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GENERALIZATION OF EXPANSIVE OPERATORS

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Abstract. In this paper, we study $[m,C]$-expansive operators and $[m,\widehat{C}]$-expansive operators on a Banach space. More precisely, after exploring their properties as operators, we examine the spectral properties of these two operator classes.

1. Introduction and preliminaries

1.1. Introduction

Let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathscr{H}$ with the inner product $\langle\cdot,\cdot\rangle$. In [1], J. Agler and M. Stankus introduced m-*isometric* operators $T \in \mathscr{L}(\mathscr{H})$ satisfying

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*j}T^j = 0 \tag{1}$$

(where T^* is the adjoint of T) and examined their properties. After this there are generalizations of m-isometric operators in two natural ways: being isometric in a general sense or being expansive, i.e., weakening the equality '$=$' in (1) to the inequaliy '$\leqslant$'.

Among them we are interested in the ones which are relevant to a so-called conjugation. A *conjugation* C is an antilinear isometric involution, that is, $C : \mathscr{H} \to \mathscr{H}$ satisfying $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathscr{H}$. See [13, 14] for more details. For a given positive integer m, an operator $T \in \mathscr{L}(\mathscr{H})$ is called

- m-*expansive* if T satisfies $\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*j}T^j \leqslant 0$,
- (m,C)-*isometric* if T satisfies $\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{*j}CT^j = 0$,

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Paper MIA-27-12

- (m,C)-*expansive* if T satisfies $\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{*j}CT^j \leqslant 0$,
- $[m,C]$-*isometric* if T satisfies $\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^jCT^j = 0$.

For more details see [11] for m-expansive operators, [8] for (m,C)-isometric ones, and [9] for $[m,C]$-isometric ones.

In this manuscript we discuss both $[m,C]$-*expansive* and $[m,\widehat{C}]$-*expansive* operators as a natural generalization of $[m,C]$-isometric ones. See Definition 2.3 and Definition 3.3 below. Let us emphasize that this generalization will be done on Banach spaces. After exploring several operator properties, we will examine their spectral properties.

This paper is organized as follows. After reviewing numerical ranges and their relations to spectra in the next section, we study $[m,C]$-expansive operators on a Banach space in Section 2. In particular, if T is an $[m,C]$-expansive operator on a Banach space, then so is CTC and T^* is an $[m,C^*]$-expansive operator on the dual Banach space. Moreover, in this case, T is invertible. In Section 3, we study spectral properties and the single-valued extension property of $[m,\widehat{C}]$-expansive operators under some condition.

1.2. Preliminaries

In order to deal with positive operators and study spectral properties on a Banach space, a numerical range is necessary. Here is a brief summary of numerical ranges and their relation to spectra, which will be used later.

The concept of a numerical range was introduced by O. Toeplitz [26] on a Hilbert space and then Bauer [2] and Lumer [20] extended his concept to a Banach space. For $T \in \mathcal{L}(\mathcal{H})$ the *numerical range* $W(T)$ is the set given by

$$W(T) := \{\langle Tx,x\rangle : x \in \mathcal{H}, \|x\| = 1\}. \tag{2}$$

Most importantly, O. Toeplitz and F. Hausdorff [16, 17, 26] discovered the convexity of $W(T)$. See [12, 25] for basic properties and further developments of $W(T)$.

For a natural extension of $W(T)$ to Banach spaces, let $\mathcal{X}$ be a Banach space, $\mathcal{X}^*$ the dual space of $\mathcal{X}$, and T^* the adjoint operator of $T \in \mathcal{L}(\mathcal{X})$, where $\mathcal{L}(\mathcal{X})$ is the algebra of all bounded linear operators on $\mathcal{X}$. Then the (spatial) *numerical range* of T denoted by $V(T)$ is the subset of $\mathbb{C}$,

$$V(T) := \{\, g(Tx) \,:\, (x,g) \in \Pi \,\}$$

where the set Π is

$$\Pi = \{\, (x,g) \in \mathcal{X} \times \mathcal{X}^* \,:\, \|g\| = g(x) = \|x\| = 1 \,\}. \tag{3}$$

In contrast to the Hilbert setting, $V(T)$ is, in general, not convex (see [7, Example 1 on page 98]), but it is connected since the set Π is connected (see [6] and [7, Corollary 5 on page 102]). It is still open if $V(T)$ is path-connected (see [7, (7) on page 129]).

