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NEW RESULTS ON SYMMETRIZED CONVEX SEQUENCES

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Abstract. In this article, we establish new inequalities involving symmetrized convex sequences. The obtained results involve a new range of applications that contains the set of convex sequences. Some applications are given at the end of this paper.

1. Introduction and auxiliary results

Real-valued symmetrized convex sequences and positive sequences that are symmetric about a point are two essential classes of sequences that have been extensively studied in mathematical analysis. Symmetric sequences have numerous applications in different branches of mathematics, such as number theory, approximation theory, and signal processing. They have also been used in various applied fields, such as physics, engineering, and computer science.

The investigation of convex sequences probably started in the book Mitrinovic [8]. In recent years, there has been a growing interest in the study of symmetric sequences, and several important results have been obtained. For example, in [6], the author studied the properties of convex sequences and got a characterization of them in terms of their generating functions. In [1], the author investigated the asymptotic behavior of symmetric sequences and proved several important results related to their growth rates. Other results can be found in [3], [4], [9], [10], [11], [12], [13].

In this article, we will extend some of the existing results on real-valued symmetrized convex sequences and positive sequences that are symmetric about a point. We will study the properties of these sequences and investigate their connections to other important classes of sequences.

Throughout this paper, we denote by I the set $\{1,2,\ldots,n\}$, and we denote by σ the integer part of $\frac{n+1}{2}$ (or $\sigma = \left[\frac{n+1}{2}\right]$). For citing some extension results, we need to define a new class of real sequences

$$E(I) = \left\{ (a_k)_{k=1}^{n} : \frac{a_\sigma + a_{n+1-\sigma}}{2} \leqslant \frac{a_k + a_{n+1-k}}{2} \leqslant \frac{a_1 + a_n}{2} \text{ for all } k = 1,2,\ldots,n \right\}$$

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where, $(a_k)_{k=1}^{n}$ is a real sequence and

$$E_{-}(I) = \{(a_k)_{k=1}^{n} : -(a_k)_{k=1}^{n} \in E(I)\}.$$

DEFINITION 1. [5] Let $(a_k)_{k=1}^{n}$ be a sequence of real numbers, a is called a convex sequence if for all $k = 2, \ldots, n-1$, we have

$$a_{k-1} + a_{k+1} \geqslant 2a_k.$$

If the opposite inequality holds, the sequence a is said to be concave.

We need to define the following type of sequences for proving our results:

DEFINITION 2. A real-valued sequence $(a_k)_{k=1}^{n}$ is said to be a symmetrized convex sequence, if $A_k = \frac{1}{2}(a_{n+1-k} + a_k)$ is convex. Conversely, if $A_k = \frac{1}{2}(a_{n+1-k} + a_k)$ is concave, the sequence $(a_k)_{k=1}^{n}$ is said to be symmetrized concave.

We denote respectively by $Scon(I)$ $(con_{-}(I))$ the set of all symmetrized convex (concave) sequences for all $k \in I$.

DEFINITION 3. A real-valued sequence $(a_k)_{k=1}^{n}$ is said to be symmetrized decreasing (increasing) for $k \in I$, if the sequence $A_k = \frac{1}{2}(a_{n+1-k} + a_k)$ is decreasing (increasing) for all $k = 1, \ldots, \left[\frac{n+1}{2}\right]$.

We denote by $Sd(I)$ (respectively $Sd_{-}(I)$) the set of all symmetrized decreasing (respectively increasing) sequences for all $k \in I$.

EXAMPLE 1. We consider the following sequence:

$$a_k = \begin{cases} e^k & \text{for } 1 \leqslant k \leqslant \frac{n+1}{2} \\ 0 & \text{for } \frac{n+1}{2} < k \leqslant n \end{cases} \quad n \text{ is a larger integer.}$$

It is easy to see that the sequence $(a_k)_{k=1}^{n}$ is not symmetrized increasing but the sequence $(A_k)_{k=1}^{n}$ defined by

$$A_k = \frac{1}{2}(a_{n+1-k} + a_k) = \begin{cases} \frac{1}{2}e^k & \text{for } 1 \leqslant k \leqslant \frac{n+1}{2} \\ \frac{1}{2}e^{n+1-k} & \text{for } \frac{n+1}{2} < k \leqslant n \end{cases}$$

is symmetrized increasing.

