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# APPROXIMATION BY NÖRLUND MEANS WITH RESPECT TO WALSH SYSTEM IN LEBESGUE SPACES

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*Abstract.* In this paper we improve and complement a result by Móricz and Siddiqi [12]. In particular, we prove that their inequality of the Nörlund means with respect to the Walsh system holds also without their additional condition. Moreover, we prove some new approximation results and inequalities in Lebesgue spaces for any $1 \leqslant p < \infty$.

## 1. Introduction

Concerning some definitions and notations used in this introduction we refer to Section 2. Fejér's theorem shows that (see e.g. [9] and [10]) if one replaces ordinary summation by Fejér means $\sigma_n$, defined by

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f,$$

then, for any $1 \leqslant p \leqslant \infty$, there exists an absolute constant $C_p$, depending only on $p$ such that the inequality

$$\|\sigma_n f\|_p \leqslant C_p \|f\|_p$$

holds. Moreover, (see e.g. [16]) let $1 \leqslant p \leqslant \infty$, $2^N \leqslant n < 2^{N+1}$, $f \in L^p(G)$ and $n \in \mathbb{N}$. Then the following inequality holds:

$$\|\sigma_n f - f\|_p \leqslant 3\sum_{s=0}^{N} \frac{2^s}{2^N}\omega_p\left(1/2^s, f\right). \tag{1}$$

It follows that if $f \in lip(\alpha, p)$, i.e.

$$\omega_p\left(1/2^n, f\right) = O\left(1/2^{n\alpha}\right), \text{ as } n \to \infty,$$

where

$$\omega_p\left(1/2^k, f\right) := \sup_{0 \leqslant |h| \leqslant 1/2^k} \|f(x+h) - f(x)\|_p.$$

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then

$$\|\sigma_n f - f\|_p = \begin{cases} O\left(1/2^N\right), & \text{if} \quad \alpha > 1, \\ O\left(N/2^N\right), & \text{if} \quad \alpha = 1, \\ O\left(1/2^{N\alpha}\right), & \text{if } \alpha < 1. \end{cases}$$

Moreover, (see [16]) if $1 \leqslant p < \infty$, $f \in L^p(G)$ and

$$\|\sigma_{2^n} f - f\|_p = o\left(1/2^n\right), \text{ as } n \to \infty,$$

then $f$ is a constant function.

Boundedness of maximal operators of Vilenkin-Fejer means and weak-$(1,1)$ type inequality

$$\mu\left(\sigma^* f > \lambda\right) \leqslant \frac{c}{\lambda} \|f\|_1, \qquad \left(f \in L^1(G), \ \lambda > 0\right)$$

can be found in Zygmund [21] for trigonometric series, in Schipp [17] for Walsh series and in Pál, Simon [14] and Weisz [19, 20] for bounded Vilenkin series.

Convergence and summability of Nörlund means were studied by several authors. We mention Baramidze, Persson and G. Tephnadze [2] (see also [1], [3], [4] and [5]), Fridli, Manchanda, Siddiqi [8], Persson, Tephnadze and Weisz [16] (see also [15]), Blahota and Nagy [6] (see also [7] and [13]). Móricz and Siddiqi [12] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of $L^p$ functions in norm. In particular, they proved that if $f \in L^p(G)$, $1 \leqslant p \leqslant \infty$, $n = 2^j + k$, $1 \leqslant k \leqslant 2^j$ $(n \in \mathbb{N}_+)$ and $(q_k, k \in \mathbb{N})$ is a sequence of non-negative numbers, such that

$$\frac{n^{\gamma-1}}{Q_n^{\gamma}} \sum_{k=0}^{n-1} q_k^{\gamma} = O(1), \quad \text{for some} \quad 1 < \gamma \leqslant 2, \tag{2}$$

then

$$\|t_n f - f\|_p \leqslant \frac{C_p}{Q_n} \sum_{i=0}^{j-1} 2^i q_{n-2^i} \omega_p\left(\frac{1}{2^i}, f\right) + C_p \omega_p\left(\frac{1}{2^j}, f\right), \tag{3}$$

when $(q_k, k \in \mathbb{N})$ is non-decreasing, while

$$\|t_n f - f\|_p \leqslant \frac{C_p}{Q_n} \sum_{i=0}^{j-1} \left(Q_{n-2^i+1} - Q_{n-2^{i+1}+1}\right) \omega_p\left(\frac{1}{2^i}, f\right) + C_p \omega_p\left(\frac{1}{2^j}, f\right),$$

when $(q_k, k \in \mathbb{N})$ is non-increasing.