To see another way to extend a numerical range to a Banach space $\mathscr{X}$, let I be the identity operator on $\mathscr{X}$ and let

$$\Phi = \{g \in \mathscr{L}(\mathscr{X})^* \ : \ \|g\| = g(I) = 1 \}.$$

Then the so-called (algebra) *numerical range* $V\big(T, \mathscr{L}(\mathscr{X})\big)$ is the set given by

$$V\big(T, \mathscr{L}(\mathscr{X})\big) := \{g(T) : g \in \Phi\}.$$

For $T \in \mathscr{L}(\mathscr{X})$, even though they are different, $V(T)$ and $V\big(T, \mathscr{L}(\mathscr{X})\big)$ are both called the *numerical range* of T. This is because we do not think there is any ambiguity.

These numerical ranges have several useful relations with spectra and themselves. Let $\sigma(T)$ be the spectrum and let $\overline{M}$ be the closure of M for any subset M of $\mathbb{C}$. For $T \in \mathscr{L}(\mathscr{X})$, one of the most basic but crucial relations is

$$\sigma(T) \subset \overline{V(T)}$$

(see [27] or [7, Theorem 1 on page 88]). Since the following relations will be used several times, we recall the following:

PROPOSITION 1.1. ([7, Theorem 4 in §9]) *For $T \in \mathscr{L}(\mathscr{X})$,*

$$\overline{\mathrm{co}}\, V(T) = V\big(T, \mathscr{L}(\mathscr{X})\big),$$

where $\overline{\mathrm{co}}\, V(T)$ is the closed convex hull of $V(T)$.

Note that, when $\mathscr{X}$ is a complex Hilbert space $\mathscr{H}$, then $V(T) = W(T)$ and hence $\overline{\mathrm{co}}\, W(T) = V(T, \mathscr{L}(\mathscr{X}))$.

PROPOSITION 1.2. ([7, Corollary 6 in §9]) *For $T \in \mathscr{L}(\mathscr{X})$,*

$$V(T) \subseteq V(T^*) \subseteq \overline{V(T)}.$$

On a Banach space $\mathscr{X}$, a hermitian operator H (i.e., $V(H) \subset \mathbb{R}$) is convexoid, which means that $V(H) = \mathrm{co}\, \sigma(H)$ (see [7, Corollary 11, page 53]), where $\mathrm{co}\star$ is the convex hull of $\star$. It is notable that, even if H is hermitian, H^2 may not be hermitian (see [7, Example 1 on page 58]).

2. $[m,C]$-expansive operators

In this section we discuss $[m,C]$-expansive operators. For this let us start with conjugations.

DEFINITION 2.1. An antilinear isometric involution C on a Banach space $\mathscr{X}$ is called a *conjugation* on $\mathscr{X}$, i.e., C satisfies

$$\|C\| \leqslant 1, C^2 = I,\ C(\alpha x + \beta y) = \overline{\alpha} Cx + \overline{\beta} Cy \ \ (x, y \in \mathscr{X},\ \alpha, \beta \in \mathbb{C}),$$

where $\|C\| = \sup_{\|x\| \leqslant 1} \{\|Cx\| \ : \ x \in \mathscr{X}\}$.

Note that this is a natural generalization of a conjugation on a Hilbert space, as shown in [10].

DEFINITION 2.2. For $T \in \mathscr{L}(\mathscr{X})$, we write $T \leqslant 0$ if $V\big(T, \mathscr{L}(\mathscr{X})\big) \subset (-\infty, 0]$.

Observe that Proposition 1.1 implies that, if $V(T) \subset (-\infty, 0]$, then

$$V(T, \mathscr{L}(\mathscr{X})) \subset (-\infty, 0].$$

Hence $T \leqslant 0$ if and only if $V(T) \subset (-\infty, 0]$. Of course, for a complex Hilbert space $\mathscr{H}$ and an operator $T \in \mathscr{L}(\mathscr{H})$, if $W(T) \subset (-\infty, 0]$, then $T \leqslant 0$.

We are now ready to define an $[m, C]$-expansive operator on $\mathscr{X}$.

DEFINITION 2.3. Let C be a conjugation on $\mathscr{X}$. For $T \in \mathscr{L}(\mathscr{X})$ and $m \in \mathbb{N}$, an operator T is called $[m, C]$*-expansive* if

$$\beta_m(T, C) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^jCT^j \leqslant 0,$$

or equivalently, $V\big(\beta_m(T, C), \mathscr{L}(\mathscr{X})\big) \subset (-\infty, 0]$.

