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EMBEDDINGS AND RELATED TOPICS IN GRAND VARIABLE EXPONENT HAJŁASZ–MORREY–SOBOLEV SPACES

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Abstract. Embeddings in the framework of grand variable exponent function spaces are studied. In particular, embeddings from grand variable exponent Hajłasz-Sobolev-Morrey spaces to variable exponent Hölder spaces are established. The regularity of a fractional integral operator defined with respect to a non-doubling measure is also investigated. In particular, mapping properties of this operator from a grand variable exponent Morrey space to a grand variable parameter Hölder space are studied. The results are proved under the log-Hölder continuity condition on the exponents. The spaces are defined, generally speaking, on quasi-metric measure spaces, however, the results are new even for Euclidean spaces.

1. Introduction

Our aim is to study problems related to embeddings from grand variable exponent Hajłasz-Morrey-Sobolev spaces ($GVEHMSS$ briefly) $(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ to variable parameter Hölder spaces $H^{\lambda(\cdot)}(X)$ ($VPHS$ briefly) under the log-Hölder continuity condition on exponents and parameters. We treat also the regularity of a fractional integral operator in appropriate spaces. In particular, mapping properties of fractional-type integral operators defined on an open set Ω in $\mathbb{R}^n$ with Ahlfors upper N-regular Borel measure μ on Ω, from grand variable exponent Morrey spaces ($GVEMS$ briefly) to $VPHS$ are also studied.

The study of Hajłasz-Sobolev embeddings in the variable exponent setting was initiated in [1]. Later, a similar problem from the grand variable exponent viewpoint was investigated in [7], where the authors also studied Sobolev–type embeddings in the framework of these spaces defined on open sets in $\mathbb{R}^n$.

In the last two decades it was realized that classical function spaces are no longer adequate for solving a number of contemporary problems arising naturally in various mathematical models of applied sciences. It thus became necessary to introduce and study the new nonstandard function spaces ($NSFS$) from various viewpoints. We emphasize that in recent years the following function spaces were studied: variable exponent Lebesgue and Sobolev spaces, "grand" function spaces, Morrey-type spaces, etc.

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$NSFS$s are extensively investigated by many authors nowadays. We emphasize some recent books and surveys published in this area, and recall, for example, the monographs [3], [5], [22], [23], the survey paper [17], etc.

Classical Morrey spaces were introduced by C. Morrey in 1938 and applied to the regularity problems of solutions to partial differential equations. We mention, for example, the recent two-volume monograph [27] for properties of Morrey-type spaces, and related topics.

Classical grand Lebesgue spaces $L^{p_c)}(\Omega)$, where Ω is a bounded open set in $\mathbb{R}^n$, naturally arise, for example, when studying integrability problems of the Jacobian under minimal hypotheses (see [18]), while $L^{p_c),\theta}(\Omega)$, $\theta > 0$, is related to the investigation of the nonhomogeneous n-harmonic equation $\operatorname{div} A(x, \nabla u) = \mu$ (see [14]). It is known (see, e.g., [12]) that the space $L^{p_c),\theta}(\Omega)$ is non-reflexive and non-separable.

Grand Morrey spaces were introduced in [25], where the boundedness of integral operators in these spaces was also established. Later, H. Rafeiro [26] considered the space, where the author "grandified" the parameter of the space as well.

Grand variable exponent Lebesgue spaces were introduced in [19] (see also [6] for more precise spaces). These spaces unify two non-standard spaces: variable and grand Lebesgue spaces. In the present paper we are interested in Hajłasz-Sobolev space based on $GVEMS$ defined over quasi-metric measure spaces. The latter spaces were introduced in [21].

Sobolev embeddings in variable exponent Lebesgue spaces were studied in the papers [4], [9], [10] (see also the monograph [5] and references cited therein).

Finally we mention that the results of this paper were announced in [8].

2. Preliminaries

In this section we recall the definition and some properties of a quasi-metric measure space.

Let X be a topological space endowed with a locally finite complete measure μ and quasi-metric $d : X \times X \mapsto \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(iii) there exists a constant $\kappa \geqslant 1$ such that for all $x, y, z \in X$,

$$d(x,y) \leqslant \kappa[d(x,z) + d(z,y)];$$

(iv) for every neighborhood V of a point $x \in X$ there exists $r > 0$ such that the ball $B(x,r) = \{y \in X : d(x,y) < r\}$ with center x and radius r is contained in V.