We propose to the reader the following proposition. We have:

PROPOSITION 1. *If $(a_k)_{k=1}^n$ is a convex sequence. Therefore the sequence $(A_k)_{k=1}^n$ is convex too. The inverse is false.*

For the proof of the above proposition 1, see Lemma 2 below. Note that if we take the sequence $(a_k)_{k=1}^n$ defined by

$$a_k = (k-\alpha)^3, \text{ for } k \in \{1,2,\ldots,n\} \text{ with } 1 \leqslant \alpha \leqslant \frac{n+1}{2}.$$

It is easy to see that $(a_k)_{k=1}^n$ is not convex sequence, but $(A_k)_{k=1}^n$, where

$$A_k = \frac{1}{2}\left((k-\alpha)^3 + (n+1-k-\alpha)^3\right), \text{ for } k \in \{1,2,\ldots,n\}$$

is convex.

In [4], the authors proved the following result:

THEOREM 1. *Let $a = (a_k)_{k=1}^n$ be a convex sequences of real numbers, and $p = (p_k)_{k=1}^n$ be a positive sequence symmetric about $\sigma = \left[\frac{n+1}{2}\right]$. Then we have*

$$\left(\frac{a_\sigma + a_{n+1-\sigma}}{2}\right)\sum_{k=1}^n p_k \leqslant \sum_{k=1}^n p_k a_k \leqslant \left(\frac{a_1+a_n}{2}\right)\sum_{k=1}^n p_k. \tag{1}$$

If $a = (a_k)_{k=1}^n$ is concave sequence of real numbers, then the inequality (1) is reversed.

It is very natural to ask: *Can we obtain the inequality (1), if $(a_k)_{k=1}^n$ is not convex sequence and $(p_k)_{k=1}^n$ is not necessarily positive, and not necessarily symmetric sequence?*

2. Main results

In this paper, we give an answer to the above question, when $(a_k)_{k=1}^n$ is convex or symmetrized convex sequence or $(a_k)_{k=1}^n$ is a real sequence in the set $E(I)$. So, our aim is to extend the validity of (1) for the case where $(a_k)_{k=1}^n$ is an element in $E(I)$, and to extend the result when $(p_k)_{k=1}^n$ is a real sequence, not necessarily positive and not necessarily symmetric with respect to $\sigma = \left[\frac{n+1}{2}\right]$. We are in measure to prove the following “several” theorems:

THEOREM 2. *Let $(a_k)_{k=1}^n$ be a real sequence in the set $E(I)$. Then we have*

$$\frac{a_\sigma + a_{n+1-\sigma}}{2} \leqslant \frac{1}{n}\sum_{k=1}^n a_k \leqslant \frac{a_1+a_n}{2}, \text{ where } \sigma = \left[\tfrac{n+1}{2}\right]. \tag{2}$$

If $(a_k)_{k=1}^n \in E_-(I)$, then the inequality (2) is reversed.

THEOREM 3. *Let $(a_k)_{k=1}^n$ be a real sequence in the set $E(I)$, and let $(p_k)_{k=1}^n$ be a positive sequence symmetric about $\sigma = \left[\frac{n+1}{2}\right]$ (or $p_k = p_{n+1-k}$ for all $k \in I$). Then we have*

$$\left(\frac{a_\sigma + a_{n+1-\sigma}}{2}\right)\sum_{k=1}^n p_k \leqslant \sum_{k=1}^n p_k a_k \leqslant \left(\frac{a_1+a_n}{2}\right)\sum_{k=1}^n p_k. \tag{3}$$

If $(a_k)_{k=1}^n \in E_-(I)$, then the inequality (3) is reversed.

COROLLARY 1. *Let $(a_k)_{k=1}^n$ be a real sequences in the set $Sd(I)$, and let $(p_k)_{k=1}^n$ be a positive sequence symmetric about $\sigma = \left[\frac{n+1}{2}\right]$ (or $p_k = p_{n+1-k}$ for all $k \in I$). Then the inequality (3) holds. If $(a_k)_{k=1}^n \in Sd_-(I)$, then the inequality (3) is reversed.*

COROLLARY 2. *Let $(a_k)_{k=1}^n$ be a real sequences in the set $Scon(I)$, and let $(p_k)_{k=1}^n$ be a positive sequence symmetric about $\sigma = \left[\frac{n+1}{2}\right]$ (or $p_k = p_{n+1-k}$ for all $k \in I$). Then the inequality (3) holds. If $(a_k)_{k=1}^n \in Scon_-(I)$, then the inequality (3) is reversed.*

