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COUNTER–EXAMPLES CONCERNING BRECKNER–CONVEXITY

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Abstract. In this paper, we examine convexity type inequalities. Let D be a nonempty convex subset of a linear space, $c > 0$ and $\alpha : D - D \to \mathbb{R}$ be a given even function. The inequality

$$f\left(\frac{x+y}{2}\right) \leqslant cf(x) + cf(y) + \alpha(x-y) \qquad (x,y \in D)$$

is the focus of our examinations. We will construct an example to show that for $c = 1$, this Jensen type inequality does not imply the convexity of the function. Then, we compare this inequality with Hermite–Hadamard type inequalities.

1. Introduction

Denote by $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{R}_+$ the sets of real numbers, positive integers, and nonnegative real numbers, respectively. Let D be a nonempty convex subset of a linear space X and denote by D^* the set $\{x - y : x, y \in D\}$. Let $\alpha : D^* \to \mathbb{R}$ be a nonnegative even error function.

The convexity has many applications and many generalization. In the first step, we consider the following. We say that a function $f : D \to \mathbb{R}$ is α-Jensen convex, if for all $x, y \in D$,

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} + \alpha(x-y). \tag{1}$$

Many authors examined this inequality from many context. For example, Házy and Páles ([8, 10, 11]), Makó and Páles ([15, 17, 18]), Tabor and Tabor ([24, 25]), Tabor, Tabor and Zoldak ([27]). If α is constant zero, we have the notion of classical Jensen-convexity.

In this paper, we will examine the following Jensen type inequality, which is a kind of generalization of the previous notion. Let $c > 0$. We say that a function $f : D \to \mathbb{R}$ is (c, α)*-Jensen convex* if for all $x, y \in D$,

$$f\left(\frac{x+y}{2}\right) \leqslant cf(x) + cf(y) + \alpha(x-y). \tag{2}$$

When $c \neq \frac{1}{2}$ this inequality was examined by Breckner ([2, 3]), Breckner and Orbán ([4]), Házy [9], Burai and Házy [5], Burai, Házy and Juhász [6].

The following theorem is the famous Bernstein-Doetsch theorem ([1]).

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THEOREM A. *Let I be a nonvoid interval, $f : I \to \mathbb{R}$ be locally bounded from above on I and assume that f is Jensen-convex, then f is convex.*

In Section 2, we will prove that if $c \geqslant 1$, this connection is not valid between (c,α)-Jensen convexity and convexity type inequality.

Now let us recall the theorem of Nikodem, Riedel, and Sahoo from [22]. They proved that from an approximate convexity on an interval I, that is

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)+\varepsilon \qquad (x,y \in I),$$

we can get Hermite–Hadamard type inequalities, namely,

$$f\left(\frac{x+y}{2}\right) \leqslant \int_0^1 f(tx+(1-t)y)dt+\varepsilon \qquad (x,y \in I),$$

and

$$\int_0^1 f(tx+(1-t)y)dt \leqslant \frac{f(x)+f(y)}{2}+\varepsilon \qquad (x,y \in I).$$

But the converse implications are not true. In fact they constructed some counter-example. In Section 3, we would like to comprise the new generalized Jensen-convexity type inequality ((c,α)-Jensen convexity) and Hermite–Hadamard type inequalities and we will also construct some counter-examples.

2. Counter-examples concerning Bernstein–Doetsh theorem

For the sake of simplicity, assume that $X=\mathbb{R}$ and $D=I$ is a real interval of $\mathbb{R}$ and $\alpha=0$. Then (2) reduces to,

$$f\Big(\frac{x+y}{2}\Big) \leqslant cf(x)+cf(y) \qquad (x,y \in I). \tag{3}$$

In the following, we will call this inequality *c-Jensen inequality*. With the substitution $x=y$, we have that $0 \leqslant (2c-1)f(x)$. This means that if $c>\frac{1}{2}$ then $f(x) \geqslant 0\,(x \in I)$ and if $c<\frac{1}{2}$ then $f(x) \leqslant 0\,(x \in I)$. We will consider the first case. We are looking for functions $\varphi : [0,1[\to \mathbb{R}$ such that, for all $t \in [0,1[$ and $x,y \in I$, f satisfies the following convexity type inequality:

$$f(tx+(1-t)y) \leqslant \varphi(t)f(x)+\varphi(1-t)f(y) \tag{4}$$

In the sequel, we will construct an example, which shows, there are no such functions in the case $c \geqslant 1$

