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THE SMALLEST EIGENVALUE OF LARGE HANKEL MATRICES ASSOCIATED WITH A SEMICLASSICAL LAGUERRE WEIGHT

DAN WANG, MENGKUN ZHU* AND YANG CHEN

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Abstract. The smallest eigenvalue of large Hankel matrices generated by a semiclassical Laguerre weight, $z^{\alpha}\mathrm{e}^{-z^2+tz}$, where $z\in[0,\infty)$, $\alpha>-1$, and $t\in\mathbf{R}$, can be obtained through the asymptotics of the orthonormal polynomials $\mathscr{P}_n(z)$ with respect to this weight.

1. Introduction

One of the most important properties of the Hankel matrices $\mathbf{H}_n$ is to analyse the largest or smallest eigenvalues of the Hankel matrices generated by a given weight function $w(z)$, $z\in I\subseteq\mathbf{R}$, where

$$u_j := \int_I z^j w(z)dz, \quad j=0,1,2,\cdots, \tag{1}$$

are the moments and

$$\mathbf{H}_n := \Big(u_j\Big)_{j=0}^{n}, \quad n=0,1,2,\cdots.$$

The asymptotic behavior of the smallest eigenvalue, as $n\to\infty$, has been investigated by many authors. We refer to [2, 6, 9, 10, 11, 12, 14, 16, 18, 19, 20, 21] for more information about the smallest eigenvalue. Based on the support, the study of the smallest eigenvalue can be divided into two parts: finite and infinite. For finite cases, we list a few examples of such a connection between the weight with their supports and their associated references.

- $z\in[-1,1]$, $w(z)=1$, [16].
- $z\in[0,1]$, $w(z)=z^{\alpha}(1-z)^{\beta}$, $\alpha>-1$, $\beta>-1$, [19]; if $\alpha=\beta=0$, [16, 17].

The similarly for infinite cases,

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* Corresponding author.

- $z \in [0,\infty)$, $w(z) = z^{\alpha}\mathrm{e}^{-z^{\beta}}$, $\alpha > -1$, $\beta \geqslant \frac{1}{2}$, [21]; if $\alpha = 0$, $\beta = \frac{1}{2}$, [10]; if $\alpha = 0$, $\beta > \frac{1}{2}$, [6, 11].

- $z \in [0,\infty)$, $w(z) = z^{\alpha}\mathrm{e}^{-z-\frac{t}{z}}$, $\alpha > -1$, $t \geqslant 0$, [20].

- $z \in (-\infty,\infty)$, $w(z) = z^{\alpha}\mathrm{e}^{-|z|^{\alpha}}$, $\alpha > 1$, [9].

Let λ_n denote the smallest eigenvalue of $\mathbf{H}_n$. Widom and Wilf [17] were to sum up a 'universal' law for λ_n, i.e. if $w(z) > 0$, $z \in [a,b]$, and the Szegö condition is satisfied, the asymptotic form of the smallest eigenvalue is $An^{\frac{1}{2}}B^{-n}$, where $A > 0$, $0 < B < 1$. There were also a new criteria for the determinacy of the Hamburger moment problem. Berg, Chen and Ismail [1] have proved that the moment sequence (1) is determinate if and only if $\lambda_n \to 0$, as $n \to \infty$.

It is well known that λ_n is posed by the classical Rayleigh quotient

$$\lambda_n = \min\left\{ \frac{\sum\limits_{j,k=0}^{n} \bar{x}_j u_{j+k} x_k}{\sum\limits_{k=0}^{n} |x_k|^2} \,\middle|\, X := (x_0, x_1, \cdots, x_n)^T \in \mathbf{C}^{n+1} \setminus \{0\} \right\}. \tag{2}$$

There were many manuscripts mentioned how to compute λ_n and showed a lower bound for λ_n,

$$\lambda_n \geqslant \frac{2\pi}{\sum\limits_{j=0}^{n} \mathbb{K}_{jj}}, \tag{3}$$

where

$$\mathbb{K}_{jk} := \int_0^{2\pi} \mathscr{P}_j(\mathrm{e}^{i\phi}) \mathscr{P}_k(\mathrm{e}^{-i\phi}) d\phi,$$

and $\mathscr{P}_n(z)$ is a polynomials of degree n, orthonormal with respect to a given weight, see [6, 9, 16, 19, 20].

