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A QUANTITATIVE POPOVICIU TYPE INEQUALITY

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Abstract. In this paper, we prove a general quantitative multiple Popoviciu type inequality for positive definite matrices. As corollaries, we obtained generalized multiple Hartfiel's inequalities.

1. Introduction

Positive definite (or positive semi-definite) matrices have similar properties with positive (or nonnegative) numbers, especially about inequalities, please see [6, 7, 12, 15]. One of the fundamental inequalities is the following: for any two positive definite matrices with the same order, we have (e.g. [7, p. 511])

$$\det(A+B) \geqslant \det(A) + \det(B). \tag{1}$$

In 1970, E. V. Haynsworth [4] made the first improvement of (1) by using the Schur complement method. Please see [13] for more about the Schur complement and its application.

To be precise, E. V. Haynsworth [4] proved that:

THEOREM 1. [4, Theorem 3] *Let A,B be positive definite $n\times n$ matrices. Then*

$$\det(A+B) \geqslant \left(1+\sum_{k=1}^{n-1}\frac{\det(B_k)}{\det(A_k)}\right)\det(A) + \left(1+\sum_{k=1}^{n-1}\frac{\det(A_k)}{\det(B_k)}\right)\det(B),$$

where A_k, B_k, $k=1,2,\cdots,n-1$ denote the k-th leading principal submatrices of A,B respectively.

Later on, D. J. Hartfiel [3] proved a quantitative and sharp version of Theorem 1 in 1973 as follows.

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THEOREM 2. [3] *Let A, B be positive definite $n \times n$ matrices. Then*

$$\det(A+B) \geqslant \left(1+\sum_{k=1}^{n-1} \frac{\det(B_k)}{\det(A_k)}\right)\det(A) + \left(1+\sum_{k=1}^{n-1} \frac{\det(A_k)}{\det(B_k)}\right)\det(B) + (2^n-2n)\sqrt{\det(AB)},$$

where $A_k, B_k, k=1,2,\cdots,n-1$ denote the k-th leading principal submatrices of A, B respectively. And the equality holds if and only if $A=B$.

Along this line, M. Lin [9], Hou-Dong [8] and Hong-Qi [5] generalized above results to three positive definite matrices.

THEOREM 3. [9, Theorem 1.1] *Let A, B, C be positive definite $n \times n$ matrices. Then*

$$\det(A+B+C) + \det(A) + \det(B) + \det(C) \geqslant \det(A+B) + \det(B+C) + \det(A+C).$$

THEOREM 4. [8, Theorem 1] *Let A, B, C be positive definite $n \times n$ matrices. Then*

$$\begin{aligned}\det(A+B+C) \geqslant &\left(1+\sum_{k=1}^{n-1} \frac{\det(B_k)+\det(C_k)}{\det(A_k)}\right)\det(A)\\ &+\left(1+\sum_{k=1}^{n-1} \frac{\det(A_k)+\det(C_k)}{\det(B_k)}\right)\det(B)\\ &+\left(1+\sum_{k=1}^{n-1} \frac{\det(A_k)+\det(B_k)}{\det(C_k)}\right)\det(C)\\ &+(2^n-2n)\left(\sqrt{\det(AB)}+\sqrt{\det(BC)}+\sqrt{\det(AC)}\right).\end{aligned}$$

where $A_k, B_k, C_k, k=1,2,\cdots,n-1$ denote the k-th leading principal submatrices of A, B, C respectively.

THEOREM 5. [5, Theorem 3] *Let A, B, C be positive definite $n \times n$ matrices. Then*

$$\begin{aligned}&\det(A+B+C) + \det(A) + \det(B) + \det(C)\\ \geqslant &\det(A+B) + \det(B+C) + \det(A+C) + (3^n+3-3\cdot 2^n)\left[\det(ABC)\right]^{\frac{1}{3}}.\end{aligned}$$

According to the conclusion for two or three positive definite matrices, it is natural to search for its multiple version. Recently, in a remarkable work of W. Berndt and S. Sra [1], a Popoviciu type inequality for positive operators was obtained. When restricted to determinants, they proved the following multiple version theorem.