In this paper we improve and complement a result by Móricz and Siddiqi [12]. In particular, we prove that their estimate of the Nörlund means with respect to the Walsh system holds also without their additional condition. Moreover, we prove a similar approximation result in Lebesgue spaces for any $1 \leqslant p < \infty$.

The paper is organized as follows: The main results are presented, proved and discussed in Section 3. In particular, Theorems 1, 2 and 3 are parts of this new approach. In order not to disturb the presentations in Section 3, we use Section 2 for some necessary preliminaries.

## 2. Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by $Z_2 := \{0,1\}$ the additive group of integers modulo 2. Define the group $G$ as the complete direct product of the group $Z_2$ with the product of the discrete topologies of $Z_2$'s. The direct product $\mu$ of the measures $\mu^*(\{j\}) := 1/2 \ (j \in Z_2)$ is the Haar measure on $G$ with $\mu(G) = 1$. The elements of $G$ are represented by the sequences

$$x := (x_0, x_1, \ldots, x_k, \ldots) \qquad (x_k \in Z_2).$$

It is easy to give a base for the neighborhood of $G$, namely

$$I_0(x) := G, \quad I_n(x) := \{y \in G \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \ (x \in G, \ n \in \mathbb{N}).$$

Denote $I_n(0)$ by $I_n$ i.e $I_n := I_n(0)$. It is well-known that every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_j 2^j, \quad \text{where} \quad n_j \in Z_2 \quad (j \in \mathbb{N})$$

and only a finite number of $n_j$'s differ from zero.

First define the Rademacher functions as

$$r_k(x) := (-1)^{x_k}, \quad (k \in \mathbb{N}).$$

Now we define the Walsh system $w := (w_n : n \in \mathbb{N})$ on $G$ as

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L^2(G)$ (see e.g. [18]).

If $f \in L^1(G)$, then we can define the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Walsh system in the usual manner:

$$\widehat{f}(k) := \int_G f w_k d\mu, \quad (k \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \quad (n \in \mathbb{N}_+, \ S_0 f := 0),$$

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f, \quad (n \in \mathbb{N}_+),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+),$$

$$K_n := \frac{1}{n} \sum_{k=1}^{n} D_k, \quad (n \in \mathbb{N}_+).$$

Recall that for Dirichlet and Fejér kernels $D_n$ and $K_n$ we have that (see e.g. [9])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{4}$$

$$D_{2^n - m}(x) = D_{2^n}(x) - w_{2^n - 1}(x) D_m(x), \; 0 \leqslant m < 2^n \tag{5}$$

$$n|K_n| \leqslant 3 \sum_{l=0}^{|n|} 2^l |K_{2^l}|, \tag{6}$$

where $|n| =: \max\{j \in \mathbb{N}, n_j \neq 0\}$ and

$$\int_G K_n(x) d\mu(x) = 1, \qquad \sup_{n\in\mathbb{N}} \int_G |K_n(x)| d\mu(x) \leqslant 2. \tag{7}$$

Moreover, if $n > t, \; t, n \in \mathbb{N}$, then

$$K_{2^n}(x) = \begin{cases} 2^{t-1}, & x \in I_t \backslash I_{t+1}, \quad x - e_t \in I_n, \\ \frac{2^n+1}{2}, & x \in I_n, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

The $n$-th Nörlund mean $t_n$ of the Fourier series of $f$ is defined by

$$t_n f := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \quad \text{where} \quad Q_n := \sum_{k=0}^{n-1} q_k. \tag{9}$$

We always assume that $\{q_k : k \geqslant 0\}$ is a sequence of nonnegative numbers, where $q_0 > 0$ and $\lim_{n\to\infty} Q_n = \infty$. Then the summability method (9) generated by $\{q_k : k \geqslant 0\}$ is regular if and only if (see [11])

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

The following representation play central roles in the sequel

$$t_n f(x) = \int_G f(t) F_n(x-t) d\mu(t), \quad \text{where} \quad F_n =: \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k$$

is called the kernels of the Nörlund means.