The paper is structured as follows. In Section 2, we present the main results of this paper. The proofs of these results are provided in Section 3 and Section 4, respectively.

2. The main results

In this paper, we consider the asymptotic behavior of the smallest eigenvalue λ_n of $\mathbf{H}_n$ generated by the semiclassical Laguerre weight

$$w(z) = z^{\alpha}\mathrm{e}^{-z^2+tz}, \quad \alpha > -1, \; t \in \mathbf{R}, \; z \in [0,\infty). \tag{4}$$

The moments associated with (4) are

$$\begin{aligned}u_k =& \int_0^{\infty} z^{k+\alpha} \mathrm{e}^{-z^2+tz} dz\\ =&\frac{1}{2}\Gamma\left(\frac{1+k+\alpha}{2}\right) {}_1F_1\left(\frac{1+k+\alpha}{2};\frac{1}{2};\frac{t^2}{4}\right)\\ &+\frac{t}{2}\Gamma\left(\frac{2+k+\alpha}{2}\right) {}_1F_1\left(\frac{2+k+\alpha}{2};\frac{3}{2};\frac{t^2}{4}\right),\end{aligned}$$

where ${}_1F_1(a;b;c)$ is the Kummer confluent hypergeometric function.

The main results of this paper are as follow:

THEOREM 1. *For $n \to \infty$, the asymptotic expression of the orthonormal polynomials associated with the weight* (4) *are*

$$\begin{aligned}\mathscr{P}_n(z) \sim (-1)^n (2\pi)^{-\frac{1}{2}} (-z)^{\frac{1}{4}-\frac{\alpha}{2}} b^{-\frac{1}{4}} \exp\left\{\frac{z^2}{2} + \frac{1}{2} b^{\frac{1}{2}} (-z)^{\frac{1}{2}} (2b+z) - \frac{tz}{2}\right.\\ \left. - \frac{1}{6} t b^{-\frac{1}{2}} (-z)^{\frac{1}{2}} (6b+z)\right\}, \quad z \notin [0,b],\end{aligned} \tag{5}$$

where

$$b = \frac{1}{3}(t + \sqrt{24n + t^2 + 12\alpha}). \tag{6}$$

THEOREM 2. *The smallest eigenvalue λ_n of large Hankel matrices $\boldsymbol{H}_n$ associated with the weight* (4) *can be given by*

$$\begin{aligned}\lambda_n \sim 4\pi^{\frac{3}{2}} b^{\frac{1}{2}} \left[\frac{t}{2} + \frac{1}{2} t b^{-2}\left(\frac{3}{4} - \frac{b}{2}\right) + 4b^{\frac{1}{2}}\right]^{\frac{1}{2}} \exp\left\{ -1 - 2b^{\frac{3}{2}} + b^{\frac{1}{2}}\right.\\ \left. + \frac{1}{3} t b^{-\frac{1}{2}} (6b-1) - t\right\}, \quad n \to \infty,\end{aligned} \tag{7}$$

with

$$b = \frac{1}{3}(t + \sqrt{24n + t^2 + 12\alpha}).$$

REMARK 1. For this problem, using the theory of Coulomb fluid [4, 5, 7],

$$v(z) = -\ln w(z) = z^2 - tz - \alpha \ln z, \tag{8}$$

supported on $[a,b]$, following the discussions of Chen and Lawrence [5, 6], Zhu, Emmart, Chen and Weems[21], here $a = 0$ and b satisfies the supplementary condition

$$\int_0^b \frac{z v'(z)}{\sqrt{(b-z)z}} dz = 2\pi n,$$

so we have (6) and

$$3b^2 - 2tb = 4\alpha + 8n. \tag{9}$$

Kindly reminder, $b := b_n$ depends on n, which is used in section 4.