THEOREM 6. [1, Theorem 4.3] *Let $A_1, A_2, \cdots, A_m (m \geqslant 3)$ be positive definite $n \times n$ matrices. Then*

$$\det\left(\sum_{j=1}^{m} A_j\right) + (m-2)\sum_{j=1}^{m} \det(A_j) \geqslant \sum_{1 \leqslant i < j \leqslant m} \det(A_i + A_j).$$

In this paper, we first extend the Popoviciu type inequality Theorem 6 to a quantitative version, which is Theorem 7. And as corollaries, we will obtain some generalized multiple Hartfiel's inqualities, which are Corollary 1 and Corollary 2.

2. Lemmas

At first, we introduce some lemmas needed for the proof of our Main Theorem 7.

Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, then rearrange the components of x in decreasing order and obtain a vector $x^{\downarrow} = \left(x_1^{\downarrow}, x_2^{\downarrow}, \cdots, x_n^{\downarrow}\right)$, where

$$x_1^{\downarrow} \geqslant x_2^{\downarrow} \geqslant \cdots \geqslant x_n^{\downarrow}.$$

Given $x, y \in \mathbb{R}^n$, we say that x majorizes y, written $x \succ y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \geqslant \sum_{i=1}^{k} y_i^{\downarrow} \text{ for } 1 \leqslant k \leqslant n-1 \text{ and } \sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}.$$

LEMMA 1. [9, Lemma 2.2] *The function*

$$f(x) = \prod_{i=1}^{n} (1 + x_i) - \prod_{i=1}^{n} x_i,$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$, is Schur convave, that is to say if $x, y \in \mathbb{R}_+^n, x \succ y$, then

$$f(x) \leqslant f(y).$$

For an $n \times n$ Hermitian matrix X, we denote the vertor of eigenvalues of X by

$$\lambda(X) = (\lambda_1(X), \lambda_2(X), \cdots, \lambda_n(X))$$

with $\lambda_1(X) \geqslant \lambda_2(X) \geqslant \cdots \geqslant \lambda_n(X)$. We also need the following classical result of Fan for $n \times n$ Hermitian matrices $A_1, A_2, \cdots A_m$.

LEMMA 2. [15, p. 356] *Let $A_1, A_2, \cdots, A_m$ be $n \times n$ Hermitian matrices, then*

$$\sum_{j=1}^{m} \lambda(A_i) \succ \lambda\left(\sum_{j=1}^{m} A_i\right).$$

LEMMA 3. *Let $A_1, A_2, \cdots, A_m$ be $n \times n$ positive definite matrices, then*

$$\prod_{i=1}^{n}\left(1+\lambda_i\left(\sum_{j=1}^{m} A_j\right)\right)-\prod_{i=1}^{n}\lambda_i\left(\sum_{j=1}^{m} A_j\right) \geqslant \prod_{i=1}^{n}\left(1+\sum_{j=1}^{m}\lambda_i(A_j)\right)-\prod_{i=1}^{n}\sum_{j=1}^{m}\lambda_i(A_j).$$

Proof. It is straight from Lemma 1 and Lemma 2. □

The following Lemma is diagonal case for Theorem 7.

LEMMA 4. *Suppose $x_{ji} > 0$ for any $1 \leqslant j \leqslant m-1, 1 \leqslant i \leqslant n$, then*

$$\prod_{i=1}^{n}\left(1+\sum_{j=1}^{m-1} x_{ji}\right)-\prod_{i=1}^{n}\sum_{j=1}^{m-1} x_{ji}+\sum_{j=1}^{m-1}\prod_{i=1}^{n} x_{ji}-\sum_{j=1}^{m-1}\prod_{i=1}^{n}(1+x_{ji})+(m-2)$$
$$+\left[(m-1)^n-(m-1)-(2^{n-1}-1)(m-2)(m-1)\right]\left[\prod_{i=1}^{n} x_{1i}\cdot\prod_{i=1}^{n} x_{2i}\cdots\prod_{i=1}^{n} x_{(m-1)i}\right]^{\frac{1}{m-1}}$$
$$\geqslant \left[m^n-m-(2^{n-1}-1)(m-1)m\right]\left[\prod_{i=1}^{n} x_{1i}\cdot\prod_{i=1}^{n} x_{2i}\cdots\prod_{i=1}^{n} x_{mi}\right]^{\frac{1}{m}}.$$