It is well-known (see e.g. [16]) that every Nörlund summability method generated by non-increasing sequence $(q_k, k \in \mathbb{N})$ is regular, but Nörlund means generated by non-decreasing sequence $(q_k, k \in \mathbb{N})$ is not always regular. In this paper we investigate regular Nörlund means only.

If we invoke Abel transformation we get the following identities:

$$Q_n := \sum_{j=0}^{n-1} q_j = \sum_{j=1}^{n} q_{n-j} \cdot 1 = \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n \tag{10}$$

and

$$t_n f = \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j \sigma_j f + q_0 n \sigma_n f \right). \tag{11}$$

## 3. Main results

Based on estimate (1) we can prove our first main results:

THEOREM 1. *Let $2^N \leqslant n < 2^{N+1}$ and $t_n$ be a regular Nörlund mean generated by non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$, in sign $q_k \uparrow$. Then, for any $f \in L^p(G)$, where $1 \leqslant p < \infty$, the following inequality holds:*

$$\|t_n f - f\|_p \leqslant \frac{18}{Q_n} \sum_{i=0}^{N-1} 2^i q_{n-2^i} \omega_p \left( \frac{1}{2^i}, f \right) + 12 \omega_p \left( \frac{1}{2^N}, f \right).$$

*Proof.* Let $2^N \leqslant n < 2^{N+1}$. Since $t_n$ are regular Nörlund means, generated by sequences of non-decreasing numbers $\{q_k : k \in \mathbb{N}\}$ by combining (10) and (11), we can conclude that

$$\begin{aligned} &\|t_n f(x) - f(x)\|_p \\ \leqslant\ & \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} \left(q_{n-j} - q_{n-j-1}\right) j \|\sigma_j f(x) - f(x)\|_p + q_0 n \|\sigma_n f(x) - f(x)\|_p \right) \\ :=\ & I + II, \end{aligned}$$

Furthermore,

$$\begin{aligned} I = & \frac{1}{Q_n} \sum_{j=1}^{2^N - 1} \left(q_{n-j} - q_{n-j-1}\right) j \|\sigma_j f(x) - f(x)\|_p \\ & + \frac{1}{Q_n} \sum_{j=2^N}^{n-1} \left(q_{n-j} - q_{n-j-1}\right) j \|\sigma_j f(x) - f(x)\|_p := I_1 + I_2. \end{aligned}$$

Now we estimate each terms separately. By using (1) for $I_1$ we obtain that

$$\begin{aligned} I_1 \leqslant & \frac{3}{Q_n} \sum_{k=0}^{N-1} \sum_{j=2^k}^{2^{k+1}-1} \left(q_{n-j} - q_{n-j-1}\right) j \sum_{s=0}^{k} \frac{2^s}{2^k} \omega_p \left(1/2^s, f\right) \qquad (12) \\ \leqslant & \frac{3}{Q_n} \sum_{k=0}^{N-1} 2^{k+1} \sum_{j=2^k}^{2^{k+1}-1} \left(q_{n-j} - q_{n-j-1}\right) \sum_{s=0}^{k} \frac{2^s}{2^k} \omega_p \left(1/2^s, f\right) \\ \leqslant & \frac{6}{Q_n} \sum_{k=0}^{N-1} \left(q_{n-2^k} - q_{n-2^{k+1}}\right) \sum_{s=0}^{k} 2^s \omega_p \left(1/2^s, f\right) \\ \leqslant & \frac{6}{Q_n} \sum_{s=0}^{N-1} 2^s \omega_p \left(1/2^s, f\right) \sum_{k=s}^{N-1} \left(q_{n-2^k} - q_{n-2^{k+1}}\right) \\ \leqslant & \frac{6}{Q_n} \sum_{s=0}^{N-1} 2^s q_{n-2^s} \omega_p \left(1/2^s, f\right). \end{aligned}$$