REMARK 2.1. For $T \in \mathscr{L}(\mathscr{H})$, T is called C*-symmetric* if there exists a conjugation C such that $CTC = T^*$. See [15] for details of C-symmetric operators. One of the motivations to study $[m, C]$-expansive operators is that, if T is any C-symmetric operator, then T is m-expansive if and only if it is $[m, C]$-expansive. However, it is not (m, C)-expansive in general. For instance, assume that $\mathscr{H} = \mathbb{C}^2$ and C is a conjugation on $\mathscr{H}$ defined by $C\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] := \left[\begin{smallmatrix} \bar{x} \\ \bar{y} \end{smallmatrix}\right]$. Then $C\begin{bmatrix} a & b \\ c & d \end{bmatrix}C = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$ for any 2×2-matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let $T = \begin{bmatrix} 1+i & 0 \\ 0 & 1 \end{bmatrix} \in \mathscr{L}(\mathscr{H})$. Then $CTC = T^*$ and so T is C-symmetric. By direct computation, we have

$$I - CTCT = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1-i & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1+i & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \leqslant 0$$

and

$$I - CT^*CT = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1+i & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1+i & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-2i & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence we conclude that T is $[1, C]$-expansive but not $(1, C)$-expansive.

For an example of an $[m, C]$-expansive operator on a Banach space, some known result (which is Lemma 2.1) is needed. For a fixed $A \in \mathscr{L}(\mathscr{X})$ the left multiplicative operator by A on $\mathscr{L}(\mathscr{X})$ is defined by

$$L_A(X) = AX \quad (X \in \mathscr{L}(\mathscr{X})).$$

LEMMA 2.1. ([19, Lemma 2.1]) *For $A \in \mathscr{L}(\mathscr{X})$, it holds that*

$$V\big(A, \mathscr{L}(\mathscr{X})\big) = V\big(L_A, \mathscr{L}(\mathscr{L}(\mathscr{X}))\big).$$

EXAMPLE 2.1. Let T be $[m,C]$-expansive on a Hilbert space $\mathscr{H}$ with

$$W(\beta_m(T,C)) = W\left(\sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^jCT^j\right) \subset (-\infty,0].$$

Note that, for a conjugation C of $\mathscr{H}$, M_C defined by $M_C(T) = CTC$ is a a conjugation on $\mathscr{L}(\mathscr{H})$. Then $L_{\beta_m(T,C)}$ is an $[m,M_C]$-expansive operator on the Banach space $\mathscr{L}(\mathscr{H})$. Since it is easy to see

$$\big(M_C L_T^j M_C L_T^j\big)(S) = CT^jCT^jS,$$

we have

$$\left(\sum_{j=0}^{m}(-1)^j \binom{m}{j} M_C L_T^j M_C L_T^j\right)(S) = \left(\sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^jCT^j\right)(S),$$

which implies that

$$\beta_m(L_T, M_C) = L_{\beta_m(T,C)}.$$

By Lemma 2.1,

$$\begin{aligned} V(L_{\beta_m(T,C)}, \mathscr{L}(\mathscr{L}(\mathscr{H}))) &= V\big(\beta_m(T,C), \mathscr{L}(\mathscr{H})\big) \\ &= \overline{\mathrm{co}}\, W\big((\beta_m(T,C)\big) \subset (-\infty,0]. \end{aligned}$$

Hence $L_{\beta_m(T,C)}$ is an $[m,M_C]$-expansive operator on $\mathscr{L}(\mathscr{H})$.

For $T \in \mathscr{L}(\mathscr{X})$, T^* denotes the *dual operator* of T defined by

$$(T^*f)(x) := f(Tx)$$

for all $x \in \mathscr{X}$ and $f \in \mathscr{X}^*$. Similarly C^* on $\mathscr{X}^*$ is the dual operator of C defined by

$$\big(C^*(f)\big)(x) := \overline{f(Cx)} \quad (x \in \mathscr{X},\ f \in \mathscr{X}^*).$$

Then C^* is a conjugation on $\mathscr{X}^*$ (see [10, Theorem 2.6]). Note also that

$$(x,f) \in \Pi \text{ if and only if } (Cx, C^*f) \in \Pi \tag{4}$$

and

$$V(CTC) = \{\overline{z} : z \in V(T)\} \ (=: V(T)^*). \tag{5}$$

See [23, Lemma 2.5 and Theorem 2.6] for more details.

THEOREM 2.1. *If T is an $[m,C]$-expansive operator on a Banach space $\mathscr{X}$, then CTC is also $[m,C]$-expansive on $\mathscr{X}$.*

Proof. By Proposition 1.1 it suffices to show that, for any $(x,f) \in \Pi$,

$$f\big(\beta_m(CTC,C)x\big) \leqslant 0.$$

Observe that

$$\begin{aligned}
\beta_m(CTC,C) &= \sum_{j=0}^{m}(-1)^j \binom{m}{j} C(CTC)^j C\,(CTC)^j \\
&= \sum_{j=0}^{m}(-1)^j \binom{m}{j} T^j C T^j C \\
&= C\beta_m(T,C)C.
\end{aligned}$$

Choose any $(x,f) \in \Pi$. Since $(Cx, C^*f) \in \Pi$ and $\beta_m(T,C) \leqslant 0$, it follows that

$$f\big(\beta_m(CTC,C)x\big) = (C^*f)\big(\beta_m(T,C)Cx\big) \leqslant 0.$$

Hence $V\big(\beta_m(CTC,C)\big) \subset (-\infty,0]$ and then CTC is $[m,C]$-expansive on $\mathscr{X}$. $\square$

For the next result, recall Proposition 1.2, that is, $V(T) \subseteq V(T^*) \subseteq \overline{V(T)}$.