It is also assumed that all balls $B(x,r) := \{y \in X : d(x,y) < r\}$ in X are measurable with finite measure, $\mu\{x\}$ =0 for all $x \in X$, and that the class of continuous functions with compact supports is dense in the space of integrable functions on X.

In this case we say that (X, d, μ) is a quasi-metric measure space. Further, we say that the measure μ of the quasi-metric measure space (X, d, μ) is Ahlfors upper α-regular (or satisfies the growth condition) if there is a positive constant C such that for all $x \in X$ and $R > 0$,

$$\mu\big(B(x,R)\big) \leqslant CR^{\alpha}. \tag{1}$$

A quasi-metric measure space with this growth condition is also called a space of non-homogeneous type.

The measure μ on X is said to satisfy a doubling condition ($\mu \in DC(X)$) if there is a constant $D_\mu > 0$ such that

$$\mu B(x,2r) \leqslant D_\mu \cdot \mu B(x,r) \tag{2}$$

for every $x \in X$ and $r > 0$. The best possible constant in (2) is called the doubling constant for μ and will be denoted again by D_μ.

Denote by d_X the diameter of X. Throughout the paper we will assume that $d_X < \infty$. In this case μ is a finite measure, i.e. $\mu(X) < \infty$.

Further, it can be checked (see also [16], Lemma 14.6) that there is a positive constant C such that whenever $0 < r \leqslant \rho < d_X$, $x \in X$ and $y \in B(x,r)$,

$$\frac{\mu B(x,\rho)}{\mu B(y,r)} \leqslant C\Big(\frac{\rho}{r}\Big)^N,$$

where

$$N = \log_2 D_\mu \tag{3}$$

and D_μ is the doubling constant. Consequently, since $d_X < \infty$, there is a positive constant C_N such that

$$\mu(B(x,r)) \geqslant C_N r^N \tag{4}$$

whenever $x \in X$ and $0 < r < d_X$, where N is defined by (3).

A quasi-metric measure space (X,d,μ) with doubling measure μ is called a space of homogeneous type (SHT).

Recall that for a quasi-metric measure space (X,d,μ) with condition (1), the doubling condition might be not satisfied.

Examples of SHT are: (a) domain Ω in $\mathbb{R}^d$ satisfying the condition: there is a positive constant $C > 0$ such that $|\Omega \cap B(x,r)| \geqslant Cr^d$, where $|E|$ is the Lebesgue measure induced on Ω; here $N = d$; (b) regular curves, i.e. rectifiable curves Γ satisfying the condition: $\nu\big(\Gamma \cap D(x,r)\big) \leqslant Cr$, where $D(x,r)$ is the disc with center x and radius $r > 0$ and ν is the arc-length measure on Γ (in this case $N = 1$); (c) nilpotent Lie groups G with appropriate distance and Haar measure, where $N = Q$ is a homogeneous dimension of G. In particular, the Heisenberg group $\mathscr{H}^n$ is a special case of such a group with $Q = 2n+2$).

For basic properties and examples of an SHT we refer e.g., to [2].

To introduce grand variable exponent Hajłasz–Morrey spaces we need to recall some auxiliary definitions.

We denote by $\mathbf{P}_0(X)$ (resp. $\mathbf{P}(X)$) the family of all real-valued μ-measurable functions $p(\cdot)$ on X such that

$$0 < p_- \leqslant p_+ < \infty, \quad \text{(resp. } 1 < p_- \leqslant p_+ < \infty,)$$

where

$$p_- := p_-(X) := \inf_X p(x), \quad p_+ := p_+(X) := \sup_X p(x).$$

It is clear that $\mathbf{P}_0(X) \subset \mathbf{P}(X)$.

We say that a function $p(\cdot) \in \mathbf{P}_0(X)$ belongs to the class $\mathscr{P}^{\log}(X)$ (or $p(\cdot)$ satisfies the log-Hölder continuity condition) if there is a positive constant ℓ such that for all $x, y \in X$ with $0 < d(x,y) \leqslant 1/2$,

$$|p(x) - p(y)| \leqslant \frac{\ell}{-\ln(d(x,y))}. \tag{5}$$

The best possible constant in (5) is called the log-Hölder continuity constant and will be denoted again by ℓ.