It is evident that

$$I_2 \leqslant \frac{3}{Q_n} \sum_{j=2^N}^{n-1} \left(q_{n-j} - q_{n-j-1}\right) j \sum_{s=0}^{N} \frac{2^s}{2^N} \omega_p \left(1/2^s, f\right) \tag{13}$$

$$\leqslant \frac{3 \cdot 2^{N+1}}{Q_n} \sum_{j=2^N}^{n-1} \left(q_{n-j} - q_{n-j-1}\right) \sum_{s=0}^{N} \frac{2^s}{2^N} \omega_p \left(1/2^s, f\right)$$

$$\leqslant \frac{6q_{n-2^N}}{Q_n} \sum_{s=0}^{N} 2^s \omega_p \left(1/2^s, f\right)$$

$$\leqslant \frac{6}{Q_n} \sum_{s=0}^{N} 2^s q_{n-2^s} \omega_p \left(1/2^s, f\right)$$

$$\leqslant \frac{6}{Q_n} \sum_{s=0}^{N-1} 2^s q_{n-2^s} \omega_p \left(1/2^s, f\right) + 6\omega_p \left(1/2^N, f\right).$$

For $II$ we have that

$$II \leqslant \frac{3q_0 2^{N+1}}{Q_n} \sum_{s=0}^{N} \frac{2^s}{2^N} \omega_p \left(1/2^s, f\right) \leqslant \frac{6}{Q_n} \sum_{s=0}^{N-1} 2^s q_{n-2^s} \omega_p \left(1/2^s, f\right) + 6\omega_p \left(1/2^N, f\right).$$

The proof is complete. □

Our next main result reads:

THEOREM 2. *Let $t_n$ be Nörlund mean generated by a non-increasing sequence $\{q_k : k \in \mathbb{N}\}$, in sign $q_k \downarrow$. Then, for any $f \in L^p(G)$, where $1 \leqslant p < \infty$, the inequality*

$$\|t_{2^n} f - f\|_p \leqslant \sum_{s=0}^{n-1} \frac{2^s}{2^n} \omega_p \left(1/2^s, f\right) + 3 \sum_{s=0}^{n-1} \frac{n-s}{2^{n-s}} \frac{q_{2^s}}{q_{2^n}} \omega_p \left(1/2^s, f\right) + 3\omega_p \left(1/2^n, f\right)$$

*holds.*

*Proof.* By using (5) we find that

$$t_{2^n} f = D_{2^n} * f - \frac{1}{Q_{2^n}} \sum_{k=0}^{2^n-1} q_k \left(\left(w_{2^n-1} D_k\right) * f\right). \tag{14}$$

By using Abel transformation we get that

$$t_{2^n} f = D_{2^n} * f - \frac{1}{Q_{2^n}} \sum_{j=0}^{2^n-2} \left(q_j - q_{j+1}\right) j\left(\left(w_{2^n-1} K_j\right) * f\right) \tag{15}$$

$$- \frac{1}{Q_{2^n}} q_{2^n-1} (2^n - 1) \left(w_{2^n-1} K_{2^n-1} * f\right)$$

$$= D_{2^n} * f - \frac{1}{Q_{2^n}} \sum_{j=0}^{2^n-2} \left(q_j - q_{j+1}\right) j\left(\left(w_{2^n-1} K_j\right) * f\right)$$

$$- \frac{1}{Q_{2^n}} q_{2^n-1} 2^n \left(w_{2^n-1} K_{2^n} * f\right) + \frac{q_{2^n-1}}{Q_{2^n}} \left(w_{2^n-1} D_{2^n} * f\right)$$