THEOREM 2.2. *If T is an $[m,C]$-expansive operator on $\mathscr{X}$, then T^* is $[m,C^*]$-expansive on $\mathscr{X}^*$.*

Proof. Let us show that $V\big(\beta_m(T^*,C^*)\big) \subset (-\infty,0]$. Due to the relation

$$C^*\beta_m(T,C)^*C^* = \beta_m(T^*,C^*),$$

we have $V\big(\beta_m(T^*,C^*)\big) = V\big(C^*\beta_m(T,C)^*C^*\big)$. Since C^* is a conjugation on $\mathscr{X}^*$, (5) implies that

$$V\big(C^*\beta_m(T,C)^*C^*\big) = V\big(\beta_m(T,C)^*\big)^*,$$

where $V\big(\beta_m(T,C)\big)^* = \{\overline{z} : z \in V\big(\beta_m(T,C)\big)\}$. Proposition 1.2 leads that

$$V\big(\beta_m(T,C)^*\big) \subseteq \overline{V\big(\beta_m(T,C)\big)}$$

(where $\overline{V\big(\beta_m(T,C)\big)}$ is the closure of $V\big(\beta_m(T,C)\big)$ and we therefore have

$$\begin{aligned}
V\big(\beta_m(T^*,C^*)\big) &= V\big(C^*\beta_m(T,C)^*C^*\big) \\
&= V\big(\beta_m(T,C)^*\big) \subseteq \overline{V\big(\beta_m(T,C)\big)} \subset (-\infty,0].
\end{aligned}$$

Hence T^* is an $[m,C^*]$-expansive operator on $\mathscr{X}^*$. $\square$

The following corollary is clear from the previous results.

COROLLARY 2.1. *If T is $[m,C]$-expansive on $\mathscr{X}$, then $C^*T^*C^*$ is $[m,C^*]$-expansive on $\mathscr{X}^*$.*

Based on the theorem above, we now show the invertibility of $[m,C]$-expansive operators, which is Theorem 2.3. Let $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ be the spectrum, the point spectrum and the approximate point spectrum of T, respectively.

PROPOSITION 2.1. *Let T be $[m,C]$-expansive on $\mathscr{X}$. Then $0 \notin \sigma_{ap}(T)$.*

Proof. If $0 \in \sigma_{ap}(T)$, then there exists a sequence $\{x_n\}$ of unit vectors in $\mathscr{X}$ such that $Tx_n \to 0$. Then note that, for any $g \in \mathscr{X}^*$ and $j \in \mathbb{N}$, $g(CT^jCT^jx_n) \to 0$. By Hahn-Banach separation theorem, for x_n there exists $f_n \in \mathscr{X}^*$ such that $f_n(x_n) = 1 = \|f_n\|$. Since $\beta_m(T,C) \leqslant 0$,

$$f_n\big(\beta_m(T,C)x_n\big) \leqslant 0. \tag{6}$$

But

$$f_n\big(\beta_m(T,C)x_n\big) = f_n(x_n) + \sum_{j=1}^{m}(-1)^j \binom{m}{j} f_n\left(CT^jCT^jx_n\right) \to 1,$$

which contradicts (6). Therefore $0 \notin \sigma_{ap}(T)$. $\square$

For the next result, we need the following lemma.

LEMMA 2.2. ([24, Propostion 1.3.1]) *For $T \in \mathscr{L}(\mathscr{X})$, $\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T^*)$.*

THEOREM 2.3. *If T is $[m,C]$-expansive on $\mathscr{X}$, then $0 \notin \sigma(T)$, i.e., T is invertible.*

Proof. By Proposition 2.1, $0 \notin \sigma_{ap}(T)$. By Theorem 2.2, since T^* is an $[m,C^*]$-expansive operator on $\mathscr{X}^*$, similarly $0 \notin \sigma_{ap}(T^*)$ and hence $0 \notin \sigma_p(T^*)$. Therefore Lemma 2.2 implies $0 \notin \sigma(T)$. $\square$

REMARK 2.2. It is worthwhile to emphasize the invertibility of $[m,C]$-expansive operators. Let S be the unilateral shift on ℓ^2-space. Since $I - S^*S = 0$ and $I - SS^* \gneqq 0$, S is $(1,I)$-expansive (and S^* is not), but S is not invertible. However, due to the theorem above, any $[m,C]$-expansive operators are always invertible. This basically depends on perserving $\cdot^*$ on the property of operators. Even though S is $(1,I)$-expansive, S^* is not. In contrast, whenever T is $[m,C]$-expansive, T^* is $[m,C^*]$-expansive due to Theorem 2.2.