Proof. Expand polynomial

$$\prod_{i=1}^{n}\left(1+\sum_{j=1}^{m-1} x_{ji}\right)-\prod_{i=1}^{n}\sum_{j=1}^{m-1} x_{ji}+\sum_{j=1}^{m-1}\prod_{i=1}^{n} x_{ji}-\sum_{j=1}^{m-1}\prod_{i=1}^{n}(1+x_{ji}).$$

After cancelling all negative terms, we only have $m^n-(m-1)^n+(m-1)-(m-1)2^n$ terms left, which are all with coefficient 1. And there are $m^{n-1}-(m-1)^{n-1}+1-2^{n-1}$ terms of them contain x_{11}. Written

$$m-2=\underbrace{1+1+\cdots+1}_{m-2}$$

and

$$\left[(m-1)^n-(m-1)-(2^{n-1}-1)(m-2)(m-1)\right]\left[\prod_{i=1}^{n} x_{1i}\cdot\prod_{i=1}^{n} x_{2i}\cdots\prod_{i=1}^{n} x_{(m-1)i}\right]^{\frac{1}{m-1}}$$
$$=\underbrace{\left[\prod_{i=1}^{n} x_{1i}\cdot\prod_{i=1}^{n} x_{2i}\cdots\prod_{i=1}^{n} x_{(m-1)i}\right]^{\frac{1}{m-1}}+\cdots+\left[\prod_{i=1}^{n} x_{1i}\cdot\prod_{i=1}^{n} x_{2i}\cdots\prod_{i=1}^{n} x_{(m-1)i}\right]^{\frac{1}{m-1}}}_{(m-1)^n-(m-1)-(2^{n-1}-1)(m-2)(m-1)},$$

then we have

$$\begin{aligned}&[m^n-(m-1)^n+(m-1)-(m-1)2^n]+(m-2)\\&+\left[(m-1)^n-(m-1)-(2^{n-1}-1)(m-2)(m-1)\right]\\=&m^n-m-(2^{n-1}-1)(m-1)m\end{aligned}$$

terms which with coefficient 1, and the power of x_{11} in the product of them is

$$\begin{aligned} &\left[(m-1)^n-(m-1)-(2^{n-1}-1)(m-2)(m-1)\right]\cdot\frac{1}{m-1}\\ &+m^{n-1}-(m-1)^{n-1}+1-2^{n-1}\\ =&\,m^{n-1}-1-(2^{n-1}-1)(m-1)\\ =&\,\frac{1}{m}\left[m^n-m-(2^{n-1}-1)(m-1)m\right]. \end{aligned}$$

Therefore, by the symmetry of x_{ji} and the Arithmetic Mean-Geometric Mean Inequality, we obtain that

$$\begin{aligned} &\prod_{i=1}^{n}\left(1+\sum_{j=1}^{m-1}x_{ji}\right)-\prod_{i=1}^{n}\sum_{j=1}^{m-1}x_{ji}+\sum_{j=1}^{m-1}\prod_{i=1}^{n}x_{ji}-\sum_{j=1}^{m-1}\prod_{i=1}^{n}(1+x_{ji})+(m-2)\\ &+\left[(m-1)^n-(m-1)-(2^{n-1}-1)(m-2)(m-1)\right]\left[\prod_{i=1}^{n}x_{1i}\cdot\prod_{i=1}^{n}x_{2i}\cdots\prod_{i=1}^{n}x_{(m-1)i}\right]^{\frac{1}{m-1}}\\ \geqslant&\left[m^n-m-(2^{n-1}-1)(m-1)m\right]\left[\prod_{i=1}^{n}x_{1i}\cdot\prod_{i=1}^{n}x_{2i}\cdots\prod_{i=1}^{n}x_{(m-1)i}\right]^{\frac{1}{m}}. \quad\square \end{aligned}$$

3. Main results

Now, we are in a position to extend the Popoviciu type inequality Theorem 6 to a quantitative version.

