order m, $m=1,2,\dots,$ with step h, $0<h<\pi$ by:

$$\Delta_h^m f(t)=(\mathscr{T}^h+\mathscr{T}^{-h}-2I_{\mathbb{L}_2})^m f(t),$$

where $I_{\mathbb{L}_2}$ is the identity operator in $\mathbb{L}_2^{(\alpha,\beta)}$.

3. Auxiliary results

In order to get our results, we will need some auxiliary results.

Throughout the paper c_1, c_2, $c_3,\dots$ are positive constants, which may be different in different formulas and may depend on α,β and other parameters (we usually indicate them)

We note that the procedure for proving the results in this Section is similar to that in Platonov's paper [18].

We denote by $\mathscr{E}=\mathscr{E}(I)$, the set of all infinitely differentiable 2π-periodic functions on $\mathbb{R}$ such that for all $k=0,1,\dots,$

$$N_k(f):=\sum_{j=0}^{k}\sup_{t\in I}|\partial_t^j f(t)|<+\infty,$$

where $f\in\mathscr{E}$ and ∂_t is the operator of differentiation with respect to t.

The topology of $\mathscr{E}$ is defined by the semi-norms N_k, $k\in\mathbb{N}$.

We define another system of seminorms on $\mathscr{E}$ by putting

$$\widetilde{N}_k(f):=\sum_{j=0}^{k}\sup_{t\in I}|\partial_t^j(\Lambda f)(t)|,\quad k\in\mathbb{N}.$$

Let $\mathscr{E}'=\mathscr{E}'(I)$ be the set of distributions on I (that is, continuous linear functionals on $\mathscr{E}$). The spaces $\mathbb{L}_2^{(\alpha,\beta)}$ are embedded in $\mathscr{E}'$ by the formula

$$\langle f,\varphi\rangle_2:=\int_{-\pi}^{\pi}f(t)\overline{\varphi(t)}A(t)dt,$$

for all $f\in\mathbb{L}_2^{(\alpha,\beta)}$ and $\varphi\in\mathscr{E}$.

LEMMA 3. *For every $k\in\mathbb{N}$, there is a number $c_1=c_1(k)>0$ such that for all $f\in\mathscr{E}$, we have*

$$\sup_{t\in I}|\partial_t^k(\Lambda f)(t)|\leqslant c_1N_{k+1}(f).\tag{21}$$

Proof. Let $f\in\mathscr{E}$ and $k\in\mathbb{N}$. It follows from (9) that

$$\begin{aligned}&\partial_t^k(\Lambda f)(t)\\&=\partial_t^{k+1}(f)(t)+\left(\alpha+\frac{1}{2}\right)\partial_t^k\left(\left(\cot\frac{t}{2}\right)f_o(t)\right)-\left(\beta+\frac{1}{2}\right)\partial_t^k\left(\left(\tan\frac{t}{2}\right)f_o(t)\right).\end{aligned}$$

Thus, we have the inequality

$$\sup_{t\in I}|\partial_t^k(\Lambda f)(t)| \leqslant \sup_{t\in I}|\partial_t^{k+1}f(t)| + \left(\alpha+\frac{1}{2}\right)\sup_{t\in I}\left|\partial_t^k\left(\left(\cot\frac{t}{2}\right)f_o(t)\right)\right| + \left(\beta+\frac{1}{2}\right)\sup_{t\in I}\left|\partial_t^k\left(\left(\tan\frac{t}{2}\right)f_o(t)\right)\right|.$$

Let us estimate each term on the right-hand side of the above inequality. Clearly,

$$\sup_{t\in I}|\partial_t^{k+1}f(t)| \leqslant N_{k+1}(f).$$

Since $f_o(0)=0$, one can represent $f_o(t)$ in the form

$$f_o(t)=\int_0^t [f_o(s)]'ds=\int_0^t f_e'(s)ds = t\int_0^1 f_e'(tu)du.$$

Then

$$\left(\cot\frac{t}{2}\right)f_o(t)=t\left(\cot\frac{t}{2}\right)\int_0^1 f_e'(tu)du. \tag{22}$$

We put for a moment

$$a(t):=\begin{cases} t\cot\dfrac{t}{2} & \text{for} \quad t\in I_0\setminus\{0\}, \\ 2 & \text{for} \quad t=0, \\ 0 & \text{for} \quad t=-\pi \text{ or } t=\pi. \end{cases}$$

Clearly $a(t)\in\mathscr{C}^\infty(I)$. Put

$$A_j=\max_{t\in I}|\partial_t^j a(t)|.$$

By Leibniz's formula and relation (22), we have

$$\partial_t^k\left(\left(\cot\frac{t}{2}\right)f_o(t)\right)=\sum_{j=0}^k\binom{k}{j}\partial_t^{k-j}(a(t))\left(\int_0^1 u^j f_{e/o}^{(j+1)}(tu)du\right), \tag{23}$$

where $\binom{k}{j}$ is the binomial coefficient and

$$f_{e/o}^{(j+1)}(tu)=\begin{cases} f_e^{(j+1)}(tu) & \text{if} \quad j \text{ is even}, \\ f_o^{(j+1)}(tu) & \text{if} \quad j \text{ is odd}. \end{cases}$$

It is also clear that,

$$\left|\int_0^1 u^j f_{e/o}^{(j+1)}(tu)du\right| \leqslant \sup_{t\in I}|f_{e/o}^{(j+1)}(t)| \leqslant \frac{1}{2}\left(\sup_{t\in I}|f^{(j+1)}(t)|+\sup_{t\in I}|f^{(j+1)}(-t)|\right) = \sup_{t\in I}|f^{(j+1)}(t)| \leqslant \sum_{j=0}^{k+1}\sup_{t\in I}|f^{(j)}(t)| = N_{k+1}(f).$$