PROPOSITION 1. *Assume that a function $f : I \to \mathbb{R}$ is nonnegative and monotone increasing, then it is also 1-Jensen convex.*

Proof. Let $x \leqslant y$ be elements of I, then $x \leqslant \frac{x+y}{2} \leqslant y$. Since f is nondecreasing and nonnegative, we have that,

$$f\left(\frac{x+y}{2}\right) \leqslant f(y) \leqslant f(x)+f(y),$$

which means that f is 1-Jensen convex. $\square$

The following easy-to-prove propositions will be useful in the sequel.

PROPOSITION 2. *Assume that* $f : I \to \mathbb{R}$ *is* 1*-Jensen convex, then, for all* $d > 0$, $f + d$ *is also* 1*-Jensen convex.*

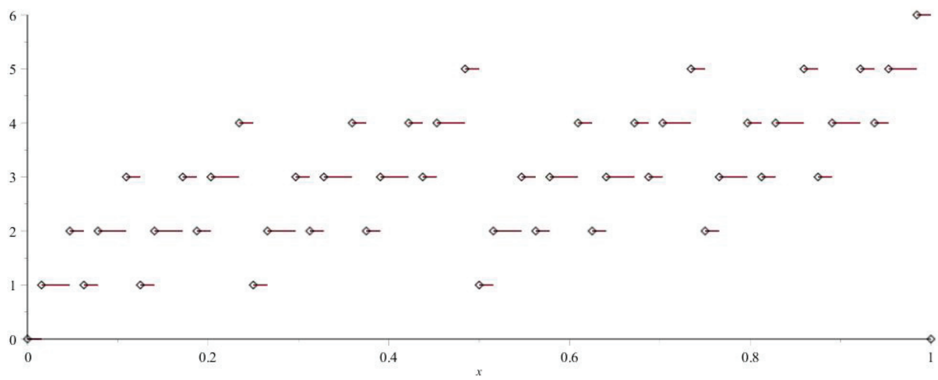
PROPOSITION 3. *Let* $\frac{1}{2} \leqslant c \leqslant d$. *Assume that* $f : I \to \mathbb{R}$ *is* c*-Jensen convex, then, it is also* d*-Jensen convex.*

Let's consider our first example, namely, for $n \in \mathbb{N}$ and $x \in [0,1[$ let,

$$f_n(x) := \sum_{k=0}^{n} ([2^{k+1}x] - 2[2^k x]), \qquad 0 \leqslant x < 1. \tag{5}$$

REMARK. It is easy to see that the function $f_n(x)$ is the number of 1's of the binary form of $[2^{n+1}x]$.

The following picture will show the graph of f_n, when $n = 5$.



THEOREM 4. *The function* $f_n : [0,1[\to \mathbb{R}$ *defined by* (5) *is* 1*-Jensen convex, but not convex in the sense of* (4)*, i.e., for all* $n \in \mathbb{N}$*, there exist* $\lambda_n \in \mathbb{R}$*, with* $\lim_{n\to\infty} \lambda_n = \infty$*,* $t_n \in]0,1[$ *and* $x_n, y_n \in [0,1[$*, such that*

$$f_n(t_n x_n + (1-t_n)y_n) > \lambda_n f_n(x_n) + \lambda_n f_n(y_n). \tag{6}$$

Proof. By the definition, it can be seen that

$$f_{n+1}(x) = \begin{cases} f_n(2x) & \text{if} \quad 0 \leqslant x < \frac{1}{2} \\ f_n(2x-1)+1 & \text{if} \quad \frac{1}{2} \leqslant x < 1. \end{cases} \tag{7}$$