THEOREM 7. *Let* $A_1,A_2,\cdots,A_m (m\geqslant 3)$ *be positive definite* $n\times n$ *matrices. Then*

$$\begin{aligned} &\det\left(\sum_{j=1}^{m}A_j\right)+(m-2)\sum_{j=1}^{m}\det(A_j)\\ \geqslant&\sum_{1\leqslant i<j\leqslant m}\det(A_i+A_j)+\left[m^n-m-(2^{n-1}-1)(m-1)m\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}. \end{aligned}$$

Proof. The proof of the theorem is by induction on m.

Step 1: For $m=3$, the inequality reduces to

$$\begin{aligned} &\det(A_1+A_2+A_3)+\det(A_1)+\det(A_2)+\det(A_3)\\ \geqslant&\det(A_1+A_2)+\det(A_2+A_3)+\det(A_3+A_1)+(3^n-3\cdot 2^n+3)\left[\det(A_1A_2A_3)\right]^{\frac{1}{3}}, \end{aligned}$$

which is clearly true by Theorem 5.

Step 2: Suppose that the inequality holds for $m-1$. Denote

$$\hat{A}_j=A_m^{-\frac{1}{2}}A_jA_m^{-\frac{1}{2}}, 1\leqslant j\leqslant m-1,$$

then

$$
\begin{aligned}
&\left[\det\left(\sum_{j=1}^{m} A_j\right)+(m-2)\sum_{j=1}^{m}\det\left(A_j\right)-\sum_{1\leqslant i<j\leqslant m}\det\left(A_i+A_j\right)\right]\cdot\det\left(A_m^{-1}\right)\\
=&\det\left(\sum_{j=1}^{m-1}\hat{A}_j+I_n\right)+(m-2)\left(\sum_{j=1}^{m-1}\det\left(\hat{A}_j\right)+\det(I_n)\right)\\
&-\sum_{1\leqslant i<j\leqslant m-1}\det\left(\hat{A}_i+\hat{A}_j\right)-\sum_{1\leqslant j\leqslant m-1}\det\left(\hat{A}_j+I_n\right)\\
=&\det\left(\sum_{j=1}^{m-1}\hat{A}_j+I_n\right)-\det\left(\sum_{j=1}^{m-1}\hat{A}_j\right)+\det\left(\sum_{j=1}^{m-1}\hat{A}_j\right)\\
&+(m-2)\left(\sum_{j=1}^{m-1}\det\left(\hat{A}_j\right)+\det(I_n)\right)\\
&-\sum_{1\leqslant i<j\leqslant m-1}\det\left(\hat{A}_i+\hat{A}_j\right)-\sum_{1\leqslant j\leqslant m-1}\det\left(\hat{A}_j+I_n\right)\\
=&\det\left(\sum_{j=1}^{m-1}\hat{A}_j+I_n\right)-\det\left(\sum_{j=1}^{m-1}\hat{A}_j\right)\\
&+\det\left(\sum_{j=1}^{m-1}\hat{A}_j\right)+(m-3)\sum_{j=1}^{m-1}\det\left(\hat{A}_j\right)-\sum_{1\leqslant i<j\leqslant m-1}\det\left(\hat{A}_i+\hat{A}_j\right)\\
&+\sum_{j=1}^{m-1}\det\left(\hat{A}_j\right)+(m-2)\det(I_n)-\sum_{1\leqslant j\leqslant m-1}\det\left(\hat{A}_j+I_n\right)\\
\geqslant&\prod_{i=1}^{n}\left(1+\lambda_i\left(\sum_{j=1}^{m-1}\hat{A}_j\right)\right)-\prod_{i=1}^{n}\lambda_i\left(\sum_{j=1}^{m-1}\hat{A}_j\right)\\
&+\left[(m-1)^n-(m-1)-(2^{n-1}-1)(m-2)(m-1)\right]\left[\det(\hat{A}_1\hat{A}_2\cdots\hat{A}_{m-1})\right]^{\frac{1}{m-1}}\\
&+\sum_{j=1}^{m-1}\prod_{i=1}^{n}\lambda_i\left(\hat{A}_j\right)+(m-2)-\sum_{1\leqslant j\leqslant m-1}\prod_{i=1}^{n}\left(1+\lambda_i\left(\hat{A}_j\right)\right)\\
\geqslant&\prod_{i=1}^{n}\left(1+\sum_{j=1}^{m-1}\lambda_i\left(\hat{A}_j\right)\right)-\prod_{i=1}^{n}\sum_{j=1}^{m-1}\lambda_i\left(\hat{A}_j\right)\\
&+\left[(m-1)^n-(m-1)-(2^{n-1}-1)(m-2)(m-1)\right]\left[\det(\hat{A}_1\hat{A}_2\cdots\hat{A}_{m-1})\right]^{\frac{1}{m-1}}\\
&+\sum_{j=1}^{m-1}\prod_{i=1}^{n}\lambda_i\left(\hat{A}_j\right)+(m-2)-\sum_{1\leqslant j\leqslant m-1}\prod_{i=1}^{n}\left(1+\lambda_i\left(\hat{A}_j\right)\right)\\
\geqslant&\left[\left(m^n-m-(2^{n-1}-1)(m-1)m\right)\right]\left[\det(\hat{A}_1\hat{A}_2\cdots\hat{A}_{m-1})\right]^{\frac{1}{m}}\\
=&\left[\left(m^n-m-(2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_{m-1})\right]^{\frac{1}{m}}\cdot\left[\det(A_m)\right]^{-\frac{m-1}{m}},
\end{aligned}
$$