Then, it follows from (23) that

$$\left|\partial_t^k\left(\left(\cot\frac{t}{2}\right)f_o(t)\right)\right| \leqslant c_2 N_{k+1}(f), \tag{24}$$

where $c_2=\sum_{j=0}^{k}\binom{k}{j}A_{k-j}$. On the other hand, we note that

$$\begin{aligned}&\sup_{t\in I}\left|\partial_t^k\left(\left(\tan\frac{t}{2}\right)f_o(t)\right)\right| \\ &\leqslant \sup_{t\in[-\pi,0]}\left|\partial_t^k\left(\left(\tan\frac{t}{2}\right)f_o(t)\right)\right| + \sup_{t\in[0,\pi]}\left|\partial_t^k\left(\left(\tan\frac{t}{2}\right)f_o(t)\right)\right|.\end{aligned} \tag{25}$$

We estimate each term on the right-hand side of (25) separately. Since the function $f_o(t)$ is odd and 2π-periodic, we have $f_o(-\pi)=-f_o(\pi)$ and $f_o(-\pi)=f_o(\pi)$, whence $f_o(\pm\pi)=0$. One can represent the function $f_o(t)$ in the form

$$f_o(t)=-\int_t^{\pi} f_e'(s)ds=(t-\pi)\int_0^1 f_e'(\pi+(t-\pi)u)du,$$

Then

$$\left(\tan\frac{t}{2}\right)f_o(t)=b^+(t)\int_0^1 f_e'(\pi+(t-\pi)u)du, \tag{26}$$

where

$$b^+(t):=\begin{cases}(t-\pi)\tan\dfrac{t}{2} & \text{for} \quad t\in[0,\pi),\\ -2 & \text{for} \quad t=\pi.\end{cases}$$

Clearly, $b^+(t)\in\mathscr{C}^\infty([0,\pi])$.

Using (26) and arguing as in the proof of (24), we get

$$\sup_{t\in[0,\pi]}\left|\partial_t^k\left(\left(\tan\frac{t}{2}\right)f_o(t)\right)\right| \leqslant c_3 N_{k+1}(f), \tag{27}$$

where $c_3=c_3(k)$ is a constant.

On the other side, since $f_o(-\pi)=0$, then we can represent the function $f_o(t)$ in the form

$$f_o(t)=\int_{-\pi}^{t} f_e'(s)ds=(t+\pi)\int_0^1 f_e'(-\pi+(t+\pi)u)du.$$

Then

$$\left(\tan\frac{t}{2}\right)f_o(t)=b^-(t)\int_0^1 f_e'(-\pi+(t+\pi)u)du, \tag{28}$$

where

$$b^-(t):=\begin{cases}(t+\pi)\tan\dfrac{t}{2} & \text{for} \quad t\in(-\pi,0],\\ -2 & \text{for} \quad t=-\pi.\end{cases}$$

Clearly, $b^-(t)\in\mathscr{C}^\infty([-\pi,0])$.

Using (28) and arguing as in the proof of (24), we get

$$\sup_{t\in[-\pi,0]}\left|\partial_t^k\left(\left(\tan\frac{t}{2}\right)f_o(t)\right)\right| \leqslant c_4 N_{k+1}(f), \tag{29}$$

where $c_4 = c_4(k)$ is a constant.

Finally, inequality (21) follows from (24), (25), (27) and (29). □

Lemma 3 and the definition of the seminorms N_k yield the following corollary.

COROLLARY 1. *For all $k \in \mathbb{N}$ and $f \in \mathscr{E}$, we have*

$$N_k(\Lambda f) \leqslant c_5 N_{k+1}(f), \tag{30}$$

where $c_5 = c_5(k)$ is a constant.

LEMMA 4. *For every $k \in \mathbb{N}$, there is a number $c_6 = c_6(k) > 0$ such that for all $f \in \mathscr{E}$, we have*

$$\sup_{t\in I}|\Lambda^k f(t)| \leqslant c_6 N_k(f). \tag{31}$$

Proof. It follows from Corollary 1 that

$$\sup_{t\in I}|\Lambda^k f(t)| = N_0(\Lambda^k f) \leqslant c_5(0)N_1(\Lambda^{k-1}f) \leqslant \cdots \leqslant c_5(0)c_5(1)\cdots c_5(k-1)N_k(f).$$

This proves (31) with $c_6 = \prod_{j=0}^{k-1} c_5(j)$. □

LEMMA 5. *For every $k \in \mathbb{N}$, there is a number $c_7 = c_7(k) > 0$ such that we have*

$$\sup_{t\in I}|\partial_t^k f(t)| \leqslant c_7 N_{k-1}(\Lambda f), \tag{32}$$

for all $f \in \mathscr{E}$.

Proof. Let $f = f_e + f_o \in \mathscr{E}$. We note first that

$$\sup_{t\in I}|\partial_t^k f(t)| \leqslant \sup_{t\in[-\pi,-\pi/2]}|\partial_t^k f(t)| + \sup_{t\in[-\pi/2,\pi/2]}|\partial_t^k f(t)| + \sup_{t\in[\pi/2,\pi]}|\partial_t^k f(t)|. \tag{33}$$

We estimate each term on the right-hand side of (33) separately.

If f is even, then $\Lambda f_e = f_e'$ and we have

$$\sup_{t\in[-\pi,-\pi/2]}|\partial_t^k f_e(t)| = \sup_{t\in[-\pi,-\pi/2]}|\partial_t^{k-1}(f_e'(t))| = \sup_{t\in[-\pi,-\pi/2]}|\partial_t^{k-1}(\Lambda f_e)(t)|. \tag{34}$$

Since $\Lambda f_e = 1/2(\Lambda f + \Lambda \widetilde{f})$ and $\Lambda \widetilde{f} = -\Lambda f$ where $\widetilde{f}(t) = f(-t)$, then it follows from (34) that

$$\begin{aligned} \sup_{t\in[-\pi,-\pi/2]} |\partial_t^k f_e(t)| &\leqslant \frac{1}{2}\left(\sup_{t\in I}|\partial_t^{k-1}(\Lambda f)(t)| + \sup_{t\in I}|\partial_t^{k-1}(\Lambda \widetilde{f})(t)|\right) \\ &\leqslant \sup_{t\in I}|\partial_t^{k-1}(\Lambda f)(t)| \leqslant \sum_{j=0}^{k-1}\sup_{t\in I}|\partial_t^{j}(\Lambda f)(t)| \\ &= N_{k-1}(\Lambda f). \end{aligned} \tag{35}$$