and

$$\begin{aligned}
t_{2^n}f(x)-f(x) &= \int_G (f(x+t)-f(x))D_{2^n}(t)dt \qquad (16)\\
&-\frac{1}{Q_{2^n}}\sum_{j=0}^{2^n-2}\left(q_j-q_{j+1}\right)j\int_G (f(x+t)-f(x))\,w_{2^n-1}(t)K_j(t)dt\\
&-\frac{1}{Q_{2^n}}q_{2^n-1}2^n\int_G (f(x+t)-f(x))w_{2^n-1}(t)K_{2^n}(t)dt\\
&+\frac{q_{2^n-1}}{Q_{2^n}}\int_G (f(x+t)-f(x))w_{2^n-1}(t)D_{2^n}(t)dt\\
&:= I+II+III+IV.
\end{aligned}$$

By combining generalized Minkowski's inequality and equality (4) we find that

$$\|I\|_p \leqslant \int_{I_n}\|f(x+t)-f(x)\|_p D_{2^n}(t)dt \leqslant \omega_p\left(1/2^n,f\right).$$

and

$$\|IV\|_p \leqslant \int_{I_n}\|f(x+t)-f(x)\|_p D_{2^n}(t)dt \leqslant \omega_p\left(1/2^n,f\right).$$

Since

$$2^n q_{2^n-1}\leqslant Q_{2^n},\ n\in\mathbb{N}, \qquad (17)$$

If we combine (8), (17) and generalized Minkowski's inequality, then we get that

$$\begin{aligned}
\|III\|_p &\leqslant \int_G \|f(x+t)-f(x)\|_p K_{2^n}(t)\,d\mu(t)\\
&= \int_{I_n}\|f(x+t)-f(x)\|_p K_{2^n}(t)\,d\mu(t)\\
&+\sum_{s=0}^{n-1}\int_{I_n(e_s)}\|f(x+t)-f(x)\|_p K_{2^n}(t)\,d\mu(t)\\
&\leqslant \int_{I_n}\|f(x+t)-f(x)\|_p \frac{2^n+1}{2}d\mu(t)\\
&+\sum_{s=0}^{n-1}2^s\int_{I_n(e_s)}\|f(x+t)-f(x)\|_p\,d\mu(t)\\
&\leqslant \omega_p\left(1/2^n,f\right)\int_{I_n}\frac{2^n+1}{2}d\mu(t)+\sum_{s=0}^{n-1}2^s\int_{I_n(e_s)}\omega_p\left(1/2^s,f\right)d\mu(t)\\
&\leqslant \omega_p\left(1/2^n,f\right)+\sum_{s=0}^{n-1}\frac{2^s}{2^n}\omega_p\left(1/2^s,f\right)\\
&\leqslant \sum_{s=0}^{n}\frac{2^s}{2^n}\omega_p\left(1/2^s,f\right).
\end{aligned}$$

From this estimate also it follows that

$$2^n \int_G \|f(x+t)-f(x)\|_p K_{2^n}(t)\,d\mu(t) \leqslant \sum_{s=0}^{n} 2^s \omega_p(1/2^s, f). \tag{18}$$