Let $q(\cdot) \in \mathbf{P}(X)$. The variable exponent Lebesgue space $L^{q(\cdot)}(X)$ (or $L^{q(x)}(X)$) ($VELS$ briefly) is also called a Nakano space. It is a special case of more general spaces called Musielak–Orlicz spaces. $L^{q(\cdot)}(X)$ is the class of all μ-measurable functions f on X for which

$$S_{q(\cdot)}(f) := \int_X |f(x)|^{q(x)} d\mu(x) < \infty.$$

$L^{q(\cdot)}(X)$ is a Banach function space when given the norm defined by

$$\|f\|_{L^{q(\cdot)}(X)} = \inf\left\{\lambda > 0 : S_{q(\cdot)}\left(f/\lambda\right) \leqslant 1\right\}.$$

The class of exponents $\mathscr{P}^{\log}(X)$ plays an important role in the theory mapping properties of integral operators in $L^{q(\cdot)}$ spaces. For example, maximal, fractional and singular integral operators are bounded in $L^{q(\cdot)}$ under the condition $q(\cdot) \in \mathscr{P}^{\log}(X)$ (see, e.g., the monographs [3], [5], [22] and references cited therein).

The following relations hold for $VELS$s (see, e.g., [24] and p. 3 of [22]):

$$\|f\|^{q_+}_{L^{q(\cdot)}} \leqslant S_{q(\cdot)}(f) \leqslant \|f\|^{q_-}_{L^{q(\cdot)}}, \quad \|f\|_{L^{q(\cdot)}} \leqslant 1,$$

$$\|f\|^{q_-}_{L^{q(\cdot)}} \leqslant S_{q(\cdot)}(f) \leqslant \|f\|^{q_+}_{L^{q(\cdot)}}, \quad \|f\|_{L^{q(\cdot)}} \geqslant 1.$$

Recall that (see e.g., [24]) Hölder's inequality in $VELS$s has the following form:

$$\|fg\|_{L^1} \leqslant C_{q(\cdot)} \|f\|_{L^{q(\cdot)}} \|g\|_{L^{q'(\cdot)}}, \tag{6}$$

where

$$C_{q(\cdot)} = \frac{1}{q_-} + \frac{1}{(q')_-}, \quad q'(\cdot) = \frac{q(\cdot)}{q(\cdot)-1}.$$

Further, the following statement is valid (see, e.g., [22], p. 9):

LEMMA 1. *Let $s(\cdot)$ and $r(\cdot)$ be variable exponents on X such that $1 < s_- \leqslant s(x) \leqslant r(x) \leqslant r_+ < \infty$ μ-a.e. We set*

$$\frac{1}{p(x)} = \frac{1}{s(x)} - \frac{1}{r(x)}.$$

If $1 \in L^{p(\cdot)}$, then

$$\|f\|_{L^{s(\cdot)}} \leqslant 2^{1/s_-} \|1\|_{L^{p(\cdot)}} \|f\|_{L^{r(\cdot)}}.$$

We say that $\varphi(\cdot) \in A_{q(\cdot)}$, where $1 < q_- \leqslant q_+ < \infty$, if $\varphi(\cdot)$ is defined and bounded on $(0, q_- - 1)$, is non-decreasing on $(0, \delta)$ for some small positive constant δ, and

$$\lim_{x \to 0^+} \varphi(x) = 0.$$

Further, we write that the pair of variable exponents $\Big(p(\cdot), q(\cdot)\Big) \in \widetilde{\mathbf{P}}(X)$ if $1 < q_- \leqslant q(\cdot) \leqslant p(\cdot) \leqslant p_+ < \infty$.

Let $\Big(p(\cdot), q(\cdot)\Big) \in \widetilde{\mathbf{P}}(X)$ and let $\varphi(\cdot) \in A_{q(\cdot)}$. We recall the definitions of the spaces $L^{p(\cdot)}_{q(\cdot)}(X)$ and $L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ determined by the norms

$$\|f\|_{L^{p(\cdot)}_{q(\cdot)}(X)} = \sup_{\substack{x \in X \\ 0 < r < d_X}} (\mu B(x,r))^{\frac{1}{p(x)} - \frac{1}{q(x)}} \|f\|_{L^{q(\cdot)}(B(x,r))}$$

and

$$\|f\|_{L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)} = \sup_{0 < c < q_- - 1} \varphi(c)^{\frac{1}{q_- - c}} \|f\|_{L^{p(\cdot)}_{q(\cdot)-c}(X)},$$

respectively, where c is a constant.

The spaces $L^{p(\cdot)}_{q(\cdot)}(X)$ and $L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ are variable exponent Morrey spaces ($VEMS$ briefly) and $GVEMS$, respectively.

If $p(\cdot) = q(\cdot)$, then $L^{p(\cdot)}_{q(\cdot)}(X)$ is the $VELS$ $L^{q(\cdot)}(X)$.