Similarly, we show that

$$\sup_{t\in[-\pi/2,\pi/2]} |\partial_t^k f_e(t)| \leqslant N_{k-1}(\Lambda f) \ \text{ and } \sup_{t\in[\pi/2,\pi]} |\partial_t^k f_e(t)| \leqslant N_{k-1}(\Lambda f). \tag{36}$$

The other side, if f is odd, then

$$\Lambda f_o = f_o' + \frac{A'}{A} f_o = \frac{(Af_o)'}{A}. \tag{37}$$

From (37), we can represent $f_o(t)$ as

$$f_o(t) = -\frac{1}{A(t)}\int_t^{\pi} \Lambda f_o(s)A(s)ds, \tag{38}$$

by virtue of $f_o(\pi) = 0$. So,

$$\begin{aligned} f_o'(t) &= \int_t^{\pi} \frac{A'(t)A(s)}{A^2(t)}\Lambda f_o(s)ds + \Lambda f_o(t) \\ &= (\pi - t)\int_0^1 \frac{A'(t)}{A^2(t)}A(\pi+(t-\pi)u)\Lambda f_o(\pi+(t-\pi)u)du + \Lambda f_o(t). \end{aligned} \tag{39}$$

We put for a moment

$$a(t,u) := \begin{cases} \dfrac{A'(t)}{A^2(t)}(\pi - t)A(\pi+(t-\pi)u) & \text{for } t \in [\pi/2,\pi), \\ -(2\beta+1)u^{2\beta+1} & \text{for } \quad t = \pi. \end{cases}$$

We also put

$$\sigma(x) := \begin{cases} \dfrac{\sin x}{x} & \text{for} \quad x \neq 0, \\ 1 & \text{for} \quad x = 0, \end{cases}$$

$$r_{\alpha,\beta}(t,u) := \left(\frac{\sigma(tu/2)}{\sigma(t/2)}\right)^{2\alpha+1}\left(\frac{\cos(tu/2)}{\cos(t/2)}\right)^{2\beta+1}. \tag{40}$$

Since $\sigma(x) \in \mathscr{C}^\infty(\mathbb{R})$, we see that $r_{\alpha,\beta}(t,u) \in \mathscr{C}^\infty((-\pi,\pi)\times[0,1])$.

One can represent the function $a(t,u)$ in the form

$$a(t,u)=\left[\left(\alpha+\frac{1}{2}\right)(\pi-t)\cot\frac{t}{2}-\left(\beta+\frac{1}{2}\right)(\pi-t)\tan\frac{t}{2}\right]u^{2\beta+1}r_{\beta,\alpha}(\pi-t,u).$$

It follows that $a(t,u)$ is defined on the rectangle $[\pi/2,\pi]\times[0,1]$ and is infinitely differentiable with respect to t. For every $k\in\mathbb{N}$, the function $\partial_t^k a(t,u)$ is continuous on the rectangle $[\pi/2,\pi]\times[0,1]$.

For every $j\in\mathbb{N}$, we put

$$\overline{A}_j=\max\left\{|\partial_t^j a(t,u)| : (t,u)\in[\pi/2,\pi]\times[0,1]\right\}.$$

It follows from (39) that

$$f_o'(t)=\int_0^1 a(t,u)\Lambda f_o(\pi+(t-\pi)u)du+\Lambda f_o(t). \tag{41}$$

Then, we have

$$\begin{aligned}\partial_t^k f_o(t) &= \partial_t^{k-1}(f_o'(t))\\ &=\int_0^1\left(\sum_{j=0}^{k-1}\binom{k-1}{j}(\partial_t^{k-1-j}a(t,u))(\Lambda f_o)^{(j)}(\pi+(t-\pi)u)u^j\right)du\\ &\quad+\partial_t^{k-1}(\Lambda f_o)(t).\end{aligned}$$

So,

$$\sup_{t\in[\pi/2,\pi]}|\partial_t^k f_o(t)|\leqslant\sum_{j=0}^{k-1}\binom{k-1}{j}\overline{A}_{k-1-j}\sup_{t\in I}|\partial_t^j(\Lambda f_o)(t)|+\sup_{t\in I}|\partial_t^{k-1}(\Lambda f_o)(t)|.$$

Therefore,

$$\sup_{t\in[\pi/2,\pi]}|\partial_t^k f_o(t)|\leqslant\left(\sum_{j=0}^{k-1}\binom{k-1}{j}\overline{A}_{k-1-j}\right)N_{k-1}(\Lambda f_o)+N_{k-1}(\Lambda f_o).$$

Using the same technique as in (35), we get

$$\sup_{t\in[\pi/2,\pi]}|\partial_t^k f_o(t)|\leqslant c_8 N_{k-1}(\Lambda f), \tag{42}$$

with $c_8=c_8(k)=(1+\sum_{j=0}^{k-1}\binom{k-1}{j}\overline{A}_{k-1-j})$.

To estimate the first term on the right-hand side of (33), we deduce from the equality $f_o(-\pi)=0$ that

$$f_o(t)=\frac{1}{A(t)}\int_{-\pi}^{t}\Lambda f_o(s)A(s)ds.$$

Thus,

$$f_o'(t) = -\int_{-\pi}^{t} \frac{A'(t)A(s)}{A^2(t)} \Lambda f_o(s)ds + \Lambda f_o(t) \tag{43}$$
$$= -(\pi+t)\int_0^1 \frac{A'(t)}{A^2(t)} A(-\pi+(t+\pi)u)\Lambda f_o(-\pi+(t+\pi)u)du + \Lambda f_o(t).$$

We put for a moment

$$b(t,u) := \begin{cases} -\dfrac{A'(t)}{A^2(t)}(\pi+t)A(-\pi+(t+\pi)u) & \text{for } t \in (-\pi,-\pi/2], \\ -(2\beta+1)u^{2\beta+1} & \text{for } \quad t=-\pi. \end{cases}$$

One can represent the function $b(t,u)$ in the form

$$b(t,u) = \left[-\left(\alpha+\frac{1}{2}\right)(\pi+t)\cot\frac{t}{2} + \left(\beta+\frac{1}{2}\right)(\pi+t)\tan\frac{t}{2}\right] u^{2\beta+1} r_{\beta,\alpha}(\pi+t,u). \tag{44}$$

where $r_{\beta,\alpha}(t,u)$ is the function (40) with and interchanged. We easily see from (44) that the function $b(t,u)$ is defined on the rectangle $[-\pi,-\pi/2]\times[0,1]$ and is infinitely differentiable with respect to t. For every $k \in \mathbb{N}$, the function $\partial_t^k b(t,u)$ is continuous on the rectangle $[-\pi,-\pi/2]\times[0,1]$.