Let $2^k \leqslant j \leqslant 2^{k+1}-1$. By applying (6) and (18) we find that

$$\left\| j\int_G |f(x+t)-f(x)|\, K_j(t)\,d\mu(t) \right\|_p \tag{19}$$

$$\leqslant 3\sum_{s=0}^{k} 2^s \int_G \|f(x+t)-f(x)\|_p K_{2^s}(t)\,d\mu(t)$$

$$\leqslant 3\sum_{l=0}^{k}\sum_{s=0}^{l} 2^s \omega_p(1/2^s, f). \tag{20}$$

Hence, by applying (6) and (19) we find that

$$\begin{aligned}
\|II\|_p &\leqslant \frac{1}{Q_{2^n}} \sum_{j=0}^{2^n-1} (q_j - q_{j+1})\, j \int_G \|f(x+t)-f(x)\|_p |K_j(t)|dt \\
&\leqslant \frac{1}{Q_{2^n}} \sum_{k=0}^{n-1} \sum_{j=2^k}^{2^{k+1}-1} (q_j - q_{j+1})\, j \int_G \|f(x+t)-f(x)\|_p |K_j(t)|dt \\
&\leqslant \frac{3}{Q_{2^n}} \sum_{k=0}^{n-1} \sum_{j=2^k}^{2^{k+1}-1} (q_j - q_{j+1}) \sum_{l=0}^{k}\sum_{s=0}^{l} 2^s \omega_p(1/2^s, f) \\
&\leqslant \frac{3}{Q_{2^n}} \sum_{k=0}^{n-1} (q_{2^k} - q_{2^{k+1}}) \sum_{l=0}^{k}\sum_{s=0}^{l} 2^s \omega_p(1/2^s, f) \\
&\leqslant \frac{3}{Q_{2^n}} \sum_{l=0}^{n-1}\sum_{k=l}^{n-1} (q_{2^k} - q_{2^{k+1}}) \sum_{s=0}^{l} 2^s \omega_p(1/2^s, f) \\
&\leqslant \frac{3}{Q_{2^n}} \sum_{l=0}^{n-1} q_{2^l} \sum_{s=0}^{l} 2^s \omega_p(1/2^s, f) \\
&\leqslant \frac{3}{Q_{2^n}} \sum_{s=0}^{n-1} 2^s \omega_p(1/2^s, f) \sum_{l=s}^{n-1} q_{2^l} \\
&\leqslant \frac{3}{Q_{2^n}} \sum_{s=0}^{n-1} 2^s \omega_p(1/2^s, f)\, q_{2^s}(n-s) \\
&\leqslant 3\sum_{s=0}^{n-1} \frac{n-s}{2^{n-s}} \frac{q_{2^s}}{q_{2^n}} \omega_p(1/2^s, f).
\end{aligned}$$

The proof is complete. □

Finally, we state and prove the third main result.

THEOREM 3. *Let* $2^N \leqslant n < 2^{N+1}$ *and* $t_n$ *be Nörlund mean generated by non-increasing sequence* $\{q_k : k \in \mathbb{N}\}$ *(in sign* $q_k \downarrow$*) satisfying the condition*

$$\frac{1}{Q_n} = O\left(\frac{1}{n}\right), \quad as \quad n \to \infty \tag{21}$$

*Then, for any* $f \in L^p(G)$, *where* $1 \leqslant p < \infty$, *we have the following inequality*

$$\|t_n f - f\|_p \leqslant C \sum_{j=0}^{N} \frac{2^j}{2^N} \omega_p \left(1/2^j, f\right),$$

*where* $C$ *is a constant only depending on* $p$.

*Proof.* Let $2^N \leqslant n < 2^{N+1}$. Since $t_n$ is a regular Nörlund means, generated by a sequence of non-increasing numbers $\{q_k : k \in \mathbb{N}\}$ by combining (10) and (11), we can conclude that

$$\begin{aligned}
&\|t_n f(x) - f(x)\|_p \\
&\leqslant \frac{1}{Q_n}\left(\sum_{j=1}^{n-1} \left(q_{n-j-1} - q_{n-j}\right) j \|\sigma_j f(x) - f(x)\|_p + q_0 n \|\sigma_n f(x) - f(x)\|_p\right) \\
&:= I + II.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I &= \frac{1}{Q_n} \sum_{j=1}^{2^N-1} \left(q_{n-j-1} - q_{n-j}\right) j \|\sigma_j f(x) - f(x)\|_p \\
&\quad + \frac{1}{Q_n} \sum_{j=2^N}^{n-1} \left(q_{n-j-1} - q_{n-j}\right) j \|\sigma_j f(x) - f(x)\|_p \\
&:= I_1 + I_2.
\end{aligned}$$