3. Proof of Theorem 1

In this section, we will utilize the approach, established by Chen and Lawrence [5], which is based on the Coulomb fluid linear statistics, to derive the asymptotic behavior of orthonormal polynomials with respect to (4). In this case, the asymptotic expression of the monic orthogonal polynomials $\{P_n(z)\}$, [5, 8], are approximated by

$$P_n(z) \sim \mathrm{e}^{-S_1(z)-S_2(z)}, \tag{10}$$

where

$$\mathrm{e}^{-S_1(z)} = \frac{1}{2}\left[\left(\frac{z-b}{z}\right)^{\frac{1}{4}} + \left(\frac{z}{z-b}\right)^{\frac{1}{4}}\right], \quad z \notin [0,b] \tag{11}$$

$$\begin{aligned} S_2(z) =& \frac{1}{2\pi}\int_0^b \frac{v(x)}{\sqrt{(b-x)x}}\left[\frac{\sqrt{(z-b)z}}{x-z}+1\right]dx \\ &- n\ln\left(\frac{\sqrt{z}+\sqrt{z-b}}{2}\right)^2, \quad z \notin [0,b], \end{aligned} \tag{12}$$

and the relationship between the orthogonal polynomials $\{P_n(z)\}$ and the orthonormal polynomials $\{\mathscr{P}_n(z)\}$, [5], can be given by

$$\mathscr{P}_n(z) = \frac{\sqrt{2}\mathrm{e}^{\frac{A}{2}}}{\sqrt{b\pi}} P_n(z), \tag{13}$$

with

$$A = \int_0^b \frac{v(x)}{\pi\sqrt{(b-x)x}}dx - 2n\ln\frac{b}{4}, \tag{14}$$

where the orthonormal polynomials $\mathscr{P}_n(z)$ with respect to (4),

$$\int_0^b \mathscr{P}_n^2(z)w(z)dz = 1.$$

It mainly divides into two steps to prove the theorem 1.

Step 1*: The asymptotic of the monic orthogonal polynomials associated with* (4)
Firstly, let $\eta := -\frac{z}{b}$, $z \notin [0,b]$ and $|\eta| \ll 1$, it handles the below integral

$$\begin{aligned} &\frac{\sqrt{z(z-b)}}{2\pi}\int_0^b \frac{x^2-tx-\alpha\ln x}{(x-z)\sqrt{(b-x)x}}dx \\ =&\frac{\sqrt{z(z-b)}}{2\pi}\int_0^b \frac{x^2}{(x-z)\sqrt{(b-x)x}}dx - \frac{t\sqrt{z(z-b)}}{2\pi}\int_0^b \frac{x}{(x-z)\sqrt{(b-x)x}}dx \\ &- \frac{\alpha\sqrt{z(z-b)}}{2\pi}\int_0^b \frac{\ln x}{(x-z)\sqrt{(b-x)x}}dx \\ =&-\frac{3b^2\sqrt{z(z-b)}}{16z}\,{}_2F_1\left(1,\frac{5}{2};3;\frac{b}{z}\right) + \frac{tb\sqrt{z(z-b)}}{4z}\,{}_2F_1\left(1,\frac{3}{2};2;\frac{b}{z}\right) + \frac{\alpha}{2}\ln(-z) \\ &- \alpha\ln(\sqrt{\eta}+\sqrt{\eta+1}). \end{aligned} \tag{15}$$

We give the explanations for the above three integrals. The detailed evaluation of the first integral using the basic properties of the hypergeometric function is as follows:

$$\begin{aligned}
&\frac{\sqrt{z(z-b)}}{2\pi}\int_0^b \frac{x^2}{(x-z)\sqrt{(b-x)x}}dx \qquad (x:=by)\\
=&-\frac{b^2\sqrt{z(z-b)}}{2\pi z}\int_0^1 y^{\frac{3}{2}}(1-y)^{-\frac{1}{2}}\left(1-\frac{by}{z}\right)^{-1}dy\\
=&-\frac{b^2\sqrt{z(z-b)}}{2\pi z}\mathscr{B}\left(\frac{5}{2},\frac{1}{2}\right)\,{}_2F_1\left(1,\frac{5}{2};3;\frac{b}{z}\right),
\end{aligned}$$