It follows from (43) that

$$f_o'(t) = \int_0^1 b(t,u)\Lambda f_o(-\pi+(t+\pi)u)du + \Lambda f_o(t). \tag{45}$$

Using (45) and arguing as in the proof of (41), we get

$$\sup_{t\in[-\pi,-\pi/2]} |\partial_t^k f_o(t)| \leqslant c_9 N_{k-1}(\Lambda f). \tag{46}$$

To estimate the second term on the right-hand side of (33), we deduce from the equality $f_o(0)=0$ that

$$f_o(t) = \frac{1}{A(t)}\int_0^t \Lambda f_o(s)A(s)ds.$$

Thus,

$$f_o'(t) = -\int_0^t \frac{A'(t)A(s)}{A^2(t)}\Lambda f_o(s)ds + \Lambda f_o(t)$$
$$= -\int_0^1 \frac{tA'(t)}{A^2(t)} A(tu)\Lambda f_o(tu)du + \Lambda f_o(t). \tag{47}$$

We put for a moment

$$c(t,u) := \begin{cases} -\dfrac{tA'(t)}{A^2(t)}A(tu) & \text{for } t \in [-\pi/2,\pi/2]\setminus\{0\}, \\ -(2\alpha+1)u^{2\alpha+1} & \text{for } \quad t=0. \end{cases}$$

One can represent the function $c(t,u)$ in the form

$$c(t,u)=\left[-\left(\alpha+\frac{1}{2}\right)t\cot\frac{t}{2}+\left(\beta+\frac{1}{2}\right)t\tan\frac{t}{2}\right]u^{2\alpha+1}r_{\alpha,\beta}(t,u). \tag{48}$$

We easily see from (48) that the function $c(t,u)$ is defined on the rectangle $[-\pi/2,\pi/2]$ $\times[0,1]$ and is infinitely differentiable with respect to t. For every $k\in\mathbb{N}$, the function $\partial_t^k c(t,u)$ is continuous on the rectangle $[-\pi/2,\pi/2]\times[0,1]$.

It follows from (47) that

$$f_o'(t)=\int_0^1 c(t,u)\Lambda f_o(tu)du+\Lambda f_o(t). \tag{49}$$

Using (49) and arguing as in the proof of (41) and (45), we get

$$\sup_{t\in[-\pi/2,\pi/2]}|\partial_t^k f_o(t)|\leqslant c_{10}N_{k-1}(\Lambda f). \tag{50}$$

Then, by combining relations (34), (35) (36), (42), (46) and (50), we have

$$\sup_{t\in I}|\partial_t^k f(t)|\leqslant \sup_{t\in I}|\partial_t^k f_e(t)|+\sup_{t\in I}|\partial_t^k f_o(t)|\leqslant c_{11}N_{k-1}(\Lambda f), \tag{51}$$

where $c_{11}=c_{11}(k)$ is a constant. □

Lemma 5 and the definition of the seminorm N_k yield the following corollary.

COROLLARY 2. *For all $k\in\mathbb{N}$ and $f\in\mathscr{E}$, we have*

$$N_k(f)\leqslant c_{12}(N_{k-1}(\Lambda f)+N_{k-1}(f)), \tag{52}$$

where $c_{12}=c_{12}(k)$ is a constant.

THEOREM 3. *For every $k\in\mathbb{N}$, there are positive numbers $C_1=C_1(k)$ and $C_2=C_2(k)$ such that for all functions $f\in\mathscr{E}$, we have*

$$\widetilde{N}_k(f)\leqslant C_1N_k(f), \tag{53}$$

$$N_k(f)\leqslant C_2\widetilde{N}_{k+1}(f). \tag{54}$$

Proof. Using Lemma 4, we get

$$\widetilde{N}_k(f)=\sum_{j=0}^{k}\sup_{t\in I}|\Lambda_t^j f(t)|\leqslant\sum_{j=0}^{k}c_6(j)N_j(f)\leqslant\left(\sum_{j=0}^{k}c_6(j)\right)N_k(f). \tag{55}$$

This proves inequality (53) with $C_1=\sum_{j=0}^{k}c_6(j)$.

The proof of inequality (54) is by induction on k. For $k=0$ this inequality holds with $C_2(0)=1$. If it holds for $k-1$, then we use (52) to obtain that

$$\begin{aligned}N_k(f)&\leqslant c_{12}(k)(N_{k-1}(\Lambda f)+N_{k-1}(f))\\&\leqslant c_{12}(k)C_2(k-1)(\widetilde{N}_k(\Lambda f)+\widetilde{N}_k(f))\\&\leqslant 2c_{12}(k)C_2(k-1)\widetilde{N}_{k+1}(f).\end{aligned}$$

This proves (54) with $C_2(k)=2c_{12}(k)C_2(k-1)$. □

COROLLARY 3. *The systems of seminorms* $\{N_k(f), k \in \mathbb{N}\}$ *and* $\{\widetilde{N}_k(f), k \in \mathbb{N}\}$ *determine the same topology on the vector space* $\mathscr{E}$.

COROLLARY 4. *The Jacobi-Dunkl differential operator* Λ *is a continuous linear operator on the space* $\mathscr{E}$.

We extend the action of Λ to distributions in $\mathscr{E}'$ by the formula

$$\langle \Lambda f, \varphi \rangle_2 := \langle f, \Lambda\varphi \rangle_2, \quad f \in \mathscr{E}', \quad \varphi \in \mathscr{E}. \tag{56}$$

In particular, the action of Λ is defined on every function $f \in \mathbb{L}_2^{(\alpha,\beta)}$, but Λ is a distribution in general.