COROLLARY 3. *Let $(a_k)_{k=1}^n$ be a convex sequence of real numbers and let $(p_k)_{k=1}^n$ be a positive sequence symmetric about $\sigma = \left[\frac{n+1}{2}\right]$ (or $p_k = p_{n+1-k}$ for all $k \in I$). Then the inequality (3) holds. If $(a_k)_{k=1}^n$ is concave sequence of real numbers, then the inequality (3) is reversed.*

THEOREM 4. *Let $(a_k)_{k=1}^n$ be a convex sequence of real numbers and let $(p_k)_{k=1}^n$ be a positive sequence symmetric about $\sigma = \left[\frac{n+1}{2}\right]$ (or $p_k = p_{n+1-k}$ for all $k \in I$). Then we have*

$$\begin{aligned}\frac{a_\alpha + a_{\sigma+1-\alpha} + a_\beta + a_{n+2-\sigma-\beta}}{4}\left(\sum_{k=1}^n p_k\right) &\leqslant \sum_{k=1}^n \left(\frac{p_k}{k}\sum_{i=1}^k a_i\right)\\ &\leqslant \frac{3a_1+a_n}{4}\left(\sum_{k=1}^n p_k\right),\end{aligned} \tag{4}$$

where, $\alpha = \left[\frac{\sigma+1}{2}\right]$ and $\beta = \left[\frac{n+2-\sigma}{2}\right]$. If $(a_k)_{k=1}^n$ is a concave sequence of real numbers, then the inequality (4) is reversed.

Taking $p_k = 1$ for all $k \in I$, in Theorem 4, we get the following result:

THEOREM 5. *Let $(a_k)_{k=1}^n$ be a convex sequence of real numbers. Then we have*

$$\frac{a_\alpha + a_{\sigma+1-\alpha} + a_\beta + a_{n+2-\sigma-\beta}}{4} \leqslant \frac{1}{n}\sum_{k=1}^n \left(\sum_{i=1}^k \frac{a_i}{k}\right) \leqslant \frac{3a_1+a_n}{4} \tag{5}$$

where, $\alpha = \left[\frac{\sigma+1}{2}\right]$ and $\beta = \left[\frac{n+2-\sigma}{2}\right]$. If $(a_k)_{k=1}^n$ is a concave sequence of real numbers, then the inequality (5) is reversed.

In the next part, taking $p_k = 1$ for all $k \in I$, in Theorem 3, we obtain the following corollaries:

COROLLARY 4. *Let* $(a_k)_{k=1}^n$ *be a real sequence in the set* $E(I)$, *then the inequality* (2) *holds. If* $(a_k)_{k=1}^n \in E_-(I)$, *then the inequality* (2.1) *is reversed.*

COROLLARY 5. *Let* $(a_k)_{k=1}^n$ *be a real sequence in the set* $Sd(I)$, *then the inequality* (2) *holds. If* $(a_k)_{k=1}^n \in Sd_-(I)$, *then the inequality* (2) *is reversed.*

COROLLARY 6. *Let* $(a_k)_{k=1}^n$ *be a real sequence in the set* $Scon(I)$, *then the inequality* (2) *holds. If* $(a_k)_{k=1}^n \in Scon_-(I)$, *then the inequality* (2) *is reversed.*

COROLLARY 7. *Let* $(a_k)_{k=1}^n$ *be a convex sequence of real numbers, then the inequality* (2) *holds. If* $(a_k)_{k=1}^n$ *be a concave sequence of real numbers, then the inequality* (2) *is reversed.*

A QUESTION. *Is it possible to obtain analog results in the case where* $(p_k)_{k=1}^n$ *is not symmetric about* σ, *and not necessarily positive?*

In this part, we give an answer to the above question. So, we propose to the reader the following theorems.