DEFINITION 1. Let $\Big(p(\cdot), q(\cdot)\Big) \in \widetilde{\mathbf{P}}(X)$ and let $\varphi(\cdot) \in A_{q(\cdot)}$. We say that a function $f \in L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ belongs to the Hajłasz-Morrey space $(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ if there is a non-negative $g \in L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ such that

$$|f(x) - f(y)| \leqslant d(x,y)[g(x) + g(y)], \quad \mu - a.e \text{ in X}.$$

In this case $g(\cdot)$ is called a generalized gradient of f.

For $p(\cdot) \equiv q(\cdot)$ this space was introduced and studied in [7].

If $p(\cdot) \equiv q(\cdot) \equiv p_c = const$ and formally $\theta = 0$, then we have the space $(HS)^{p_c}(X)$ which was introduced by P. Hajłasz [15] as a generalization of the classical Sobolev spaces W^{1,p_c} to the general setting of quasi-metric measure spaces.

PROPOSITION 1. *The space $(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ is the Banach space with respect to the norm:*

$$\|f\|_{(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)} = \|f\|_{L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)} + \inf \|g\|_{L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)},$$

where the infimum is taken over all generalized gradients g of f.

Let $p_- > N$. We say that a bounded function f belongs to the variable exponent Hölder space ($VEHS$ briefly) $H^{p(\cdot)}(X)$, if there exists $C > 0$ such that

$$|f(x) - f(y)| \leqslant Cd(x,y)^{\max\{1-N/p(x),1-N/p(y)\}}$$

for every $x, y \in X$ (see [1] for this definition).

Norms in these spaces are defined as follows:

$$\|f\|_{H^{p(\cdot)}(X)} = \|f\|_{L^\infty(X)} + \big[f\big]_{H^{p(\cdot)}(X)}$$

where

$$\big[f\big]_{H^{p(\cdot)}(X)} := \sup_{\substack{x,y\in X \\ 0<d(x,y)\leqslant 1}} \frac{|f(x)-f(y)|}{d(x,y)^{\max\{1-N/p(x),1-N/p(y)\}}}.$$

3. Embeddings

Throughout this section it will be assumed that (X,d,μ) is an SHT and that N is defined by (3).

To prove the main result of this section we need some definitions and auxiliary statements.

LEMMA 2. (see [7]) *Let $\alpha(\cdot)$ and $\beta(\cdot)$ be μ-measurable functions on X such that $0 < \alpha_- \leqslant \alpha_+ < \infty$, $0 < \beta_- \leqslant \beta_+ < \infty$. Suppose that f is a locally integrable function on X. Then for all $x, y \in X$,*

$$|f(x)-f(y)| \leqslant C\big(\mu,\alpha(\cdot),\beta(\cdot)\big)\left[d(x,y)^{\alpha(x)} M^{\#}_{\alpha(\cdot)} f(x) + d(x,y)^{\beta(y)} M^{\#}_{\alpha(\cdot)} f(y)\right],$$

where $C(\mu,\alpha,\beta)$ is the constant defined by

$$C\big(\mu,\alpha(\cdot),\beta(\cdot)\big) := D_\mu \max\left\{\frac{1}{2^{\alpha_-}-1}; 2^{\beta_+}\left(\frac{1}{2^{\beta_-}-1} + D_\mu\right)\right\}$$

and

$$M^{\#}_{\alpha(\cdot)} f(x) = \sup_{x\in X, r>0} \frac{r^{-\alpha(x)}}{\mu B(x,r)} \int_{B(x,r)} \big|f(y) - f_{B(x,r)}\big|\, d\mu(y).$$

Denote by $M_{\lambda(\cdot)}$ the fractional maximal operator given by the formula:

$$M_{\lambda(\cdot)} f(x) = \sup_{\substack{x\in X \\ r>0}} \frac{r^{\lambda(x)}}{\mu B(x,r)} \int\limits_{B(x,r)} |f(y)|\, d\mu(y), \;\; 0 \leqslant \lambda(x) < \lambda_+ < N.$$

LEMMA 3. *Let $0 \leqslant \lambda_- \leqslant \lambda_+ < 1$, $\Big(p(\cdot), q(\cdot)\Big) \in \widetilde{\boldsymbol{P}}(X)$, $\varphi(\cdot) \in A_{q(\cdot)}$. Suppose that $f \in (HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ and that g is its gradient. Then*

$$M^{\#}_{1-\lambda(\cdot)} f(x) \leqslant 4\kappa M_{\lambda(\cdot)} g(x),$$

where κ is the quasi-metric constant.