Let W_2^k be the Sobolev space of order $k \in \mathbb{N}$ constructed from the Jacobi-Dunkl operator Λ, that is,

$$W_2^k := \{f \in \mathbb{L}_2^{(\alpha,\beta)} : \Lambda^r f \in \mathbb{L}_2^{(\alpha,\beta)},\ r = 1, 2, \ldots, k\},$$

where

$$\Lambda^0 f = f, \quad \Lambda^r f = \Lambda(\Lambda^{r-1} f), \quad r = 1, 2, \ldots, k.$$

Here the inclusion $\Lambda^r f \in \mathbb{L}_2^{(\alpha,\beta)}$ means that the distribution $\Lambda^r f$ is regular and corresponds to an ordinary function of class $\mathbb{L}_2^{(\alpha,\beta)}$.

LEMMA 6. *If* $f \in \mathbb{L}_2^{(\alpha,\beta)}$, *then*

$$c_n(\Lambda f) = -i\lambda_n c_n(f), \tag{57}$$

for all $n \in \mathbb{Z}$.

Proof. For every distribution $f \in \mathscr{E}'$, we put $c_n(f) := \langle f, \psi_n^{(\alpha,\beta)} \rangle_2, n \in \mathbb{Z}$. It follows from (11) and (56) that

$$\begin{aligned} c_n(\Lambda f) &= \langle \Lambda f, \psi_n^{(\alpha,\beta)} \rangle_2 \\ &= \langle f, \Lambda \psi_n^{(\alpha,\beta)} \rangle_2 \\ &= \langle f, i\lambda_n \psi_n^{(\alpha,\beta)} \rangle_2 \\ &= -i\lambda_n \langle f, \psi_n^{(\alpha,\beta)} \rangle_2 \\ &= -i\lambda_n c_n(f). \end{aligned}$$

Then the equality (57) is valid in $\mathscr{E}'$, so it is also valid in $\mathbb{L}_2^{(\alpha,\beta)}$ (the spaces $\mathbb{L}_2^{(\alpha,\beta)}$ are embedded in $\mathscr{E}'$). □

COROLLARY 5. *If* $f \in W_2^k$, *then*

$$c_n(\Lambda^r f) = (-i)^r \lambda_n^r c_n(f), \tag{58}$$

for all $r = 1, 2, \ldots, k$.

4. Generalization of Titchmarsh theorem for discrete Jacobi-Dunkl of Lipschitz class

In order to give a generalized version of Titchmarsh theorem for the discrete Jacobi-Dunkl transform in $\mathbb{L}_2^{(\alpha,\beta)}$. We begin with auxiliary results interesting in themselves.

We emphasize that in this paper, the symbol '$\mathscr{O}$' always refers to a global estimate valid over $\mathbb{T}$.

LEMMA 7. *Let* $0<h<\pi$. *If* $f\in W_2^k$, *then for all* $n\in\mathbb{Z}$, *we have*

$$\sum_{n=-\infty}^{+\infty}\lambda_n^{2r}|1-\varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k}|c_n(f)|^2w_n^{(\alpha,\beta)}=A\|\Delta_h^k(\Lambda^r f)\|_2^2, \tag{59}$$

where A *is a positive constant and* $r=0,1,2,\ldots,k$.

Proof. According to the relations (14) and (18), we get

$$\begin{aligned} c_n(\Delta_h f) &= c_n(\mathscr{T}^h f)+c_n(\mathscr{T}^{-h} f)-2c_n(f)\\ &= (\psi_n^{(\alpha,\beta)}(h)+\psi_n^{(\alpha,\beta)}(-h)-2)c_n(f)\\ &= 2(\varphi_{|n|}^{(\alpha,\beta)}(h)-1)c_n(f). \end{aligned}$$

Using the proof of recurrence for k, we have

$$c_n(\Delta_h^k f)=2^k(\varphi_{|n|}^{(\alpha,\beta)}(h)-1)^k c_n(f).$$

In view of formula (58), we get

$$\mathscr{F}(\Delta_h^k(\Lambda^r f))(n)=(-i)^r 2^k\lambda_n^r(\varphi_{|n|}^{(\alpha,\beta)}(h)-1)^k c_n(f).$$

Now, by appealing the Parseval formula (17), we have the desired result. □

Now, we define the discrete Jacobi-Dunkl Lipschitz class:

DEFINITION 1. Let $0<\delta<k$. A function $f\in W_2^k$ is said to be in the discrete Jacobi-Dunkl Lipschitz class, denoted by $\mathscr{L}ip_k(\delta;2,\alpha,\beta)$, if

$$\|\Delta_h^k(\Lambda^r f)\|_2=\mathscr{O}(h^\delta)\quad \text{as } h\to 0,$$

where $r=0,1,2,\ldots,k$.

Observe that if $0<\delta<\sigma<1$, then

$$\mathscr{L}ip_k(\sigma;2,\alpha,\beta)\subset\mathscr{L}ip_k(\delta;2,\alpha,\beta).$$

Indeed, for $0<h\leqslant 1$ and $\delta<\sigma$, we get $h^\sigma<h^\delta$, whence the remark follows.

The proof of theorem 4 necessitates the following lemma:

LEMMA 8. *Suppose $b_n \geqslant 0$ and $0 < c < d$. Then*

$$\sum_{n=1}^{N} n^d b_n = \mathscr{O}(N^c),$$

is equivalent to

$$\sum_{n=N}^{+\infty} b_n = \mathscr{O}(N^{c-d}).$$

Proof. See [13, page 101]. □

THEOREM 4. *Let $0 < \delta < k$ and $f \in W_2^k$. The following two conditions are equivalent:*

(a) $f \in \mathscr{L}ip_k(\delta; 2, \alpha, \beta)$,

(b) $\sum_{|n| \geqslant N} \lambda_n^{2r} |c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathscr{O}(N^{-2\delta})$ *as* $N \to +\infty$.