where $\mathscr{B}(x,y)$ denotes the Beta function. Similarly, the second integral can be evaluated using the same approach. With the aid of the principal integral in Appendix A, the third integral is dealt with

$$-\frac{\alpha\sqrt{z(z-b)}}{2\pi}\int_0^b \frac{\ln x}{(x-z)\sqrt{(b-x)x}}dx=\frac{\alpha}{2}\ln\frac{-bz}{(\sqrt{-z}+\sqrt{b-z})^2}.$$

Subsequently, the another principal integral is conducted as

$$\frac{1}{2\pi}\int_0^b \frac{x^2-tx-\alpha\ln x}{\sqrt{(b-x)x}}dx=\frac{3b^2}{16}-\frac{tb}{4}-\frac{\alpha}{2}\ln\frac{b}{4}=\frac{\alpha}{4}+\frac{n}{2}-\frac{tb}{8}-\frac{\alpha}{2}\ln\frac{b}{4}, \tag{16}$$

where we use (9) instead of b^2.

Then, substituting $z=-b\eta$ and (8) into (12),

$$\begin{aligned}
-S_2(z)=&n\ln\frac{-b(\sqrt{\eta}+\sqrt{\eta+1})^2}{4}-\frac{1}{2\pi}\int_0^b \frac{x^2-tx-\alpha\ln x}{\sqrt{(b-x)x}}dx\\
&-\frac{\sqrt{z(z-b)}}{2\pi}\int_0^b \frac{x^2-tx-\alpha\ln x}{(x-z)\sqrt{(b-x)x}}dx;
\end{aligned} \tag{17}$$

taking (15) and (16) into (17),

$$-S_2(z)=\ln\left\{[2^{-2n-\alpha}\eta^{-\frac{\alpha}{2}}(-b)^n(\sqrt{\eta}+\sqrt{\eta+1})^{2n+\alpha}]\exp\left[\frac{tb}{8}-L(z)-\frac{\alpha+2n}{4}\right]\right\}, \tag{18}$$

where

$$L(z)=-\frac{3b^2\sqrt{z(z-b)}}{16z}\,{}_2F_1\left(1,\frac{5}{2};3;\frac{b}{z}\right)+\frac{tb\sqrt{z(z-b)}}{4z}\,{}_2F_1\left(1,\frac{3}{2};2;\frac{b}{z}\right).$$

In addition, substituting $z=-b\eta$ into (11),

$$\mathrm{e}^{-S_1(z)}=\frac{1}{2}\eta^{-\frac{1}{4}}(\eta+1)^{-\frac{1}{4}}[(\eta+1)^{\frac{1}{2}}+\eta^{\frac{1}{2}}]. \tag{19}$$

The asymptotic of the monic orthogonal polynomials associated with (4) are derived by combining (19), (18) with (10), i.e.

$$P_n(z) \sim (-1)^n b^n \eta^{-\frac{1}{4}-\frac{\alpha}{2}} 2^{-2n-\alpha-1} (\eta+1)^{-\frac{1}{4}} (\sqrt{\eta+1}+\sqrt{\eta})^{2n+\alpha+1} \times \exp\left\{\frac{tb}{8} - L(z) - \frac{\alpha+2n}{4}\right\}, \quad z \notin [0,b], \tag{20}$$

where

$$L(z) = -\frac{3b^2\sqrt{z(z-b)}}{16z}\, {}_2F_1\left(1,\frac{5}{2};3;\frac{b}{z}\right) + \frac{tb\sqrt{z(z-b)}}{4z}\, {}_2F_1\left(1,\frac{3}{2};2;\frac{b}{z}\right), \tag{21}$$
$$b = \frac{1}{3}(t+\sqrt{24n+t^2+12\alpha}).$$

Step 2: Asymptotic behaviors

We use the standard method to obtain the asymptotic orthonormal polynomials with respect to (4). Putting (8) into the identity relating (14),

$$A = \frac{3b^2}{8} - \frac{tb}{2} - (\alpha+2n)\ln\frac{b}{4} = \frac{\alpha+2n}{2} - (\alpha+2n)\ln\frac{b}{4} - \frac{tb}{4}, \tag{22}$$

where b^2 is replaced by (9).