3. $[m,\widehat{C}]$-expansive operators

In this section we study $[m,\widehat{C}]$-expansive operators on Banach spaces. We first need the following definitions and results.

DEFINITION 3.1. An operator T on $\mathscr{X}$ is called *hermitian* if $V(T) \subset \mathbb{R}$.

On a Banch space not every operator $T \in \mathscr{L}(\mathscr{X})$ can be represented by $T = H+iK$ with hermitian operators H and K. Let $\mathscr{S} \subset \mathscr{L}(\mathscr{X})$ be the set of all operators having an expression $T = H+iK$ with hermitian operators H and K. Note that such an expression $H+iK$ is unique. For $T = H+iK \in \mathscr{S}$, the mapping $\widehat{C}$ on $\mathscr{S}$ is defined by

$$\widehat{C}(T) := H - iK.$$

Of course, for $T = H+iK \in \mathscr{S}$, since the adjoint operator T^* of T on $\mathscr{X}^*$ is $T^* = H^* + iK^*$ and H^*, K^* are hermitian by Proposition 1.2, let us similarly define $\widetilde{C}$ on $\mathscr{S}^*$ by

$$\widetilde{C}(T^*) = \widetilde{C}(H^* + iK^*) := H^* - iK^*.$$

Then the relation between $\widehat{C}$ and $\widetilde{C}$ is, for any $T \in \mathscr{S}$,

$$\widetilde{C}(T^*) = (\widehat{C}(T))^*,$$

which means that $\widetilde{C} = \cdot^* \widehat{C} \cdot^*$ as in the following diagram:

$$\begin{array}{ccc} H+iK \in \mathscr{S} & \xrightarrow{\widehat{C}} & H-iK \in \mathscr{S} \\ \uparrow \cdot^* & & \downarrow \cdot^* \\ H^*+iK^* \in \mathscr{S}^* & \xrightarrow{\widetilde{C}} & H^*-iK^* \in \mathscr{S}^* \end{array}$$

Even though they are different, the roles of $\widehat{C}$ and $\widetilde{C}$ are essentially the same. Therefore, from now on, let us denote them only by $\widehat{C}$.

DEFINITION 3.2. ([22, Definition 1]) Let H and K be both hermitian. An operator $T = H+iK \in \mathscr{S}$ is called **-hyponormal* if, for all $z \in \mathbb{C}$,

$$\|e^{z\widehat{C}(T)} e^{-\bar{z}T}\| \leqslant 1.$$

Then the *-hyponormality is translation-invariant, i.e., if T is *-hyponormal, then so is $T - z$ for all $z \in \mathbb{C}$. Moreover, the following holds.

LEMMA 3.1. ([22, Theorem 3]) *Let $T = H+iK \in \mathscr{S}$ be *-hyponormal. If $Tx = 0$, then $Hx = Kx = 0$.*

In [3] and [4], de Barra constructed a larger space $\mathscr{X}^\circ$ of $\mathscr{X}$ via a Banach limit such that $\mathscr{X}^\circ$ has the properties on Lemma 3.2. In case of Hilbert spaces, Berberian showed these properties in [5].

LEMMA 3.2. ([3], [4]) *Let $\mathscr{X}$ be a Banach space. Then there exists a larger space, denoted by $\mathscr{X}^\circ$, satisfying the following properties: for $T \in \mathscr{L}(\mathscr{X})$,*

(1) the mapping $T \mapsto T^\circ$ is an isometric isomorphism of $\mathscr{L}(\mathscr{X})$ onto a closed subalgebra of $\mathscr{L}(\mathscr{X}^\circ)$, where T° is an operator on $\mathscr{X}^\circ$,

(2) $\sigma(T) = \sigma(T^\circ)$ and $\sigma_{ap}(T) = \sigma_{ap}(T^\circ) = \sigma_p(T^\circ)$,

(3) $\overline{\mathrm{co}}\, V(T) = V(T^\circ)$.

The following has been already done in [22]. For completeness we give a proof.