Proof. $(a) \Rightarrow (b)$ Let $f \in \mathscr{L}ip_k(\delta; 2, \alpha, \beta)$. Then we have

$$\|\Delta_h^k(\Lambda^r f)\|_2 = \mathscr{O}(h^\delta) \quad \text{as } h \to 0.$$

It follows from Lemma 7 that

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \lambda_n^{2r} |1 - \varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k} |c_n(f)|^2 w_n^{(\alpha,\beta)} &= A \|\Delta_h^k(\Lambda^r f)\|_2^2 \\ &\leqslant Ch^{2\delta},\end{aligned}$$

as $h \to 0$, where C is a positive constant.

If $0 \leqslant |n| \leqslant 1/h$, hence $|n|h \leqslant 1$ and from formula (7), we have

$$\lambda_n^{4k} h^{4k} \leqslant \frac{1}{k_2^{2k}} |1 - \varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k}.$$

From this, we get

$$\begin{aligned}&\sum_{1 \leqslant |n| \leqslant \left[\frac{1}{h}\right]} \lambda_n^{4k} \lambda_n^{2r} h^{4k} |c_n(f)|^2 w_n^{(\alpha,\beta)} \\ &\leqslant \frac{1}{k_2^{2k}} \sum_{1 \leqslant |n| \leqslant \left[\frac{1}{h}\right]} \lambda_n^{2r} |1 - \varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k} |c_n(f)|^2 w_n^{(\alpha,\beta)} \\ &\leqslant \frac{1}{k_2^{2k}} \sum_{n=-\infty}^{+\infty} \lambda_n^{2r} |1 - \varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k} |c_n(f)|^2 w_n^{(\alpha,\beta)} \\ &= \mathscr{O}(h^{2\delta}).\end{aligned}$$

Here $\left[\frac{1}{h}\right]$ is the integer part of $\frac{1}{h}$. Furthermore, by using the fact that λ_n^2 is greater than or equal to n^2 for all $n \in \mathbb{Z}$, we get

$$\sum_{1\leqslant|n|\leqslant\left[\frac{1}{h}\right]} n^{4k}\lambda_n^{2r}h^{4k}|c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathscr{O}(h^{2\delta}).$$

Consequently,

$$\sum_{1\leqslant|n|\leqslant\left[\frac{1}{h}\right]} n^{4k}\lambda_n^{2r}|c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathscr{O}(h^{2\delta-4k}) \quad \text{as } h \to 0. \tag{60}$$

Thus

$$\sum_{1\leqslant|n|\leqslant N} n^{4k}\lambda_n^{2r}|c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathscr{O}(N^{4k-2\delta}) \quad \text{as } N \to +\infty,$$

which is equivalent to

$$\sum_{n=1}^{N} n^{4k}\lambda_n^{2r}(|c_n(f)|^2+|c_{-n}(f)|^2) w_n^{(\alpha,\beta)} = \mathscr{O}(N^{4k-2\delta}) \quad \text{as } N \to +\infty,$$

by virtue of

$$(-n)^{4k}\lambda_{-n}^{2r}w_{-n}^{(\alpha,\beta)} = n^{4k}\lambda_n^{2r}w_n^{(\alpha,\beta)}, \quad \forall n \in \mathbb{Z}.$$

From Lemma 8, we have

$$\sum_{n=N}^{+\infty} \lambda_n^{2r}(|c_n(f)|^2+|c_{-n}(f)|^2) w_n^{(\alpha,\beta)} = \mathscr{O}(N^{4k-2\delta-4k}) \quad \text{as } N \to +\infty.$$

Therefore

$$\sum_{|n|\geqslant N} \lambda_n^{2r}|c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathscr{O}(N^{-2\delta}),$$

as $N \to +\infty$, witch complete the proof of the first implication.

$(b) \Rightarrow (a)$ Suppose now that

$$\sum_{|n|\geqslant N} \lambda_n^{2r}|c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathscr{O}(N^{-2\delta}) \quad \text{as } N \to +\infty,$$

we have to show that

$$\sum_{n=-\infty}^{+\infty} \lambda_n^{2r}|1-\varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k}|c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathscr{O}(h^{2\delta}) \quad \text{as } h \to 0.$$

We write

$$\sum_{n=-\infty}^{+\infty} \lambda_n^{2r}|1-\varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k}|c_n(f)|^2 w_n^{(\alpha,\beta)} \leqslant \mathscr{J}_1 + \mathscr{J}_2,$$

where

$$\mathscr{J}_1 = \sum_{1\leqslant|n|\leqslant\left[\frac{1}{h}\right]} \lambda_n^{2r}|1-\varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k}|c_n(f)|^2 w_n^{(\alpha,\beta)}$$

and

$$\mathscr{J}_2 = \sum_{|n| \geqslant [\frac{1}{h}]} \lambda_n^{2r} |1 - \varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k} |c_n(f)|^2 w_n^{(\alpha,\beta)}.$$

We estimate them separately. Let us now estimate $\mathscr{J}_1$. First, note that

$$\lambda_n^2 = 4n^2 \left(1 + \frac{\rho}{|n|}\right) \leqslant 4n^2(1+\rho) \quad \text{for } |n| \geqslant 1,\ n \in \mathbb{Z}. \tag{61}$$

It follows from this, the inequality (6) in Lemma 1 and formula (60) that

$$\begin{aligned}
\mathscr{J}_1 &= \sum_{1 \leqslant |n| \leqslant [\frac{1}{h}]} \lambda_n^{2r} |1 - \varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k} |c_n(f)|^2 w_n^{(\alpha,\beta)} \\
&\leqslant c_1^{2k} h^{4k} \sum_{1 \leqslant |n| \leqslant [\frac{1}{h}]} \lambda_n^{4k} \lambda_n^{2r} |c_n(f)|^2 w_n^{(\alpha,\beta)} \\
&\leqslant c_1^{2k} h^{4k} 4^{2k} (1+\rho)^{2k} \sum_{1 \leqslant |n| \leqslant [\frac{1}{h}]} n^{4k} \lambda_n^{2r} |c_n(f)|^2 w_n^{(\alpha,\beta)} \\
&= \mathscr{O}(h^{4k+2\delta-4k}) \\
&= \mathscr{O}(h^{2\delta}).
\end{aligned}$$

On the other hand, it follows from (4) that

$$\begin{aligned}
\mathscr{J}_2 = &\sum_{|n| \geqslant [\frac{1}{h}]} \lambda_n^{2r} |1 - \varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k} |c_n(f)|^2 w_n^{(\alpha,\beta)}. \\
&\leqslant 2^{2k} \sum_{|n| \geqslant [\frac{1}{h}]} \lambda_n^{2r} |c_n(f)|^2 w_n^{(\alpha,\beta)}. \\
&= \mathscr{O}(h^{2\delta}),
\end{aligned}$$

and this ends the proof of this theorem. □

We conclude this Section by the following immediate consequence.