Proof. Let g be the gradient of f. Then for $B := B(x,r)$,

$$\begin{aligned}
&\int_B |f(y) - f_B|\, d\mu(y) \\
\leqslant &\frac{1}{\mu(B)} \int_B \int_B |f(y) - f(z)| d\mu(y) d\mu(z) \\
\leqslant &\frac{1}{\mu(B)} \int_B \int_B d(y,z)[g(y) + g(z)] d\mu(y) d\mu(z) \\
\leqslant &\frac{2\kappa r}{\mu(B)} \int_B \int_B [g(y) + g(z)] d\mu(y) d\mu(z) \\
\leqslant &\frac{4\kappa r}{\mu(B)} \int_B \int_B g(y) d\mu(y) d\mu(z) = 4\kappa r \int_B g(y) d\mu(y).
\end{aligned}$$

Now the conclusion follows. $\square$

Lemmas 2 and 3 imply the next statement.

LEMMA 4. *Let $\alpha(\cdot)$ and $\beta(\cdot)$ be μ-measurable functions on X such that $0 \leqslant \alpha_- \leqslant \alpha_+ < 1$, $0 \leqslant \beta_- \leqslant \beta_+ < 1$. Suppose that $\big(p(\cdot), q(\cdot)\big) \in \widetilde{\boldsymbol{P}}(X), \varphi(\cdot) \in A_{q(\cdot)}$. Assume that $f \in (HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ and that g is its gradient. Then for all $x, y \in X$,*

$$|f(x) - f(y)| \leqslant \overline{C}\big(\mu, \alpha(\cdot), \beta(\cdot)\big) \left[d(x,y)^{1-\alpha(x)} M_{\alpha(\cdot)} g(x) + d(x,y)^{1-\beta(y)} M_{\beta(\cdot)} g(y) \right],$$

where

$$\overline{C}\big(\mu, \alpha(\cdot), \beta(\cdot)\big) := 8C\big(\mu, 1 - \alpha(\cdot), 1 - \beta(\cdot)\big)$$

and

$$C\big(\mu, 1 - \alpha(\cdot), 1 - \beta(\cdot)\big) = D_\mu \max\left\{ \frac{1}{2^{1-\alpha_+} - 1}; 2^{1-\beta_-} \left(\frac{1}{2^{1-\beta_+} - 1} + D_\mu \right) \right\}.$$

LEMMA 5. *Let $\big(r(\cdot), s(\cdot)\big) \in \widetilde{\boldsymbol{P}}(X)$ and let, in addition, $r(\cdot) \in \mathscr{P}^{\log}(X)$. Then for $f \in L^{s(\cdot)}_{r(\cdot)}$*

$$M_{N/s(\cdot)} f(x) \leqslant C_{s(\cdot), r(\cdot)} \|f\|_{L^{s(\cdot)}_{r(\cdot)}},$$

where $C_{s(\cdot), r(\cdot)}$ is such that

$$\sup_{0 < c < \sigma} C_{s(\cdot), r(\cdot) - c} < \infty \tag{7}$$

for some small positive constant σ.

Proof. We have

$$\begin{aligned}
\frac{R^{\frac{N}{s(x)}}}{\mu B(x,R)}\int_{B(x,R)}|f|d\mu &\leqslant \frac{2R^{\frac{N}{s(x)}}}{\mu B(x,R)}\|f\|_{L^{r(\cdot)}(B(x,R))}\|\,1\|_{L^{r'(\cdot)}(B(x,R))}\\
&\leqslant \frac{C_{r(\cdot)}R^{\frac{N}{s(x)}}R^{\frac{N}{r'(\cdot)}}}{R^N}\|f\|_{L^{r(\cdot)}(B(x,R))}\\
&\leqslant \frac{C_{s(\cdot),r(\cdot)}R^{\frac{N}{s(x)}}R^{\frac{N}{r'(x)}}R^{\frac{N}{r(x)}-\frac{N}{s(x)}}}{R^N}\|f\|_{L^{s(\cdot)}_{r(\cdot)}(X)}\\
&= C_{s(\cdot),r(\cdot)}\|f\|_{L^{s(\cdot)}_{r(\cdot)}(X)}.
\end{aligned}$$

It remains to observe that condition (7) holds for the constant $C_{s(\cdot),r(\cdot)}$. □