It is also easy to see that, $f_n(x + \frac{1}{2}) = f_n(x) + 1$, if $0 \leqslant x < \frac{1}{2}$. We will prove the 1-Jensen convexity of f_n by induction. If $n = 1$, the function f_1 is monotone increasing, thus it is 1-convex. Now suppose that the statement is true for $n \in \mathbb{N}$ and consider the case $n+1$. If $0 \leqslant x \leqslant y < \frac{1}{2}$, we can get 1-Jensen convexity of f_{n+1} by the induction assumption. If $\frac{1}{2} \leqslant x \leqslant y < 1$, we can get 1-Jensen convexity of f_{n+1} by the induction assumption and proposition 2. Now assume that $0 \leqslant x < \frac{1}{2}$ and $\frac{1}{2} \leqslant y < 1$. Then there are two cases: $\frac{1}{4} \leqslant \frac{x+y}{2} < \frac{1}{2}$ or $\frac{1}{2} \leqslant \frac{x+y}{2} < \frac{3}{4}$. Consider the case $\frac{1}{4} \leqslant \frac{x+y}{2} < \frac{1}{2}$ (the proof of the other case is very similar). Then, using (7)

$$\begin{aligned} f_{n+1}(x) + f_{n+1}(y) &= f_n(2x) + f_n(2y-1) + 1 \\ &\geqslant f_n\left(\frac{2x+2y-1}{2}\right) + 1 \\ &= f_n\left(x+y-\frac{1}{2}\right) + 1 = f_n(x+y) - 1 + 1 \\ &= f_n(x+y) \\ &= f_{n+1}\left(\frac{x+y}{2}\right). \end{aligned}$$

This means the 1-Jensen convexity of f_n.

Let's see the proof of the nonconvexity of f_n. We prove the inequality (6). Let $n \geqslant 3$ be an integer and let $\lambda_n = \frac{1}{2^{n-1}}$, $x_n = 0$ and $y_n = \frac{1}{2}$. Then, using $f_n(0) = 0$, $f_n(\frac{1}{2}) = 1$ and $f_n(x) \geqslant 0$ $(x \in [0,1[)$, we get

$$\begin{aligned} f_n\left(\tfrac{1}{2^{n-1}} \cdot 0 + \left(1 - \tfrac{1}{2^{n-1}}\right) \cdot \tfrac{1}{2}\right) &= f_n\left(\frac{2^{n-1}-1}{2^n}\right) \\ &= f_n\left(\frac{1+2+2^2+\cdots+2^{n-2}}{2^n}\right) \\ &= 1+1+1+\cdots+1 = (n-2) \\ &> \frac{(n-2)}{2} \cdot 0 + \frac{(n-2)}{2}, \end{aligned}$$

which proves (6) holds. □

3. Hermite–Hadamard type inequalities and (c,α)-Jensen convexity

In the sequel, we will use the following notion. We say that a function $f : D \to \mathbb{R}$ has got a *radially property*, if for all $x, y \in D$, the function $g_{x,y} : [0,1] \to \mathbb{R}$ defined by

$$g_{x,y}(t) = f(tx + (1-t)y) \qquad t \in [0,1] \tag{8}$$

has got the property. For example, f is *radially bounded*, if for $x, y \in D$, the function $g_{x,y}$ is bounded. Theorem 5 and theorem 6 show that (c,α)-Jensen convexity implies Hermite–Hadamard type inequalities.

THEOREM 5. *Let $c > 0$ and $\alpha : D^* \to \mathbb{R}$ be nonnegative even error function, with for all $u \in D^*$, the map $s \mapsto \alpha(su)$ is Lebesgue integrable on $[-\frac{1}{2}, \frac{1}{2}]$. If $f : D \to$*

$\mathbb{R}$ is radially Lebesgue integrable and (c,α)-Jensen convex, then it also satisfies the following lower Hermite–Hadamard type inequality:

$$f\left(\frac{x+y}{2}\right) \leqslant 2c\int_0^1 f(tx+(1-t)y)dt + \int_0^1 \alpha((1-2t)(x-y))dt \qquad (x,y\in D). \quad (9)$$

Proof. Assume that f is (c,α)-Jensen convex. Substituting x by $tx+(1-t)y$ and y by $(1-t)x+ty$, we have that

$$f\left(\frac{x+y}{2}\right) \leqslant cf(tx+(1-t)y) + cf((1-t)x+ty) + \alpha((1-2t)(x-y)).$$

Then, integrating with respect to t on $[0,1]$, we can get the lower Hermite–Hadamard type inequality (9) holds. □