where the first inequality comes from Lemma 3 and the second inequality follows from Lemma 4. Multiplied by $\det(A_m)$ on both sides, we obtain that

$$\det\left(\sum_{j=1}^{m} A_j\right) + (m-2)\sum_{j=1}^{m}\det(A_j) - \sum_{1\leqslant i<j\leqslant m}\det(A_i+A_j)$$
$$\geqslant \left[\left(m^n - m - (2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}. \quad \square$$

As applications, we have the following generalized multiple Hartfiel's inquality.

COROLLARY 1. *Let $A_1, A_2, \cdots, A_m (m \geqslant 3)$ be positive definite $n\times n$ matrices. Then*

$$\det\left(\sum_{j=1}^{m} A_j\right) \geqslant \sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-1}\frac{\sum_{j\neq i}\det(A_{jk})}{\det(A_{ik})}\right)\det(A_i)$$
$$+(2^n-2n)\sum_{1\leqslant i<j\leqslant m}\sqrt{\det(A_i)\det(A_j)}$$
$$+\left[\left(m^n - m - (2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}},$$

where A_{jk} denote the k-th leading principal submatrix of A_j, $1\leqslant j\leqslant m, 1\leqslant k\leqslant n$.

Proof. By Theorem 7 and Theorem 2, we have

$$\det\left(\sum_{j=1}^{m} A_j\right) + (m-2)\sum_{j=1}^{m}\det(A_j)$$
$$\geqslant \sum_{1\leqslant i<j\leqslant m}\det(A_i+A_j) + \left[\left(m^n - m - (2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}$$
$$\geqslant \sum_{1\leqslant i<j\leqslant m}\left[\left(1+\sum_{k=1}^{n-1}\frac{\det(A_{jk})}{\det(A_{ik})}\right)\det(A_i) + \left(1+\sum_{k=1}^{n-1}\frac{\det(A_{ik})}{\det(A_{jk})}\right)\det(A_j)\right.$$
$$\left. + (2^n-2n)\sqrt{\det(A_i)\det(A_j)}\right]$$
$$+\left[\left(m^n - m - (2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}$$
$$=\sum_{i=1}^{m}\sum_{j\neq i}\left(1+\sum_{k=1}^{n-1}\frac{\det(A_{jk})}{\det(A_{ik})}\right)\det(A_i) + (2^n-2n)\sum_{1\leqslant i<j\leqslant m}\sqrt{\det(A_i)\det(A_j)}$$
$$+\left[\left(m^n - m - (2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}$$
$$=\sum_{i=1}^{m}\left[(m-1)+\sum_{k=1}^{n-1}\frac{\sum_{j\neq i}\det(A_{jk})}{\det(A_{ik})}\right]\det(A_i) + (2^n-2n)\sum_{1\leqslant i<j\leqslant m}\sqrt{\det(A_i)\det(A_j)}$$
$$+\left[\left(m^n - m - (2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}.$$