LEMMA 6. *Let $p(\cdot)$ and $q(\cdot)$ are the variable exponents such that $p_- > N$ and $q(\cdot)\in\mathscr{P}^{\log}(X)$. Suppose that $f\in(HM)^{p(\cdot)}_{q(\cdot)}(X)$. Let g be a generalized gradient of f. Then*

$$|f(x)-f(y)|\leqslant \widetilde{C}_{p(\cdot),q(\cdot)}\|g\|_{L^{p(\cdot)}_{q(\cdot)}}d(x,y)^{1-N/\max\{p(x),p(y)\}},$$

where $\widetilde{C}_{p(\cdot),q(\cdot)}$ is a constant satisfying the condition

$$\sup_{0<c<\sigma}\widetilde{C}_{p(\cdot),q(\cdot)-c}<\infty \tag{8}$$

for some small positive constant σ.

Proof. Applying Lemmas 4 and 5 we have

$$\begin{aligned}
|f(x)-f(y)| &\leqslant C_{p(\cdot),q(\cdot)}\left[d(x,y)^{1-N/p(x)}M_{N/p(x)}g(x)+d(x,y)^{1-N/p(y)}M_{N/p(y)}g(y)\right]\\
&\leqslant C_{p(\cdot),q(\cdot)}\|g\|_{L^{p(\cdot)}_{q(\cdot)}(X)}\left[d(x,y)^{1-N/p(x)}+d(x,y)^{1-N/p(y)}\right]\\
&\leqslant \widetilde{C}_{p(\cdot),q(\cdot)}\|g\|_{L^{p(\cdot)}_{q(\cdot)}(X)}d(x,y)^{\max\{1-N/p(x);1-N/p(y)\}}.
\end{aligned}$$

Since condition (8) holds for the constant $\widetilde{C}_{p(\cdot),q(\cdot)}$, we are done. □

We will need some more auxiliary statements

LEMMA 7. ([23], p. 834) *Let (X,d,μ) be an SHT, $r(\cdot)\in \boldsymbol{P}(X)\cap\mathscr{P}^{\log}(X)$. Then the followng estimate holds for all balls B, $\mu(B)\leqslant 1$:*

$$\mu(B)^{r_-(B)-r_+(B)}\leqslant C_{r(\cdot)},$$

where $C_{r(\cdot)}$ is a constant such that

$$\sup_{0<c<\sigma}C_{r(\cdot)-c}<\infty$$

for some small positive constant σ.

LEMMA 8. *Let (X,d,μ) be an SHT and let σ be a small positive constant. Suppose that $q(\cdot)\in \boldsymbol{P}(X)\cap \mathscr{P}^{\log}(X)$ and $\varphi(\cdot)\in A_{q(\cdot)}$. Then there is a positive constant $C_{q(\cdot),\sigma,\varphi(\cdot)}$ such that*

$$\|f\|_{L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)} \leqslant C_{q(\cdot),\sigma,\varphi(\cdot)} \sup_{0<c\leqslant\sigma} \varphi^{1/(q_- -c)}\|f\|_{L^{p(\cdot)}_{q(\cdot)-c}(X)}.$$

Proof. Without loss of generality we can assume that $\mu(X)\leqslant 1$. Let $\sigma<c<q_- -1$. Then by Lemmas 1, 7, and the fact that $q(\cdot)\in\mathscr{P}^{\log}(X)$ we find that for a ball $B:=B(x,r)$,

$$\begin{aligned}\varphi(c)^{\frac{1}{q_- -c}}\mu(B)^{\frac{1}{p(x)}-\frac{1}{q(x)-c}}\|f\|_{L^{q(\cdot)-c}(B)} &\leqslant 2\varphi(c)^{\frac{1}{q_- -c}}\mu(B)^{\frac{1}{p(x)}-\frac{1}{q(x)-c}}\|1\|_{L^{l(\cdot)}(B)}\|f\|_{L^{q(\cdot)-\sigma}(B)}\\ &\leqslant C_{q(\cdot),\sigma,\varphi(\cdot)}\varphi(\sigma)^{\frac{1}{q_- -\sigma}}\mu(B)^{\frac{1}{p(x)}-\frac{1}{q(x)-\sigma}}\|f\|_{L^{q(\cdot)-\sigma}(B)},\end{aligned}$$

where $l(\cdot)=\frac{(q(\cdot)-c)(q(\cdot)-\sigma)}{c-\sigma}$.