THEOREM 6. *Let* $(a_k)_{k=1}^n$ *be a real sequence in the set* $E(I)$, *let* $(p_k)_{k=1}^n$ *be a real sequence such that* $p_k + p_{n+1-k} \geqslant 0$ *for all* $k \in I$, *and*

$$\sum_{k=1}^{n} a_k p_k = \sum_{k=1}^{n} a_k p_{n+1-k} \tag{6}$$

then, the inequality (3) *holds. If* $(a_k)_{k=1}^n \in E_-(I)$, *then the inequality* (3) *is reversed.*

THEOREM 7. *Let* $(a_k)_{k=1}^n$ *be a real sequence in the set* $E(I)$, *let* $(p_k)_{k=1}^n$ *be a real sequence such that* $p_k + p_{n+1-k} \geqslant 0$ *for all* $k \in I$*), and*

$$\sum_{k=1}^{n} a_k p_k \leqslant \sum_{k=1}^{n} a_k p_{n+1-k} \tag{7}$$

then we have

$$\sum_{k=1}^{n} p_k a_k \leqslant \left(\frac{a_1 + a_n}{2}\right) \sum_{k=1}^{n} p_k. \tag{8}$$

If $(a_k)_{k=1}^n \in E_-(I)$, *then the inequality* (8) *is reversed.*

THEOREM 8. *Let* $(a_k)_{k=1}^n$ *be a real sequence in the set* $E(I)$, *let* $(p_k)_{k=1}^n$ *be a real sequence such that* $p_k + p_{n+1-k} \geqslant 0$ *for all* $k \in I$, *and*

$$\sum_{k=1}^{n} a_k p_{n+1-k} \leqslant \sum_{k=1}^{n} a_k p_k \tag{9}$$

then we have

$$\left(\frac{a_\sigma + a_{n+1-\sigma}}{2}\right) \sum_{k=1}^{n} p_k \leqslant \sum_{k=1}^{n} p_k a_k. \tag{10}$$

If $(a_k)_{k=1}^n \in E_-(I)$, *then the inequality* (10) *is reversed.*

In Theorem 6, if $(p_k)_{k=1}^n$ be a symmetric sequence about $\sigma = \left[\frac{n+1}{2}\right]$, then the condition (6) holds, but the inverse is false, in general.

EXAMPLE 2. We take $a_k = k$ for $k \in \{1,\dots,5\}$ and $p_1 = 1,\ p_2 = p_5 = 2,\ p_3 = p_4 = 0$. Clearly $(p_k)_{k=1}^5$ is not symmetric with respect to 3 but the condition (6) is holds.

REMARK 1. In Theorem 6, if $p_k \geqslant 0$ for $k \in \{1,\dots,n\}$, then $p_k + p_{n+1-k} \geqslant 0$ for $k \in \{1,\dots,n\}$, but the inverse is false in general.

EXAMPLE 3. We take $p_k = k^2 - 4$ for $k \in \{0,1,\dots,10\}$, then it is easy to see that p_k is not positive sequence for $k \in \{0,1,2\}$, but $p_k + p_{n+1-k} = 2k^2 - 20k + 92 = 2\left((k-5)^2 + 21\right) \geqslant 0$ for all $k \in \{1,\dots,10\}$.

From Theorem 6, and Remark 1, we can deduce the following corollaries:

COROLLARY 8. *Let $(a_k)_{k=1}^n$ be a real sequence in the set $Sd(I)$, let $(p_k)_{k=1}^n$ be real sequence such that $p_k + p_{n+1-k} \geqslant 0$ for all $k \in I$), if the condition (6) holds, then the inequality (3) holds. If $(a_k)_{k=1}^n \in Sd_-(I)$, then the inequality (3) is reversed.*

COROLLARY 9. *Let $(a_k)_{k=1}^n$ be a real sequence in the set $Scon(I)$, let $(p_k)_{k=1}^n$ be a real sequence such that $p_k + p_{n+1-k} \geqslant 0$ for all $k \in I$. If the condition (2.5) holds, then the inequality (3) holds. If $(a_k)_{k=1}^n \in Scon_-(I)$, then the inequality (3) is reversed.*

COROLLARY 10. *Let $(a_k)_{k=1}^n$ be a convex sequence of real numbers, let $(p_k)_{k=1}^n$ be a real sequence such that $p_k + p_{n+1-k} \geqslant 0$ for all $k \in I$. If the condition (6) holds, then the inequality (3) holds. If $(a_k)_{k=1}^n$ be a real concave sequence, then the inequality (3) is reversed.*

3. Proof of main results

Let us present the following lemmas.

LEMMA 1. [4] *Suppose that the sequence $(a_k)_{k=1}^n$ is convex (or concave) of real numbers, then the sequence $(c_k)_{k=1}^n$, where,*

$$c_k = \frac{a_k + a_{n+1-k}}{2}$$

is decreasing (increasing) for all $k = 1,2,\dots,\left[\frac{n+1}{2}\right]$, and increasing (decreasing) for all $k = \left[\frac{n+1}{2}\right],\dots,n$.