PROPOSITION 3.1. *Let* $T = H + iK \in \mathscr{S}$ *be* **-hyponormal. For* $z = a + ib \in \mathbb{C}$ *and a sequence* $\{x_n\}$ *of unit vectors in* $\mathscr{X}$*, if* $(T-z)x_n \to 0$*, then*

$$(H-a)x_n \to 0 \text{ and } (K-b)x_n \to 0 \qquad (as\ n \to \infty).$$

Hence $(\widehat{C}(T) - \overline{z})x_n \to 0$ *as* $n \to \infty$.

Proof. Since $T - z = H - a + i(K - b)$ is *-hyponormal, by (1) and (3) of Lemma 3.2, it holds that $(T-z)^\circ = (H-a)^\circ + i(K-b)^\circ = (H^\circ - a) + i(K^\circ - b)$ is *-hyponormal on $\mathscr{X}^\circ$. Assume that $(H-a)x_n \not\to 0$. Then there exist $\varepsilon > 0$ and a subsequence $\{x_{n_j}\}$ such that $\|(H-a)x_{n_j}\| \geqslant \varepsilon$. Let $y^\circ \in \mathscr{X}^\circ$ be the vector derived from $\{x_{n_j}\}$. Then $y^\circ \neq 0$ and $(H^\circ - a)y^\circ \neq 0$. Over all, $(T^\circ - z)y^\circ = 0$ and $(H^\circ - a)y^\circ \neq 0$, which is a contradiction by Lemma 3.1. Then it holds that $(H-a)x_n \to 0$ and similarly $(K-b)x_n \to 0$. $\square$

REMARK 3.1. (1) An operator $T \in \mathscr{S}$ is called *normal* if there exist hermitian operators H and K such that $T = H + iK$ and $HK = KH$.

(2) An operator $T = H + iK \in \mathscr{S}$ is called *hyponormal* if there exist hermitian operators H and K such that $i(HK - KH) \geqslant 0$.

Obviously it holds that normal $\Longrightarrow$ *-hyponormal $\Longrightarrow$ hyponormal. See more details in Mattila [21] and [22].

Here we define $[m,\widehat{C}]$-expansive operators.

DEFINITION 3.3. For $T = H + iK \in \mathscr{S}$ with two hermitian operators H and K, T is said to be $[m,\widehat{C}]$*-expansive* if

$$\gamma_m(T;\widehat{C}) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} \widehat{C}(T)^j T^j \leqslant 0.$$

REMARK 3.2. In the case of operators on a Hilbert space $\mathscr{H}$, $\mathscr{S} = \mathscr{L}(\mathscr{H})$. Suppose that there exists a conjugation C satisfying $\widehat{C}(T) = CTC$ (for example, see the 2×2-matrix example on Remark 2.1). Then $\big(\widehat{C}(T)\big)^j T^j = CT^jCT^j$. Let $\beta_m(T,C)$ be the operator of Definition 2.3. Then it holds that $\gamma_m(T;\widehat{C}) = \beta_m(T,C)$.

PROPOSITION 3.2. *Let* $T = H + iK \in \mathscr{S}$*. Then* T *is* $[m,\widehat{C}]$*-expansive on* $\mathscr{X}$ *if and only if* $\widehat{C}(T)^* = H^* - iK^*$ *is* $[m,\widehat{C}]$*-expansive on* $\mathscr{X}^*$.

Proof. First observe that $\gamma_m(T^*;\widehat{C}) = \gamma_m(T;\widehat{C})^*$ from

$$\gamma_m(T^*;\widehat{C}) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*j}\widehat{C}(T^*)^j = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \big(\widehat{C}(T)^j T^j\big)^* = \big(\gamma_m(T;\widehat{C})\big)^*$$

by $T^* = \left(\widehat{C}(\widehat{C}(T))\right)^*$. If T is $[m,\widehat{C}]$-expansive, that is,

$$\gamma_m(T;\widehat{C}) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\widehat{C}(T)^jT^j \leqslant 0,$$

then $V\left(\gamma_m(T^*;\widehat{C})\right) = V\left(\left(\gamma_m(T;\widehat{C})\right)^*\right) \subseteq \overline{V\left(\gamma_m(T;\widehat{C})\right)} \subset (-\infty,0]$ by Proposition 1.1. Hence $\widehat{C}(T)^*$ is an $[m,\widehat{C}]$-expansive operator on $\mathscr{X}^*$. The converse implication holds by applying the same argument on $\widehat{C}(T)^*$. $\square$

Let us denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

THEOREM 3.1. *Let $T \in \mathscr{S}$ be $[m,\widehat{C}]$-expansive and *-hyponormal on $\mathscr{X}$. Then the following statements hold.*

(i) *If m is even, then $\sigma_{ap}(T) \subseteq \mathbb{T}$. In this case, $\sigma(T) \subseteq \mathbb{T}$ or $\sigma(T) = \overline{\mathbb{D}}$.*