THEOREM 6. *Let $0<c<1$ and $\alpha : D^* \to \mathbb{R}$ is an error function, with for all $u\in D^*$, the map $s\mapsto \alpha(su)$ is Lebesgue integrable on $[0,1]$. If $f : D\to\mathbb{R}$ is radially Lebesgue integrable and (c,α)-Jensen convex, then it also satisfies the following upper Hermite–Hadamard type inequality:*

$$\int_0^1 f(tx+(1-t)y)dt \leqslant \frac{c}{1-c}f(x) + \frac{c}{1-c}f(y) + \int_0^1 \alpha(t(x-y))dt \qquad (x,y\in D). \tag{10}$$

Proof. Let $0\leqslant t\leqslant \frac{1}{2}$, then substituting x by $2tx+(1-2t)y$ in the (c,α)-Jensen convex inequality, we have that,

$$f\left(\frac{(2tx+(1-2t)y)}{2}\right) \leqslant cf(2tx+(1-2t)y)) + cf(y) + \alpha(2t(x-y))$$

Integrating the above inequality with respect to t on the interval $[0,\frac{1}{2}]$, we have that

$$\int_0^{\frac{1}{2}} f(tx+(1-t)y)dt \leqslant c\int_0^{\frac{1}{2}} f(2tx+(1-2t)y))dt + cf(y) + \int_0^{\frac{1}{2}} \alpha(2t(x-y))dt.$$

Making some natural integral substitution, we have that,

$$\int_0^{\frac{1}{2}} f(tx+(1-t)y)dt \leqslant \frac{c}{2}\int_0^1 f(tx+(1-t)y))dt + cf(y) + \frac{1}{2}\int_0^1 \alpha(t(x-y))dt. \tag{11}$$

Let $\frac{1}{2}\leqslant t\leqslant 1$, then substituting y by $(2t-1)x+(2-2t)y$ in the (c,α)-Jensen convex inequality, we have that,

$$f\left(\frac{x+((2t-1)x+(2-2t)y))}{2}\right) \leqslant cf(x)+cf((2t-1)x+(2-2t)y))+\alpha((2-2t)(x-y))$$

Integrating the above inequality with respect to t on the interval $[\frac{1}{2},1]$, we have that

$$\int_{\frac{1}{2}}^{1} f(tx+(1-t)y)dt \leqslant cf(x)+c\int_{\frac{1}{2}}^{1} f((2t-1)x+(2-2t)y)dt+\int_{\frac{1}{2}}^{1} \alpha((2-2t)(x-y))dt.$$

Making some natural integral substitution and using the symmetry of α, we have that,

$$\int_{\frac{1}{2}}^{1} f(tx+(1-t)y)dt \leqslant cf(x)+\frac{c}{2}\int_{0}^{1} f(tx+(1-t)y)dt+\frac{1}{2}\int_{0}^{1} \alpha((t(x-y))dt. \tag{12}$$

Adding the inequality (11) and (12), and rearranging the inequality, we got, we have the upper Hermite–Hadamard type inequality (10). $\square$

In the following theorem, we construct a function, which satisfies a lower Hermite–Hadamard type inequality, but, for all $n\in\mathbb{N}$, it is not n-Jensen convex.

THEOREM 7. *Let $c\geqslant\frac{3}{2}$. The following function*

$$f(x)=x(1-x), \qquad (x\in[0,1])$$

satisfies the lower Hermite–Hadamard type inequality

$$f\left(\frac{x+y}{2}\right)\leqslant\frac{c}{y-x}\int_{x}^{y} f(t)dt \qquad (x,y\in[0,1]), \tag{13}$$

but it is not n-Jensen convex, that is there exists $x,y\in[0,1]$ such that

$$f\left(\frac{x+y}{2}\right)>nf(x)+nf(y).$$

Proof. Computing the right hand side of (13), we have that

$$\frac{c}{y-x}\int_{x}^{y} t(1-t)dt=\frac{c}{y-x}\left[\frac{t^2}{2}-\frac{t^3}{3}\right]_{x}^{y}=c\left(\frac{x+y}{2}-\frac{x^2+xy+y^2}{3}\right).$$

Computing the left hand side of (13), we have that

$$f\left(\frac{x+y}{2}\right)=\frac{x+y}{2}\left(1-\frac{x+y}{2}\right)=\frac{x+y}{2}-\frac{x^2+2xy+y^2}{4}.$$