Since

$$\|f\|_{L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)} = \max\left\{\sup_{0<c\leqslant\sigma}\varphi(c)^{\frac{1}{q_- -c}}\|f\|_{L^{p(\cdot)}_{q(\cdot)-c}(X)},\ \sup_{\sigma<c<q_- -1}\varphi(c)^{\frac{1}{q_- -c}}\|f\|_{L^{p(\cdot)}_{q(\cdot)-c}(X)}\right\},$$

we have the desired result. □

A similar relation for grand Lebesgue spaces with constant exponents was first observed in [11].

THEOREM 1. *Let (X,d,μ) be an SHT with $\mu(X)<\infty$, and let N be determined by (3). Let $p(\cdot)$ and $q(\cdot)$ be variable exponents such that $p_->N$ and $\big(p(\cdot),q(\cdot)\big)\in \widetilde{\boldsymbol{P}}(X)$. Let $q(\cdot)\in\mathscr{P}^{\log}(X)$, $\varphi(\cdot)\in A_{q(\cdot)}$. Then*

$$(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)\hookrightarrow H^{p(\cdot)}(X).$$

Proof. Taking Lemma 8 into account, we deal with small positive c. Let $0<c<\sigma<q_- -1$. Then applying Lemma 5, we have that for $x\in X, R_0>0$,

$$\begin{aligned}\left|f(x)-f_{B(x,R_0)}\right| &\leqslant D_\mu R_0^{1-N/p(x)}M^{\#}_{1-N/p(\cdot)}f(x)\\ &\leqslant CR_0^{1-N/p(x)}M_{N/p(\cdot)}g(x)\\ &\leqslant \overline{C}R_0^{1-N/p(x)}\|g\|_{L^{p(\cdot)}_{q(\cdot)-c}(X)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left|f_{B(x,R_0)}\right| &\leqslant 2\mu\left(B(x,R_0)\right)^{-1}\|f\|_{L^{q(\cdot)-c}(B(x,R_0))}\|\,1\|_{L^{(q(\cdot)-c)'}(B(x,R_0))}\\ &\leqslant 2R_0^{-N/(q(x)-c)}\|f\|_{L^{q(\cdot)-c}(X)}\leqslant 2R_0^{-N/p(x)}\|f\|_{L^{p(\cdot)}_{q(\cdot)-c}(X)}.\end{aligned}$$

Thus, taking $R_0 = \min\{1, \mu(X)\}$, we find that

$$
\begin{aligned}
|f(x)| &\leqslant C\left[R_0^{1-N/p(x)} + R_0^{-N/p(x)}\right] \|f\|_{L^{p(\cdot)}_{q(\cdot)-c}(X)} \\
&\leqslant C\|f\|_{L^{p(\cdot)}_{q(\cdot)-c}(X)}.
\end{aligned}
$$

Thus, $f \in L^{\infty}(X)$.

Further, observe that

$$
|f(x) - f(y)| \leqslant \widetilde{C}_{p(\cdot),q(\cdot)-c} \|g\|_{L^{p(\cdot)}_{q(\cdot)-c}(X)} d(x,y)^{\max\{1-N/p(x);1-N/p(y)\}} \tag{9}
$$

where the constant $\widetilde{C}_{p(\cdot),q(\cdot)-c}$ is such that

$$
\sup_{0<c<\sigma} \widetilde{C}_{p(\cdot),q(\cdot)-c} < \infty.
$$

Finally, multiplying both sides of inequality (9) by $\varphi(c)^{\frac{1}{q_- - c}}$ and taking the supremum with respect to $c, 0 < c < \sigma$ (observe that the left-hand side of (9) does not depend on c) we have the desired result. $\square$

4. Regularity of potentials

Let Ω be an open set in $\mathbb{R}^d$ and let μ be a Borel measure on Ω. In this section we investigate the regularity of fractional integrals

$$
J^{\gamma}_{\Omega} f(x) = \int_{\Omega} \frac{f(y)}{|x-y|^{n-\gamma}} d\mu(y),\ 0 < \gamma < n,\ x \in \Omega
$$

for $f \in L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)$, where the measure μ on Ω satisfies the condition: there are positive constants c_0 and n such that for all $x \in \Omega$ and $R > 0$,

$$
\mu\big(D(x,R)\big) \leqslant c_0 R^n, \ \ D(x,R) := B(x,R) \cap \Omega. \tag{10}
$$

In this section we will need the following class of exponents on Ω.

DEFINITION 2. We say that $p(\cdot) \in \mathscr{P}(X)$ if there is a positive constant ℓ_1 such that

$$
\mu(B(x,R))^{p_-\left(D(x,R)\right) - p_+\left(D(x,R)\right)} \leqslant \ell_1
$$

for all $x \in X$ and small positive R.