(ii) *If m is odd, then $\sigma_{ap}(T) \subseteq \mathbb{C}\setminus\mathbb{D}$. Hence T is injective and $R(T)$ is closed, where $R(T)$ is the range of T.*

Proof. Let $z \in \sigma_{ap}(T)$. Then there is a sequence $\{x_n\}$ of unit vectors in $\mathscr{X}$ such that $(T-z)x_n \to 0$ as $n \to \infty$. Then $(T^j - z^j)x_n \to 0$ and by Proposition 3.1 it holds that $(\widehat{C}(T)^j - \overline{z}^j)x_n \to 0$ for each $j \in \mathbb{N}$. Since

$$\widehat{C}(T)^jT^j = \widehat{C}(T)^j(T^j - z^j) + z^j(\widehat{C}(T)^j - \overline{z}^j) + |z|^{2j},$$

it follows that

$$(\widehat{C}(T)^jT^j - |z|^{2j})x_n \to 0 \qquad (\text{as } n \to \infty).$$

For $(x_n, f_n) \in \Pi$,

$$\begin{aligned} f_n\left(\gamma_m((T;\widehat{C})x_n\right) &\to f_n\left(\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}|z|^{2j}\right)x_n\right) \ (\text{as } n \to \infty) \\ &= (1-|z|^2)^m f_n(x_n) = (1-|z|^2)^m. \end{aligned}$$

Since $\gamma_m(T;\widehat{C}) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\widehat{C}(T)^jT^j \leqslant 0$, it follows that $(1-|z|^2)^m \leqslant 0$.

For (i), let m be even. Then $(1-|z|^2)^m = 0$, and so $|z| = 1$. Hence $\sigma_{ap}(T) \subseteq \mathbb{T}$. Since $\partial\sigma(T) \subseteq \sigma_{ap}(T) \subseteq \mathbb{T}$, $\sigma(T) \subseteq \mathbb{T}$ or $\sigma(T) = \overline{\mathbb{D}}$ by a similar method on the proof of [18, Proposition 3.4].

For (ii), if m is odd and $|z| < 1$, then $0 < (1-|z|^2)^m \leqslant 0$, which is a contradiction. Hence $\sigma_{ap}(T) \subseteq \mathbb{C}\setminus\mathbb{D}$. So T is injective and $R(T)$ is closed since $0 \notin \sigma_{ap}(T)$. $\square$

A Banach space is called *uniformly c-convex* if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that $\|y\| < \varepsilon$ whenever $\|x\| = 1$ and $\|x + \lambda y\| \leqslant 1 + \delta$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$. See [22] for more details.

DEFINITION 3.4. For $T = H + iK \in \mathscr{S}$, T is called $[m,\widehat{C}]$*-isometric* if

$$\gamma_m(T;\widehat{C}) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} \widehat{C}(T)^j T^j = 0.$$

Since $\gamma_{m+1}(T;\widehat{C}) = \gamma_m(T;\widehat{C}) - \widehat{C}(T)\gamma_m(T;\widehat{C})T$, it is clear that if T is $[m,\widehat{C}]$-isometric, then T is $[n,\widehat{C}]$-isometric for all $n \geqslant m$. Since $\gamma_m(T;\widehat{C})^* = \gamma_m(T^*;\widehat{C})$, it follows that T is $[m,\widehat{C}]$-isometric on $\mathscr{X}$ if and only if $\widehat{C}(T)^*$ is $[m,\widehat{C}]$-isometric on $\mathscr{X}^*$.

The reason to give more geometric structure on $\mathscr{X}$ is that Proposition 3.1 may not hold for hyponormal operators on a general Banach space. With the uniform c-convexity on $\mathscr{X}$, Mattila showed the following lemma.

LEMMA 3.3. ([21, Theorem 2.7]) *Let $T = H + iK \in \mathscr{S}$ be hyponormal on uniformly c-convex $\mathscr{X}$. If $\{x_n\}$ is a bounded sequence in $\mathscr{X}$ and $Tx_n \to 0$, then $Hx_n \to 0$ and $Kx_n \to 0$.*

THEOREM 3.2. *Let $\mathscr{X}$ be uniformly c-convex. If $T = H + iK \in \mathscr{S}$ is $[m,\widehat{C}]$-isometric and hyponormal on $\mathscr{X}$, then $\sigma_{ap}(T) \subseteq \mathbb{T}$.*