Combining the two side, we have to prove that,

$$\frac{x+y}{2}-\frac{x^2+2xy+y^2}{4}\leqslant c\left(\frac{x+y}{2}-\frac{x^2+xy+y^2}{3}\right),$$

that is

$$0\leqslant 3(c-1)(x+y)+(3-2c)xy+\left(\frac{3}{2}-2c\right)(x^2+y^2).$$

Using the classical identity beetween the arithemitc and geometric means, and the fact $3-2c \leqslant 0$, then $x,y \in [0,1]$, we have that

$$\begin{aligned}
&3(c-1)(x+y)+(3-2c)xy+\left(\frac{3}{2}-2c\right)(x^2+y^2)\\
\geqslant\ &3(c-1)(x+y)+(3-2c)\cdot\frac{x^2+y^2}{2}+\left(\frac{3}{2}-2c\right)(x^2+y^2)\\
=\ &3(c-1)(x+y)-3(c-1)(x^2+y^2)\\
=\ &=3(c-1)(x(1-x)+y(1-y))\geqslant 0,
\end{aligned}$$

which proves that (13) holds. On the other hand, for all $n\in\mathbb{N}$, the function f is not n-Jensen convex, since

$$0=nf(1)+nf(0)<f\left(\frac{x+y}{2}\right)=f\left(\frac{1}{2}\right)=\frac{1}{4}. \qquad \square$$

In the following theorem, for all $n\in\mathbb{N}$, we construct a function, which satisfies an upper Hermite–Hadamard type inequality, but, for all it is not n-Jensen convex. Similarly, than in [22], but it is also useable in our case.

THEOREM 8. *For $n\in\mathbb{N}$, let*

$$f_n(x) := -\ln(|x|+e^{-2n})+1, \qquad \text{if} \qquad |x|\leqslant 1-e^{-2n}.$$

Then, for $n\in\mathbb{N}$, f_n is a continuous function which satisfies the following upper Hermite–Hadamard type inequality,

$$\frac{1}{y-x}\int_x^y f(t)dt \leqslant f(x)+f(y) \qquad x<y \tag{14}$$

but it is not n-Jensen convex, i.e. there exists x,y such that

$$f\left(\frac{x+y}{2}\right)>nf(x)+nf(y)$$

Proof. Substituting f_n in (13), we have that for all $-(1-e^{-n})<x<y<1-e^{-n}$

$$\frac{1}{y-x}\int_x^y\left(-\ln(|t|+e^{-2n})+1\right)dt \leqslant -\ln(|x|+e^{-2n})+1-\ln(|y|+e^{-2n})+1$$

This inequality is equivalent to

$$\frac{1}{y-x}\int_x^y\left(-\ln(|t|+e^{-2n}\right)dt \leqslant -\ln(|x|+e^{-2n})-\ln(|y|+e^{-2n})+1, \tag{15}$$

which is not else than the inequality, which was proved by Nikodem, Riedel and Sahoo in [22]. Since the function $x\mapsto -\ln(|x|+e^{-2n})$ is nonnegative on $[-(1-e^{-2n}),1-e^{-2n}]$, (15) implies (14). On the other hand,

$$nf(-(1-e^{-2n}))+nf(1-e^{-2n})=2n<f\left(\frac{-(1-e^{-2n})+1-e^{-2n}}{2}\right)=f(0)=2n+1,$$

which shows that our counter-example is correct. $\square$

OPEN PROBLEM. *Investigating the Hermite–Hadamard type inequalities,*

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{c_1}{y-x}\int_x^y f(t)dt \qquad (x<y,\, x,y\in I)$$

and

$$\frac{1}{y-x}\int_x^y f(t)dt \leqslant c_2 f(x)+c_2 f(y) \qquad (x<y,\, x,y\in I)$$

The case $1<c_1<\frac{3}{2}$ *and* $\frac{1}{2}<c_2<1$ *are open problems. We suspect that, counter-examples can be constructed in also these cases.*

REFERENCES

[1] F. BERNSTEIN AND G. DOETSCH, *Zur Theorie der konvexen Funktionen*, Math. Ann., **76** (4): 514–526, 1915.

[2] W. W. BRECKNER, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, Publ. Inst. Math. (Beograd) **23** (1978), 13–20.