Substituting (20) and (22) into (13), with some simplifications, $n \to \infty$,

$$\mathscr{P}_n(z) \sim 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}(-1)^n b^{-\frac{1}{2}}(-z)^{-\frac{\alpha}{2}}[\eta(\eta+1)]^{-\frac{1}{4}}\exp\{(2n+\alpha+1) \times \ln(\sqrt{\eta+1}+\sqrt{\eta}) - L(z)\}, \quad z \notin [0,b], \tag{23}$$

where

$$L(z) = -\frac{3b^2\sqrt{z(z-b)}}{16z}\, {}_2F_1\left(1,\frac{5}{2};3;\frac{b}{z}\right) + \frac{tb\sqrt{z(z-b)}}{4z}\, {}_2F_1\left(1,\frac{3}{2};2;\frac{b}{z}\right).$$

Next, we will simplify the representation of the orthonormal polynomials given by (23). Let's focus on the first part of the exponential term in (23), where $|\eta| \ll 1$ and $\eta = -\frac{z}{b}$:

$$\begin{aligned}
&(2n+\alpha+1)\ln(\sqrt{\eta+1}+\sqrt{\eta})\\
&\sim (2n+\alpha)\ln(\sqrt{\eta+1}+\sqrt{\eta}) = \frac{3b^2-2tb}{4}\sqrt{\eta}\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-\eta\right)\\
&= \left(\frac{3z^2}{4\eta^2}+\frac{tz}{2\eta}\right)\sum_{k=0}^{\infty}(-1)^k\frac{\eta^{k+1}\Gamma(k+\frac{1}{2})}{2\sqrt{\pi}(k+\frac{1}{2})\Gamma(k+1)}\\
&\sim \frac{1}{2}b^{\frac{1}{2}}(-z)^{\frac{1}{2}}(2b+z) - \frac{1}{6}tb^{-\frac{1}{2}}(-z)^{\frac{1}{2}}(6b+z).
\end{aligned} \tag{24}$$

Here, we utilize the inverse hyperbolic sine and the formula from Gradshteyn and Ryzhik ([13], cf. 9.121.26) to obtain the simplification.

As $\eta \to 0$, for $z = -b\eta$, with the aid of mathematical software, the second part of the exponential term for (23),

$$L(z) = \left[\frac{z}{2} - \frac{b}{2}\sqrt{\eta(\eta+1)}\right]t - \frac{z^2}{2} + b^2\sqrt{\eta}\sqrt{\eta+1}(2\eta-1) \sim -\frac{z^2}{2} + \frac{tz}{2}. \tag{25}$$

Substituting $\eta = -\frac{z}{b}$, (24) and (25) into (23), it arrives at (5).

REMARK 2. (5) with $t = 0$, we find $b = 2\times 3^{-\frac{1}{3}}(2n+\alpha)^{\frac{1}{2}}$; consequently, it covers the approximation result for the deformed Laguerre polynomials due to Zhu, Emmart, Chen and Weems [21] when $\beta = 2$ and $n \to \infty$,

$$\begin{aligned}\mathscr{P}_n(z) \sim (2\pi)^{-\frac{1}{2}}(-1)^n(-z)^{-\frac{\alpha}{2}-\frac{1}{4}}2^{-\frac{1}{4}}3^{\frac{1}{8}}(2n+\alpha)^{-\frac{1}{8}}\exp\Big\{&2^{-\frac{3}{2}}3^{-\frac{3}{4}}(2n+\alpha)^{\frac{3}{4}}(-z)^{\frac{1}{2}}\\ &\times[8+2\times 3^{\frac{1}{2}}(2n+\alpha)^{-\frac{1}{2}}z] + \frac{z^2}{2}\Big\}, \quad z \notin [0,\infty].\end{aligned}$$