Therefore,

$$\det\left(\sum_{j=1}^{m} A_j\right) \geqslant \sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-1}\frac{\sum_{j\neq i}\det(A_{jk})}{\det(A_{ik})}\right)\det(A_i)$$
$$+(2^n-2n)\sum_{1\leqslant i<j\leqslant m}\sqrt{\det(A_i)\det(A_j)}$$
$$+\left[\left(m^n-m-(2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}. \quad \square$$

REMARK 1. When $m=3$, Corollary 1 is an improvement of Theorem 4.

COROLLARY 2. *Let* $A_1, A_2, \cdots, A_m (m \geqslant 3)$ *be positive definite* $n\times n$ *matrices. Then*

$$\det\left(\sum_{j=1}^{m} A_j\right) \geqslant \sum_{i=1}^{n}\det(A_i)+(m^n-m)\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}.$$

Proof. By Corollary 1, we have

$$\det\left(\sum_{j=1}^{m} A_j\right) \geqslant \sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-1}\frac{\sum_{j\neq i}\det(A_{jk})}{\det(A_{ik})}\right)\det(A_i)$$
$$+(2^n-2n)\sum_{1\leqslant i<j\leqslant m}\sqrt{\det(A_i)\det(A_j)}$$
$$+\left[\left(m^n-m-(2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}$$
$$=\sum_{i=1}^{n}\det(A_i)+\sum_{i=1}^{m}\sum_{k=1}^{n-1}\sum_{j\neq i}\frac{\det(A_{jk})}{\det(A_{ik})}\cdot\det(A_i)$$
$$+(2^n-2n)\sum_{1\leqslant i<j\leqslant m}\sqrt{\det(A_i)\det(A_j)}$$
$$+\left[\left(m^n-m-(2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}$$
$$\geqslant\sum_{i=1}^{n}\det(A_i)+m(m-1)(n-1)\left[\prod_{i=1}^{m}\prod_{k=1}^{n-1}\prod_{j\neq i}\frac{\det(A_{jk})}{\det(A_{ik})}\cdot\det(A_i)\right]^{\frac{1}{m(m-1)(n-1)}}$$
$$+(2^n-2n)\cdot\frac{m(m-1)}{2}\left[\prod_{1\leqslant i<j\leqslant m}\det(A_i)\det(A_j)\right]^{\frac{1}{m(m-1)}}$$
$$+\left[\left(m^n-m-(2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \det(A_i) + m(m-1)(n-1)\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}} \\
&\quad + (2^{n-1}-n)m(m-1)\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}} \\
&\quad + \left[\left(m^n - m - (2^{n-1}-1)(m-1)m\right)\right]\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}} \\
&= \sum_{i=1}^{n} \det(A_i) + (m^n - m)\left[\det(A_1A_2\cdots A_m)\right]^{\frac{1}{m}},
\end{aligned}
$$

where the last inequality follows from the Arithmetic Mean-Geometric Mean Inequality. □

REMARK 2. In Theorem 7, Corollary 1 and Corollary 2, it is easy to check that the equality holds if and only if

$$A_1 = A_2 = \cdots = A_m,$$

which show that our conclusions are sharp.

REMARK 3. By the standard continuity method, all conclusions hold for positive semi-definite matrices.

REMARK 4. Recently, Haynsworth's inequality and Hartfiel's inequality were also extended to sector matrices, please see [2, 10, 11, 16] and the references therein.

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