[3] W. W. BRECKNER, *Rational s-convexity, a generalized Jensen-convexity*, Cluj University Press, 2011, p. 165

[4] W. W. BRECKNER AND G. ORBÁN, *Continuity properties of rationally s-convex mappings with values in ordered topological liner space*, “Babes-Bolyai” University, Kolozsvár, 1978.

[5] P. BURAI AND A. HÁZY, *On approximately h-convex functions*, J. Convex Anal., **18** (2): 447–454, 2011.

[6] P. BURAI, A. HÁZY AND T. JUHÁSZ, *On approximately Breckner s-convex functions*, J. Convex Anal., **40** (1): 91–99, 2011.

[7] A. HÁZY, *On approximate t-convexity*, Math. Inequal. Appl., **8** (3): 389–402, 2005.

[8] A. HÁZY, *On the stability of t-convex functions*, Aequationes Math., **74** (3): 210–218, 2007.

[9] A. HÁZY, *Bernstein-Doetsch type results for (k,h)-convex functions*, Miskolc Math. Notes, **13** (2): 325–336, 2012.

[10] A. HÁZY AND ZS. PÁLES, *On approximately midconvex functions*, Bull. London Math. Soc., **36** (3): 339–350, 2004.

[11] A. HÁZY AND ZS. PÁLES, *On approximately t-convex functions*, Publ. Math. Debrecen, **66**: 489–501, 2005.

[12] A. HÁZY AND ZS. PÁLES, *On a certain stability of the Hermite–Hadamard inequality*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **465** (2102): 571–583, 2009.

[13] D. H. HYERS AND S. M. ULAM, *Approximately convex functions*, Proc. Amer. Math. Soc., **3**: 821–828, 1952.

[14] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities*, vol. 489 of *Prace Naukowe Uniwersytetu Śląskiego w Katowicach*, Państwowe Wydawnictwo Naukowe – Uniwersytet Śląski, Warszawa-Kraków-Katowice, 1985.

[15] J. MAKÓ AND ZS. PÁLES, *Approximate convexity of Takagi type functions*, J. Math. Anal. Appl., **369**: 545–554, 2010.

[16] J. MAKÓ AND ZS. PÁLES, *Strengthening of strong and approximate convexity*, Acta Math. Hungar., **132** (1–2): 78–91, 2011.

[17] J. MAKÓ AND ZS. PÁLES, *On φ-convexity*, Publ. Math. Debrecen, **80**: 107–126, 2012.

[18] J. MAKÓ AND ZS. PÁLES, *On approximately convex Takagi type functions*, Proc. Amer. Math. Soc., **141**: 2069–2080, 2013.

[19] D. S. MITRINOVIĆ AND I. B. LACKOVIĆ, *Hermite and convexity*, Aequationes Math., **28**: 229–232, 1985.

[20] A. MUREŃKO, JA. TABOR, AND JÓ. TABOR, *Applications of de Rham Theorem in approximate midconvexity*, J. Diff. Equat. Appl., **18**: 335–344, 2012.

[21] C. T. NG AND K. NIKODEM, *On approximately convex functions*, Proc. Amer. Math. Soc., **118** (1): 103–108, 1993.

[22] K. NIKODEM, T. RIEDEL, AND P. K. SAHOO, *The stability problem of the Hermite-Hadamard inequality*, Math. Inequal. Appl., **10** (2): 359–363, 2007.
[23] ZS. PÁLES, *On approximately convex functions*, Proc. Amer. Math. Soc., **131** (1): 243–252, 2003.
[24] JA. TABOR AND JÓ. TABOR, *Generalized approximate midconvexity*, Control Cybernet., **38** (3): 655–669, 2009.
[25] JA. TABOR AND JÓ. TABOR, *Takagi functions and approximate midconvexity*, J. Math. Anal. Appl., **356** (2): 729–737, 2009.
[26] JA. TABOR, JÓ. TABOR, AND M. ŻOŁDAK, *Approximately convex functions on topological vector spaces*, Publ. Math. Debrecen, **77**: 115–123, 2010.
[27] JA. TABOR, JÓ. TABOR, AND M. ŻOŁDAK, *Optimality estimations for approximately midconvex functions*, Aequationes Math., **80**: 227–237, 2010.

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