4. Proof of Theorem 2

Before proving the theorem 2, we first reminder that b mentioned in (6) will be denoted as b_n, for convenience,

$$\begin{aligned} b_n := b &= \frac{1}{3}(t + \sqrt{24n + t^2 + 12\alpha}),\\ b_\mu :&= \frac{1}{3}(t + \sqrt{24\mu + t^2 + 12\alpha}). \end{aligned}$$

With $\mathscr{P}_n(z)$ having the form (5), we use the approach of [6, 16, 19, 20, 21] to determinate the asymptotic behavior of λ_n for large n. For sufficiently large μ and ν, the dominant contributions to $\mathbb{K}_{\mu\nu}$ are from the arc of the unit circle $|z| = 1$. Let $\delta > 0$ be an arbitrary positive number, which regulates the value of μ, ν to satisfy

$$n - \delta n^{\frac{1}{2}} \leqslant \mu,\ \nu \leqslant n, \quad n \to \infty; \tag{26}$$

meanwhile, considering $\theta = \phi - \pi$ and expanding $|\theta| \ll 1$, it shows

$$\begin{aligned}\mathbb{K}_{\mu\nu} &\sim \int_{-\varepsilon}^{\varepsilon}\mathscr{P}_\mu(-e^{i\theta})\mathscr{P}_\nu(-e^{-i\theta})d\theta\\ &\sim (2\pi)^{-1}(-1)^{\mu+\nu}b^{-\frac{1}{2}}\exp\left\{2b^{\frac{3}{2}} - 2b^{\frac{1}{2}} + \frac{b_\mu^{\frac{1}{2}}}{2} + \frac{b_\nu^{\frac{1}{2}}}{2} + 1 + t - \frac{tb^{-\frac{1}{2}}}{3}(6b-1)\right\}\\ &\quad\times\int_{-\varepsilon}^{\varepsilon}\exp\left\{\left[\frac{t}{2} + 4b^{\frac{1}{2}} + \frac{tb^{-\frac{1}{2}}}{2}\left(\frac{3}{4} - \frac{b}{2}\right)\right](-\theta^2)\right\}d\theta.\end{aligned} \tag{27}$$

It is emphasized that the contributions to the integral from $(-\infty,-\varepsilon)$ and (ε,∞) are sub-dominant compared to those from $[-\varepsilon,\varepsilon]$ as μ and ν tend to infinity. This is achieved by eliminating the linear terms since they remain bounded under the restriction on μ and ν.

Thus, dealing with (27) by the Laplace method, see Appendix B, $\mathbb{K}_{\mu\nu}$ can be rewritten as

$$\mathbb{K}_{\mu\nu} \sim (2\pi)^{-1}\pi^{\frac{1}{2}}(-1)^{\mu+\nu}b^{-\frac{1}{2}}\left[\frac{t}{2}+4b^{\frac{1}{2}}+\frac{tb^{-2}}{2}\left(\frac{3}{4}-\frac{b}{2}\right)\right]^{-\frac{1}{2}}$$
$$\times\exp\left\{2b^{\frac{3}{2}}-2b^{\frac{1}{2}}+\frac{b_{\mu}^{\frac{1}{2}}}{2}+\frac{b_{\nu}^{\frac{1}{2}}}{2}+1+t-\frac{tb^{-\frac{1}{2}}}{3}(6b-1)\right\}.$$

On the other hand, the leading behavior of (3) for large n is in turn found by replacing the sum by an integral, so

$$\begin{aligned}\frac{2\pi}{\lambda_n} &\sim \int_0^n \mathbb{K}_{\mu\nu}d\mu\\ &\sim (2\pi)^{-1}\pi^{\frac{1}{2}}b^{-\frac{1}{2}}\left[\frac{t}{2}+4b^{\frac{1}{2}}+\frac{tb^{-2}}{2}\left(\frac{3}{4}-\frac{b}{2}\right)\right]^{-\frac{1}{2}}\\ &\quad\times\exp\left\{2b^{\frac{3}{2}}-2b^{\frac{1}{2}}+1+t-\frac{tb^{-\frac{1}{2}}}{3}(6b-1)\right\}\int_0^n \exp\{b_{\mu}^{\frac{1}{2}}\}d\mu.\end{aligned}$$