COROLLARY 6. *Let* $0 < \delta < k$ *and* $f \in W_2^k$. *If*

$$f \in \mathscr{L}ip_k(\delta; 2, \alpha, \beta),$$

then

$$\sum_{|n| \geqslant N} |c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathscr{O}(N^{-2\delta-2r}) \quad \text{as } N \to +\infty.$$

5. Generalization of Titchmarsh theorem for the discrete Jacobi-Dunkl of Dini-Lipschitz class

In this Section, we will consider a different condition, the so-called Dini-Lipschitz condition on W_2^k and we will generalise the corresponding Titchmarsh theorems (cf. [26, Theorem 85]).

DEFINITION 2. Let $\gamma \in \mathbb{R}$ and $0 < \delta < k$. A function $f \in W_2^k$ is said to be in the discrete Jacobi-Dunkl Dini-Lipschitz class, denoted by $\mathscr{L}ip_k(\delta,\gamma;2,\alpha,\beta)$, if

$$\|\Delta_h^k(\Lambda^r f)\|_2 = \mathscr{O}\left(h^\delta\left(\log\frac{1}{h}\right)^\gamma\right) \quad \text{as } h \to 0,$$

where $r = 0,1,2,\ldots,k$.

LEMMA 9. *For all $n \in \mathbb{Z}$, we have*

$$\frac{1-\varphi_{|n|}^{(\alpha,\beta)}(t)}{\lambda_n^2 t^2} \to \frac{1}{4(\alpha+1)} \quad \text{as } t \to 0.$$

Proof. It follows from relation (2) and (3) that

$$\frac{1-\varphi_{|n|}^{(\alpha,\beta)}(t)}{\lambda_n^2 t^2} = \frac{1}{4(\alpha+1)}\left(\frac{\sin t/2}{t/2}\right)^2 + o\left(\left(\frac{\sin t/2}{t/2}\right)^4\right).$$

We immediately get the desired result when t tends to 0. □

THEOREM 5. *Let $\delta > 2k$ and $\gamma \leqslant 0$. If a function f belongs to $\mathscr{L}ip_k(\delta,\gamma;2,\alpha,\beta)$, then f is null almost everywhere on I.*

Proof. Assume that $f \in \mathscr{L}ip_k(\delta,\gamma;2,\alpha,\beta)$, and fix $r = 0,1,\ldots,k$. Then

$$\|\Delta_h^k(\Lambda^r f)\|_2 \leqslant K\frac{h^\delta}{(\log\frac{1}{h})^{-\gamma}}$$

where K is a positive constant, being the last inequality valid for sufficiently small values of h.

It follows from Lemma (7) that

$$\sum_{n=-\infty}^{+\infty} \lambda_n^{2r}|1-\varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k}|c_n(f)|^2 w_n^{(\alpha,\beta)} \leqslant K^2\frac{h^{2\delta}}{(\log\frac{1}{h})^{-2\gamma}}.$$

Therefore,

$$\frac{1}{h^{4k}}\sum_{n=-\infty}^{+\infty} \lambda_n^{2r}|1-\varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k}|c_n(f)|^2 w_n^{(\alpha,\beta)} \leqslant K^2\frac{h^{2(\delta-2k)}}{(\log\frac{1}{h})^{-2\gamma}}.$$

Since $\delta > 2k$ and $-2\gamma \geqslant 0$, we have

$$\lim_{h\to 0}\frac{h^{2(\delta-2k)}}{(\log\frac{1}{h})^{-2\gamma}} = 0.$$

Thus

$$\lim_{h\to 0}\sum_{n=-\infty}^{+\infty}\lambda_n^{2(r+2k)}\left(\frac{|1-\varphi_{|n|}^{(\alpha,\beta)}(h)|}{h^2\lambda_n^2}\right)^{2k}|c_n(f)|^2 w_n^{(\alpha,\beta)}=0.$$

Now, taking into consideration Lemma 9 and thanks to Fatou theorem, we have

$$\sum_{n=-\infty}^{+\infty}|\lambda_n^{(r+2k)}c_n(f)|^2 w_n^{(\alpha,\beta)}=0.$$

Hence $c_n(f)=0$ for all $n\in\mathbb{Z}$. the result follows from the injectivity of c_n. □

For the proof of the second Titchmarsh theorem we will be using an extension of Duren's lemma (cf. [29, p. 101]), Lemma 8 in this paper, adapted to the Dini-Lipschitz condition.

LEMMA 10. *Suppose $a\in\mathbb{R}$, $b_n\geqslant 0$ and $0<c<d$. Then*

$$\sum_{n=1}^{N}n^d b_n=\mathscr{O}(N^c(\log N)^a)\ \ \textit{as}\ N\to+\infty,$$

if and only if

$$\sum_{n=N}^{+\infty}b_n=\mathscr{O}(N^{c-d}(\log N)^a)\ \ \textit{as}\ N\to+\infty.$$

Proof. See [8, Lemma 4.1]. □

THEOREM 6. *Let $\gamma\in\mathbb{R}$, $0<\delta<k$ and $f\in W_2^k$. The following two conditions are equivalent:*

(A) $f\in\mathscr{L}ip_k(\delta,\gamma;2,\alpha,\beta)$,

(B) $\sum_{|n|\geqslant N}\lambda_n^{2r}|c_n(f)|^2 w_n^{(\alpha,\beta)}=\mathscr{O}\left(N^{-2\delta}(\log N)^{2\gamma}\right)$ *as* $N\to+\infty$.