The following lemma is necessary to prove our theorems:

LEMMA 2. *Suppose that the sequence* $(a_k)_{k=1}^n$ *is convex (concave) of real numbers, then the sequence* $(c_k)_{k=1}^n$, *where,*

$$c_k = \frac{a_k + a_{n+1-k}}{2}, \quad k = 2, 3, \ldots, n-1,$$

is convex (concave).

Proof. Assume that $(a_k)_{k=1}^n$ is convex sequence, then we have for all $k = 2, 3, \ldots, n-1$

$$\begin{aligned} 2(c_{k-1} + c_{k+1}) &= a_{k-1} + a_{n+1-(k-1)} + a_{k+1} + a_{n+1-(k+1)} \\ &\geqslant 2(a_k + a_{n+1-k}) = 4c_k \end{aligned}$$

this equivalent to $c_{k-1} + c_{k+1} \geqslant 2c_k$ for all $k = 2, 3, \ldots, n-1$. Hence, $(c_k)_{k=1}^n$ is convex sequence. If $(a_k)_{k=1}^n$ is a concave sequence, then by using a similar proof as before, we obtain the result. □

LEMMA 3. [13] *Assume that* $(a_k)_{k=1}^n$ *is convex sequence, then the sequence* $(A_k)_{k=1}^n$, *where*

$$A_k = \frac{1}{k} \sum_{i=1}^{k} a_i, \; k \in \{1, 2, \ldots, n\}$$

is convex.

LEMMA 4. *Let* $I = \{1, 2, \ldots, n\}$.

$(a:)$ *Let* $Con(I)$ *be the set of all convex sequences, then we have*

$$Con(I) \subset Scon(I) \subset Sd(I) \subset E(I).$$

$(b:)$ *Let* $Con_-(I)$ *be the set of all concave sequences, then we have*

$$Con_-(I) \subset Scon_-(I) \subset Sd_-(I) \subset E_-(I).$$

Proof. $(a:)$ *First*, we prove that $Con(I) \subset Scon(I)$. Let $(a_k)_{k=1}^n$ be a convex sequence. Then by Lemma 2, we deduce that $(a_k)_{k=1}^n$ is symmetrized convex sequence.

Second, we prove that $Scon(I) \subset Sd(I)$. Assume that $(a_k)_{k=1}^n$ be a symmetrized convex sequence, by definition the sequence $(c_k)_{k=1}^n$ where

$$c_k = \frac{a_k + a_{n+1-k}}{2}, \; k \in I = \{1, 2, \ldots, n\}$$

is convex, then by Lemma 1 the sequence $(c_k)_{k=1}^n$ is decreasing for all $k = 1, 2, \ldots, \left[\frac{n+1}{2}\right]$, and increasing for all $k = \left[\frac{n+1}{2}\right], \ldots, n$. Hence, $(a_k)_{k=1}^n \in Sd(I)$.

Finally, we will show that $Sd(I) \subset E(I)$. Assume that $(a_k)_{k=1}^{n} \in Sd(I)$, then by definition the sequence $c_k = \frac{a_k + a_{n+1-k}}{2}$ is decreasing for all $k = 1, 2, \ldots, \left[\frac{n+1}{2}\right]$, then we have

$$\frac{a_\sigma + a_{n+1-\sigma}}{2} \leqslant \frac{a_k + a_{n+1-k}}{2} \leqslant \frac{a_1 + a_n}{2}$$

and for all $k = \left[\frac{n+1}{2}\right], \ldots, n$, the sequence $c_k = \frac{a_k + a_{n+1-k}}{2}$ is increasing, then we have

$$\frac{a_\sigma + a_{n+1-\sigma}}{2} \leqslant \frac{a_k + a_{n+1-k}}{2} \leqslant \frac{a_1 + a_n}{2}$$

which implies that for all $k \in I$, we have

$$\frac{a_\sigma + a_{n+1-\sigma}}{2} \leqslant \frac{a_k + a_{n+1-k}}{2} \leqslant \frac{a_1 + a_n}{2},$$

hence, $(a_k)_{k=1}^{n} \in E(I)$.