Based on the indefinite integral $\int\exp\{x^{\frac{1}{2}}\}dx=(2\sqrt{x}-2)\exp\{\sqrt{x}\}+\text{constant}$, we can perform an analogous analysis to the above integral for large n. This leads us to the expression:

$$\frac{2\pi}{\lambda_n} \sim (2\pi)^{-1}b^{-\frac{1}{2}}\pi^{\frac{1}{2}}\left[\frac{t}{2}+4b^{\frac{1}{2}}+\frac{tb^{-2}}{2}\left(\frac{3}{4}-\frac{b}{2}\right)\right]^{-\frac{1}{2}}$$
$$\times\exp\left\{2b^{\frac{3}{2}}-b^{\frac{1}{2}}+1+t-\frac{tb^{-\frac{1}{2}}}{3}(6b-1)\right\},$$

which immediately yields (7).

REMARK 3. Taking $t=0$ for (7), $b=2\times 3^{\frac{1}{3}}(2n+\alpha)^{\frac{1}{2}}$, the result of Zhu, Emmart, Chen and Weems [21] with $\beta=2$ is covered too,

$$\lambda_n \sim 2^{\frac{15}{4}}3^{-\frac{3}{8}}\pi^{\frac{3}{2}}(2n+\alpha)^{\frac{3}{8}}\exp\{-1-2^{\frac{5}{2}}\times 3^{-\frac{3}{4}}(2n+\alpha)^{\frac{3}{4}}+2^{\frac{1}{2}}\times 3^{-\frac{1}{4}}(2n+\alpha)^{\frac{1}{4}}\}.$$

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Appendix A

$$\int_a^b \frac{1}{\sqrt{(b-z)(z-a)}}dz = \pi,$$
$$\int_a^b \frac{z}{\sqrt{(b-z)(z-a)}}dz = \frac{\pi(a+b)}{2},$$
$$\int_a^b \frac{1}{z\sqrt{(b-z)(z-a)}}dz = \frac{\pi}{\sqrt{ab}},$$
$$\int_a^b \frac{z^2}{\sqrt{(b-z)(z-a)}}dz = \frac{1}{8}(3a^2+2ab+3b^2)\pi,$$
$$\int_a^b \frac{1}{(x+z)\sqrt{(b-x)(x-a)}}dx = \frac{\pi}{\sqrt{(z+a)(z+b)}},$$
$$\int_a^b \frac{\ln(x+t)}{x\sqrt{(b-x)(x-a)}}dx = \frac{\pi}{\sqrt{ab}}\ln\left[\frac{(\sqrt{ab}+\sqrt{(t+a)(t+b)})^2-t^2}{(\sqrt{a}+\sqrt{b})^2}\right].$$

Appendix B

The Laplace method [15, 3] gives

$$\int_a^b f(t)\mathrm{e}^{-\lambda g(t)}dt \sim \mathrm{e}^{-\lambda g(c)}f(c)\sqrt{\frac{2\pi}{\lambda g''(c)}}, \quad \text{as} \quad \lambda \to \infty,$$

where g assumes a strict minimum over $[a,b]$ at an interior critical point c such that

$$g'(c)=0, \quad g''(c)>0, \quad \text{and} \quad f(c)\neq 0.$$

REFERENCES

[1] C. BERG, Y. CHEN AND M. E. H. ISMAIL, *Small eigenvalues of large Hankel matrices: The indeterminate case*, Math. Scand. **91** (2002) 67–81.

[2] C. BERG AND R. SZWARC, *The smallest eigenvalue of Hankel matrices*, Constr. Approx. **34** (2011) 107–133.

[3] N. G. DE BRUIJIN, *Asymptotic methods in Analysis*, New York: Interscience, 1958.