Analogously to (12) we get that

$$\begin{aligned}
I_1 &\leqslant \frac{2}{Q_n} \sum_{k=0}^{N-1} \left(q_{n-2^{k+1}} - q_{n-2^k}\right) \sum_{s=0}^{k} 2^s \omega_p\left(1/2^s, f\right) \\
&\leqslant \frac{2}{Q_n} \sum_{s=0}^{N-1} 2^s \omega_p\left(1/2^s, f\right) \sum_{k=s}^{N-1} \left(q_{n-2^{k+1}} - q_{n-2^k}\right) \\
&= \frac{2}{Q_n} \sum_{s=0}^{N-1} 2^s \omega_p\left(1/2^s, f\right) \left(q_{n-2^N} - q_{n-2^s}\right) \\
&\leqslant \frac{2 q_{n-2^N}}{Q_n} \sum_{s=0}^{N-1} 2^s \omega_p\left(1/2^s, f\right) \\
&\leqslant \frac{2 q_0}{Q_n} \sum_{s=0}^{N-1} 2^s \omega_p\left(1/2^s, f\right).
\end{aligned}$$

Moreover, analogously to (13) we find that

$$\begin{aligned} I_2 &\leqslant \frac{2}{Q_n}\sum_{j=1}^{n-1}\left(q_{n-j-1}-q_{n-j}\right) j\sum_{s=0}^{N}\frac{2^s}{2^N}\omega_p\left(1/2^s,f\right) \\ &= \frac{2}{Q_n}\left(nq_0-Q_n\right)\sum_{s=0}^{N}\frac{2^s}{2^N}\omega_p\left(1/2^s,f\right) \\ &\leqslant \frac{2nq_0}{Q_n2^N}\sum_{s=0}^{N}2^s\omega_p\left(1/2^s,f\right) \\ &\leqslant \frac{2q_0}{Q_n}\sum_{s=0}^{N}2^s\omega_p\left(1/2^s,f\right). \end{aligned}$$

For $II$ we have that

$$II \leqslant \frac{q_0 2^{N+1}}{Q_n}\sum_{s=0}^{N}\frac{2^s}{2^N}\omega_p\left(1/2^s,f\right) \leqslant \frac{2q_0}{Q_n}\sum_{s=0}^{N}2^s\omega_p\left(1/2^s,f\right).$$

Hence, by using (21) we obtain the required inequality above so the proof is complete. □

As a consequence we obtain the following similar result proved in Móricz and Siddiqi [12]:

COROLLARY 1. *Let* $\{q_k : k \geqslant 0\}$ *be a sequence of non-negative numbers such that in the case* $q_k \uparrow$ *condition*

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad as \quad n \to \infty. \tag{22}$$

*is satisfied, while in case* $q_k \downarrow$ *condition* (21) *is satisfied. If* $f \in lip(\alpha, p)$ *for some* $\alpha > 0$ *and* $1 \leqslant p < \infty$, *then*

$$\|t_n f - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if} \quad 0 < \alpha < 1, \\ O(n^{-1}\log n), & \text{if} \quad \alpha = 1, \\ O(n^{-1}), & \text{if} \quad \alpha > 1, \end{cases} \tag{23}$$

As a consequence we obtain the following similar result proved in Móricz and Siddiqi [12]:

COROLLARY 2. *a) Let* $t_n$ *be Nörlund means generated by non-decreasing sequence* $\{q_k : k \in \mathbb{N}\}$ *satisfying regularity condition* (22). *Then for any* $f \in L^p(G)$, *where* $1 \leqslant p < \infty$,

$$\lim_{n\to\infty}\|t_n f - f\|_p \to 0, \quad as \quad n \to \infty.$$

*b) Let* $t_n$ *be Nörlund mean generated by non-increasing sequence* $\{q_k : k \in \mathbb{N}\}$ *satisfying condition* (21). *Then for any* $f \in L^p(G)$, *where* $1 \leqslant p < \infty$,

$$\lim_{n\to\infty}\|t_n f - f\|_p \to 0, \quad as \quad n \to \infty.$$

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