The first part of Lemma 4 is thus proved. To prove $(b:)$, we use the same arguments as in the proof of (a). □

REMARK 2. The following inclusion is strict.

$$Con(I) \subsetneq Scon(I) \subsetneq Sd(I) \subsetneq E(I)$$

and

$$Con_{-}(I) \subsetneq Scon_{-}(I) \subsetneq Sd_{-}(I) \subsetneq E_{-}(I).$$

(1) Let $I = \{1, 2, \ldots, 7\}$. It sufficient to take $a_k = (k-2)^3$ for $k \in I$ is not convex sequence, but $A_k = \frac{1}{2}\left((k-2)^3 + (5-k)^3\right)$ is convex sequence for $k \in I$.

(2) We consider the sequence $a_k = \sqrt{|k-4|}$, for $k \in I$, then we have $(a_k) \notin Scon(I)$ but $(a_k) \in Sd(I)$.

(3) If we take $a_k = \cos(k\pi - \pi)$ for all $k \in I$, then we have

$$-1 \leqslant \frac{\cos(k\pi - \pi) + \cos(7\pi - k\pi)}{2} \leqslant 1$$

then $(a_k)_{k=1}^{7} \in E(I)$ but $(a_k)_{k=1}^{7} \notin Sd(I)$.

Proof of Theorem 2. Suppose that $(a_k)_{k=1}^{n} \in E(I)$, then for any $k \in I = \{1, 2, \ldots, n\}$, we have:

$$\frac{a_\sigma + a_{n+1-\sigma}}{2} \leqslant \frac{a_k + a_{n+1-k}}{2} \leqslant \frac{a_1 + a_n}{2} \tag{11}$$

Summing both sides of inequalities (11) with respect to $k \in I$, we obtain

$$\sum_{k=1}^{n} \frac{a_\sigma + a_{n+1-\sigma}}{2} \leqslant \sum_{k=1}^{n} \frac{a_k + a_{n+1-k}}{2} \leqslant \sum_{k=1}^{n} \frac{a_1 + a_n}{2} \tag{12}$$

which is equivalent to

$$\frac{a_\sigma + a_{n+1-\sigma}}{2} \leqslant \frac{1}{n}\sum_{k=1}^{n} a_k \leqslant \frac{a_1 + a_n}{2}.$$

If $(a_k)_{k=1}^n \in E_-(I)$, then by using a similar proof, we obtain the result. □

Then the proof of Theorem 2 is thus completed.

Proof of Theorem 3. Assume that $(a_k)_{k=1}^n \in E(I)$, then we have (11). Multiplying inequality (11) by positive sequence p_k, we obtain

$$\frac{a_\sigma + a_{n+1-\sigma}}{2} p_k \leqslant \frac{a_k + a_{n+1-k}}{2} p_k \leqslant \frac{a_1 + a_n}{2} p_k \quad (1 \leqslant k \leqslant n). \tag{13}$$

Summing both sides of (13) with respect to $(1 \leqslant k \leqslant n)$, we obtain

$$\frac{a_\sigma + a_{n+1-\sigma}}{2} \sum_{k=1}^{n} p_k \leqslant \sum_{k=1}^{n} \frac{a_k + a_{n+1-k}}{2} p_k \leqslant \frac{a_1 + a_n}{2} \sum_{k=1}^{n} p_k,$$

and then using the symmetry of the sequence $(p_k)_{k=1}^n$ (or $p_k = p_{n+1-k}$) with respect to $\left[\frac{n+1}{2}\right]$, yields

$$\frac{a_\sigma + a_{n+1-\sigma}}{2} \sum_{k=1}^{n} p_k \leqslant \sum_{k=1}^{n} a_k p_k \leqslant \frac{a_1 + a_n}{2} \sum_{k=1}^{n} p_k.$$

If $(a_k)_{k=1}^n \in E_-(I)$, we use a similar proof as above. □

Proof of Corollaries 1, 2 *and* 3. This can be concluded by using lemma 4 and theorem 3. □

Proof of Theorem 4. Assume that $(a_k)_{k=1}^n$ be a convex sequence, and let $(p_k)_{k=1}^n$ be a positive sequence and symmetric about σ, then by Lemma 3, the sequence $(A_k)_{k=1}^n$ is convex. Thanks to Theorem 3 for the sequences $(A_k)_{k=1}^n$ and $(p_k)_{k=1}^n$, we obtain the following inequalities

$$\frac{A_\sigma + A_{n+1-\sigma}}{2} \sum_{k=1}^{n} p_k \leqslant \sum_{k=1}^{n} A_k p_k \leqslant \frac{A_1 + A_n}{2} \sum_{k=1}^{n} p_k \tag{14}$$