[4] Y. CHEN AND M. E. H. ISMAIL, *Thermodynamic relations of the Hermitian matrix ensembles*, J. Phys. A: Math. Gen. **30** (1997) 6633–6654.

[5] Y. CHEN AND N. LAWRENCE, *On the linear statistics of Hermitian random matrices*, J. Phys. A: Math. Gen. **31** (1998) 1141–1152.
[6] Y. CHEN AND N. LAWRENCE, *Small eigenvalues of large Hankel matrices*, J. Phys. A: Math. Gen. **32** (1999) 7305–7315.
[7] Y. CHEN AND M. R. MCKAY, *Coulomb fluid, Painlevé transcendents and the information theory of MIMO systems*, IEEE Trans. Inform. Theory **58** (2012) 4594–4634.
[8] M. CHEN AND Y. CHEN, *Singular linear statistics of the Laguerre unitary ensemble and Painlevé. III. Double scaling analysis*, J. Math. Phys. **56** (2015) 063506.
[9] Y. CHEN AND D. S. LUBINSKY, *Smallest eigenvalues of Hankel matrices for exponential weights*, J. Math. Anal. Appl. **293** (2004) 476–495.
[10] Y. CHEN, J. SIKOROWSKI AND M. ZHU, *Smallest eigenvalue of large Hankel matrices at critical point: Comparing conjecture with parallelised computation*, Appl. Math. Comput. **363** (2019) 124628.
[11] N. EMMART, Y. CHEN AND C. WEEMS, *Computing the smallest eigenvalue of large ill-conditioned Hankel matrices*, Commun. Comput. Phys. **18** (2015) 104–124.
[12] C. ESCRIBANO, R. GONZALO AND E. TORRANO, *Small eigenvalues of large Hermitian moment matrices*, J. Math. Anal. Appl. **374** (2011) 470–480.
[13] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, 7th ed. (Elsevier; Academic Press, Amsterdam, 2007), p. xlviii+1171, translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).
[14] D. S. LUBINSKY, *Condition numbers of Hankel matrices for exponential weights*, J. Math. Anal. Appl. **314** (2006) 266–285.
[15] G. PÓLYA AND G. SZEGÖ, *Problems and theorems in Analysis I*, Spring-Verlag, Berlin Herdelberg, New York, 1978.
[16] G. SZEGÖ, *On some Hermitian forms associated with two given curves of the complex plane*, Trans. Amer. Math. Soc. **40** (1936) 450–461.
[17] H. WIDOM AND H. WILF, *Small eigenvalues of large Hankel matrices*, Proc. Am. Math. Soc. **17** (1966) 338–344.
[18] H. WIDOM AND H. S. WILF, *Errata: small eigenvalues of large Hankel matrices*, Proc. Am. Math. Soc. **19** (1968) 1508.
[19] M. ZHU, Y. CHEN, N. EMMART AND C. WEEMS, *The smallest eigenvalue of large Hankel matrices*, Appl. Math. Comput. **334** (2018) 375–387.
[20] M. ZHU, Y. CHEN AND C. LI, *The smallest eigenvalue of large Hankel matrices generated by a singularly perturbed Laguerre weight*, J. Math. Phys. **61** (2020) 073502.
[21] M. ZHU, N. EMMART, Y. CHEN AND C. WEEMS, *The smallest eigenvalue of large Hankel matrices generated by a deformed Laguerre weight*, Math. Methods Appl. Sci. **42** (2019) 3272–3288.

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Dan Wang
School of Computer Science and Artificial Intelligence
Changzhou University
Changzhou 213164, China
e-mail: bohewan@126.com
bohewan@cczu.edu.cn

Mengkun Zhu
School of Mathematics and Statistics
Qilu University of Technology (Shandong Academy of Sciences)
Jinan 250353, China
e-mail: zmk@qlu.edu.cn

Yang Chen
Department of Mathematics, Faculty of Science and Technology
University of Macau, Avenida da Universidade
Taipa, Macau, China
e-mail: yayangchen@um.edu.mo
chenyayang57@gmail.com