Proof. We first note that the theorem is proved in the case where $\gamma=0$, by virtue of Theorem 4 and the fact that

$$\mathscr{L}ip_k(\delta,0;2,\alpha,\beta)\ =\ \mathscr{L}ip_k(\delta;2,\alpha,\beta).$$

Let us now show the first implication $(A)\Rightarrow(B)$: Let $f\in\mathscr{L}ip_k(\delta,\gamma;2,\alpha,\beta)$, with $\gamma\neq 0$. Then we have

$$\|\Delta_h^k(\Lambda^r f)\|_2=\mathscr{O}\left(h^\delta\left(\log\frac{1}{h}\right)^\gamma\right)\quad\text{as } h\to 0.$$

It follows from Lemma 7 that

$$\sum_{n=-\infty}^{+\infty} \lambda_n^{2r} |1-\varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k} |c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathcal{O}\left(h^{2\delta}\left(\log\frac{1}{h}\right)^{2\gamma}\right) \quad \text{as } h \to 0.$$

If $0 \leqslant |n| \leqslant \frac{1}{h}$, hence $|n|h \leqslant 1$, and the second assertion of Lemma 1, we obtain

$$\lambda_n^{4k} h^{4k} \leqslant \frac{1}{k_2^{2k}} |1-\varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k}.$$

Therefore,

$$\sum_{1 \leqslant |n| \leqslant \left[\frac{1}{h}\right]} n^{4k} \lambda_n^{2r} |c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathcal{O}\left(h^{2\delta-4k}\left(\log\frac{1}{h}\right)^{2\gamma}\right),$$

by virtue of $\lambda_n^2 \geqslant n^2$. Putting $N = 1/h$, we may write this inequality in the following form:

$$\sum_{1 \leqslant |n| \leqslant N} n^{4k} \lambda_n^{2r} |c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathcal{O}\left(N^{4k-2\delta} (\log N)^{2\gamma}\right).$$

Equivalent to

$$\sum_{n=1}^{N} n^{4k} \lambda_n^{2r} (|c_n(f)|^2 + |c_{-n}(f)|^2) w_n^{(\alpha,\beta)} = \mathcal{O}\left(N^{4k-2\delta} (\log N)^{2\gamma}\right).$$

From Lemma 10, we have

$$\sum_{n=1}^{N} \lambda_n^{2r} (|c_n(f)|^2 + |c_{-n}(f)|^2) w_n^{(\alpha,\beta)} = \mathcal{O}\left(N^{-2\delta} (\log N)^{2\gamma}\right).$$

Consequently

$$\sum_{|n| \geqslant N} \lambda_n^{2r} |c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathcal{O}\left(N^{-2\delta} (\log N)^{2\gamma}\right), \tag{62}$$

Thus, the first implication is proved.

Let's show the reverse implication $(B) \Rightarrow (A)$: Suppose now that

$$\sum_{|n| \geqslant N} \lambda_n^{2r} |c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathcal{O}\left(N^{-2\delta} (\log N)^{2\gamma}\right) \quad \text{as } N \to +\infty,$$

i.e.,

$$\sum_{n=N}^{+\infty} \lambda_n^{2r} (|c_n(f)|^2 + |c_{-n}(f)|^2) w_n^{(\alpha,\beta)} = \mathcal{O}\left(N^{-2\delta} (\log N)^{2\gamma}\right),$$

as $N \to +\infty$. It follows from Lemma 10 that

$$\sum_{n=1}^{N} n^{4k} \lambda_n^{2r} (|c_n(f)|^2 + |c_{-n}(f)|^2) w_n^{(\alpha,\beta)} = \mathcal{O}\left(N^{4k-2\delta} (\log N)^{2\gamma}\right). \tag{63}$$

According to (59), we write

$$\|\Delta_h^k(\Lambda^r f)\|_2^2 = A^{-1}\sum_{n=-\infty}^{+\infty} \lambda_n^{2r}|1-\varphi_{|n|}^{(\alpha,\beta)}(h)|^{2k}|c_n(f)|^2 w_n^{(\alpha,\beta)}$$
$$\leqslant A^{-1}(\mathscr{I}_1+\mathscr{I}_2) = A^{-1}\left(\sum_{0\leqslant|n|\leqslant N} + \sum_{|n|\geqslant N}\right).$$

It follows from (6), (61) and (63) that

$$\mathscr{I}_1 \leqslant c_1^{2k}h^{4k}\sum_{0\leqslant|n|\leqslant N}\lambda_n^{4k}\lambda_n^{2r}|c_n(f)|^2 w_n^{(\alpha,\beta)}$$
$$\leqslant (4c_1(\rho+1))^{2k}h^{4k}\sum_{1\leqslant|n|\leqslant N} n^{4k}\lambda_n^{2r}|c_n(f)|^2 w_n^{(\alpha,\beta)}$$
$$= (4c_1(\rho+1))^{2k}h^{4k}\sum_{n=1}^{N} n^{4k}\lambda_n^{2r}(|c_n(f)^2+|c_{-n}(f)|^2)w_n^{(\alpha,\beta)}$$
$$= \mathscr{O}\left(N^{4k-2\delta-4k}(\log N)^{2\gamma}\right) = \mathscr{O}\left(N^{-2\delta}(\log N)^{2\gamma}\right).$$

On the other hand, it follows from (4) and (62) that

$$\mathscr{I}_2 \leqslant 2^{2k}\sum_{|n|\geqslant N}\lambda_n^{2r}|c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathscr{O}\left(N^{-2\delta}(\log N)^{2\gamma}\right).$$

Consequently,

$$\|\Delta_h^k(\Lambda^r f)\|_2 = \mathscr{O}\left(h^\delta\left(\log\frac{1}{h}\right)^\gamma\right) \quad \text{as } h\to 0,$$

and this ends the proof of this theorem. □

COROLLARY 7. *Let* $0<\delta<k$ *and* $f\in W_2^k$. *If*

$$f\in \mathscr{L}ip_k(\delta,\gamma;2,\alpha,\beta),$$

then

$$\sum_{|n|\geqslant N}|c_n(f)|^2 w_n^{(\alpha,\beta)} = \mathscr{O}(N^{-2\delta-2r}(\log N)^{2\gamma}) \quad \textit{as } N\to+\infty.$$

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