Substituting these inequalities in (14), we have

$$A_\sigma = \frac{1}{\sigma}\sum_{i=1}^{\sigma} a_i \geqslant \frac{a_\alpha + a_{\sigma+1-\alpha}}{2},$$

$$A_{n+1-\sigma} = \frac{1}{n+1-\sigma}\sum_{i=1}^{n+1-\sigma} a_i \geqslant \frac{a_\beta + a_{n+2-\sigma-\beta}}{2}$$

and

$$A_n = \frac{1}{n}\sum_{i=1}^{n} a_i \leqslant \frac{a_1 + a_n}{2}$$

where, $\alpha = \left[\frac{\sigma+1}{2}\right]$ and $\beta = \left[\frac{n+2-\sigma}{2}\right]$. Since $A_1 = a_1$, then we obtain (6). $\square$

Proof of Theorem 6. Assume that $(a_k)_{k=1}^{n} \in E(I)$, then we have (11). Multiplying (11) by positive sequence $(p_k + p_{n+1-k})$, for all $(1 \leqslant k \leqslant n)$, we obtain

$$\begin{aligned}(a_\sigma + a_{n+1-\sigma})(p_k + p_{n+1-k}) &\leqslant (a_k + a_{n+1-k})(p_k + p_{n+1-k}) \\ &\leqslant (a_1 + a_n)(p_k + p_{n+1-k})\end{aligned} \tag{15}$$

Summing both sides of inequalities (15) with respect to $(1 \leqslant k \leqslant n)$, we obtain

$$\frac{a_\sigma + a_{n+1-\sigma}}{2}\sum_{k=1}^{n} p_k \leqslant \frac{1}{2}\sum_{k=1}^{n}(a_k p_k + a_{n+1-k} p_k) \leqslant \frac{a_1 + a_n}{2}\sum_{k=1}^{n} p_k. \tag{16}$$

Thanks to the identity of (4), yields the following inequality

$$\frac{a_\sigma + a_{n+1-\sigma}}{2}\sum_{k=1}^{n} p_k \leqslant \sum_{k=1}^{n} a_k p_k \leqslant \frac{a_1 + a_n}{2}\sum_{k=1}^{n} p_k.$$

If $(a_k)_{k=1}^{n} \in E_-(I)$, we use the same arguments as in the above proof. $\square$

Proof of Theorem 7. We use inequality (16) and condition (7), we obtain the result. $\square$

Proof of Theorem 8. It is sufficient to apply (16) and condition (9). $\square$

4. Applications

Let us first recall that in 1979, S. Haber [3] proved the following inequality:

THEOREM 9. *Let a and b be non negative real numbers. Then, for every integer $n \geqslant 0$, we have*

$$\frac{1}{n+1}\left(a^n + a^{n-1}b + \ldots + b^n\right) \leqslant \frac{a^n + b^n}{2}.$$

Many authors have been interested in this inequality, see for instance [1, 2, 3, 4]. It is easy to show that for all $a \geqslant 0, b \geqslant 0$, the sequence

$$x_k = a^{n-k}b^k,\ (k = 0, 1, \ldots, n)$$

is convex.

As an application, by applying Theorem 2, we get

$$\frac{1}{n+1}\sum_{k=0}^{n} x_k \leqslant \frac{x_0+x_n}{2},$$

which is equivalent to state that

$$\frac{1}{n+1}\left(a^n + a^{n-1}b + \ldots + b^n\right) \leqslant \frac{a^n+b^n}{2}.$$

This is the upper bound of Haber inequality.

Another application can be seen by taking $a_k = e^{-(k-1)^2}$, for $k \in I = \{1,2,\ldots,100\}$. One can state that (a_k) is not convex. But, we have $(a_k) \in E(I)$. Therefore, we get

$$\left(e^{-(49)^2} + e^{-(50)^2}\right) \leqslant \frac{1}{50}\sum_{k=1}^{100} e^{-(k-1)^2} \leqslant \left(1 + e^{-(99)^2}\right).$$

AN OPEN PROBLEM. *Is it possible to prove that there exists an estimation better than* (2.1) *and* (2.2)*?*

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