Proof. Let $z = a + ib \in \sigma_{ap}(T)$ and let $\{x_n\}$ be a sequence of unit vectors such that $(T - z)x_n \to 0$. Since $T - z = (H - a) + i(K - b)$ is hyponormal and $\mathscr{X}$ is uniformly c-convex, Lemma 3.3 says that $(H - a)x_n \to 0$, $(K - b)x_n \to 0$ and hence $(\widehat{C}(T) - \overline{z})x_n \to 0$ as $n \to \infty$. Moreover, $(T^j - z^j)x_n \to 0$ and $(\widehat{C}(T)^j - \overline{z}^j)x_n \to 0$ for any $j \in \mathbb{N} \cup \{0\}$. Since

$$\left(\widehat{C}(T)^j T^j - |z|^{2j}\right)x_n = \left(\widehat{C}(T)^j(T^j - z^j) + z^j(\widehat{C}(T)^j - \overline{z}^j)\right)x_n \to 0,$$

we have that, as $n \to \infty$,

$$\left(\gamma_m(T;\widehat{C}) - (1 - |z|^2)^m\right)x_n = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(\widehat{C}(T)^j T^j - |z|^{2j}\right)x_n \to 0. \tag{7}$$

Since T is $[m,\widehat{C}]$-isometric, i.e., $\gamma_m(T;\widehat{C}) = 0$, it holds that $(1 - |z|^2)^m = 0$ and so $|z| = 1$. Hence $\sigma_{ap}(T) \subseteq \mathbb{T}$. $\square$

EXAMPLE 3.1. Let $\mathscr{C}_1(\mathscr{H})$ be the trace class of operators on a complex Hilbert space $\mathscr{H}$. Then $\mathscr{C}_1(\mathscr{H})$ is uniformly c-convex by Theorem 3·2 of Mattila [21]. For $T \in \mathscr{L}(\mathscr{H})$, define an operator L_T on $\mathscr{C}_1(\mathscr{H})$ by $L_T(X) := TX$ $(X \in \mathscr{C}_1)$. Let $T = H + iK$ be a hyponormal operator on $\mathscr{H}$ with hermitian operators H and K. Then $L_T = L_H + iL_K$ is a hyponormal operator on $\mathscr{C}_1(\mathscr{H})$ (by Corollary 4·5 in [21]) and $\widehat{C}(L_T) = L_H - iL_K$. If T is isometric, then L_T is $[1,\widehat{C}]$-isometric on the uniformly c-convex space $\mathscr{C}_1(\mathscr{H})$. Then Theorem 3.2 implies that $\sigma_{ap}(L_T) \subseteq \mathbb{T}$. For example, the operator L_S (where S is the unilateral shift) is $[1,\widehat{C}]$-isometric and hyponormal on the uniformly c-convex space $\mathscr{C}_1(\ell^2)$.

DEFINITION 3.5. An operator $T \in \mathscr{L}(\mathscr{X})$ has *the single-valued extension property* at $z_0 \in \mathbb{C}$ if for every neighborhood U of z_0 and any analytic function $f : U \to \mathscr{X}$, $f(z) \equiv 0$ whenever $(T-z)f(z) \equiv 0$. An operator T on $\mathscr{X}$ has the single-valued extension property if it has the single-valued extension property at all $z \in \mathbb{C}$.

Next we define the property (NP): for all $z \in \mathbb{C}$ and all non-zero vectors $x \in \mathscr{X}$, $(T-z)x = 0$ implies $(\widehat{C}(T) - \overline{z})x = 0$. The property (NP) means “a normal point”. An operator $T = H + iK \in \mathscr{S}$ with two hermitian operators H and K is said to have (NP) if T satisfies the property (NP).

THEOREM 3.3. *Let $T = H + iK \in \mathscr{S}$ be $[m, \widehat{C}]$-isometric on $\mathscr{X}$. If T has* (NP)*, then T has the single-valued extension property on $\mathbb{C} \setminus \mathbb{T}$.*

Proof. For any $z_0 \in \mathbb{C}$ and for every neighborhood G of z_0, let $(T-z)f(z) \equiv 0$ for $z \in G$. Then by the condition (NP), we have $(T^j - z^j)f(z) \equiv 0$ and so $(\widehat{C}(T)^j - \overline{z}^j)f(z) \equiv 0$. Since $\gamma_m(T; \widehat{C}) = 0$, it follows from the computation similar to (7) that $(1-|z|^2)^m f(z) \equiv 0$ and hence $f(z) \equiv 0$ for $z \notin \mathbb{T}$. Therefore, T has the single-valued extension property on $\mathbb{C} \setminus \mathbb{T}$. □

By Proposition 3.1 and the above, if T is *-hyponormal on a Banach space, then T has (NP). Hence we have the following corollary.

COROLLARY 3.1. *Let T be *-hyponormal on $\mathscr{X}$. If T is $[m, \widehat{C}]$-isometric, then T has the single-valued extension property on $\mathbb{C} \setminus \mathbb{T}$.*

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