It is known that (see, e.g., [23], p. 834) that if (X, d, μ) is an SHT, then $\mathscr{P}^{log}(\Omega) \subset \mathscr{P}(\Omega)$.

DEFINITION 3. Let γ be a constant such that $0<\gamma<n$ and let ε be a constant such that $0<\varepsilon\leqslant 1$. A function $k_\gamma:\Omega\times\Omega\to\mathbb{C}$ is said to be a fractional kernel of order γ if there exists a positive constant C_k such that

$$(a)\quad \left|k_\gamma(x,y)\right|\leqslant\frac{C_{k_\gamma}}{|x-y|^{n-\gamma}},\qquad x\neq y; \tag{11}$$

$$(b)\quad \left|k_\gamma(x,y)-k_\gamma\left(x',y\right)\right|\leqslant\frac{C_{k_\gamma}|x-x'|^{\varepsilon}}{|x-y|^{n-\gamma+\varepsilon}},\qquad |x-y|\geqslant 2\left|x'-x\right|. \tag{12}$$

For k_γ, let

$$K^\gamma f(x)=\int_\Omega k_\gamma(x,y)f(y)d\mu(y),\qquad x\in\Omega.$$

LEMMA 9. [13], [28]. *Let (X,d,μ) be a metric measure space and let $x,y,z\in X$ be such that $2d(x,y)\leqslant d(x,z)$. Then the following estimate holds:*

$$|\,d(x,z)^{\gamma-n}-d(y,z)^{\gamma-n}\,|\leqslant C\frac{d(x,y)}{d(x,z)^{n-\gamma+1}}$$

for $0<\gamma<n$, where the positive constant C depends only on n and γ. Consequently conditions (a) and (b) are satisfied for $k_\gamma(x,y)=|x-y|^{\gamma-n}$ and $\varepsilon=1$ with the constant C_{k_γ}.

Let $\lambda:\Omega\to(0,1]$ be a measurable function satisfying the condition $0<\lambda_-\leqslant\lambda_+\leqslant 1$. We say that a function f on Ω is in the space $H^{\lambda(\cdot)}(\Omega)$ if

$$[f]_{\lambda(\cdot)}=\sup_{\substack{x,x+h\in\Omega\\ 0<|h|\leqslant 1}}\frac{|f(x+h)-f(x)|}{|h|^{\lambda(x)}}$$

is finite. In particular, we denote

$$[f]_{\gamma-\eta/p(\cdot)}:=[f]_{\widetilde{H}^{p(\cdot)}_{\gamma,\eta}(\Omega)}.$$

LEMMA 10. *Let $p(\cdot)$ be an exponent on Ω such that $p(\cdot)\in\mathscr{P}^{\log}(\Omega)$. Then there is a positive constant C depending on the log-Hölder continuity constant ℓ for $p(\cdot)$ such that*

$$\frac{1}{C}|h|^{p(x+h)}\leqslant|h|^{p(x)}\leqslant C|h|^{p(x+h)},\quad |h|\leqslant 1;\, x,x+h\in\Omega.$$

Proof. Since $p(\cdot)\in\mathscr{P}^{\log}(\Omega)$ we have that for all x and h such that if $x,x+h\in\Omega$, $|h|\leqslant 1$,

$$|p(x+h)-p(x)|\leqslant\frac{\ell}{-\ln|h|}$$

holds. Hence,

$$|h|^{|p(x+h)-p(x)|}\leqslant e^{-\ell}$$

from which the desired relation follows. $\square$

LEMMA 11. *Let* $q(\cdot)$ *be an exponent such that* $q(\cdot) \in \mathscr{P}(\Omega)$ *with appropriate constant* ℓ_1. *Then* $\frac{1}{q'(\cdot)} \in \mathscr{P}(\Omega)$ *with constant* $(\max\{1, \ell_1\})^{\frac{1}{(q_- - 1)^2}}$.

Proof. It is enough to observe that for a set $D := B \cap \Omega$, $\mu(D) \leqslant 1$, where B is a ball with center in Ω, we have

$$\mu(D)^{\left(\frac{1}{q'}\right)_-(D) - \left(\frac{1}{q'}\right)_+(D)} = \left(\mu(D)^{q_-(D) - q_+(D)}\right)^{\frac{1}{(q_- - 1)^2}} \leqslant \ell_1^{\frac{1}{(q_- - 1)^2}}. \quad \square$$

Lemma 11 implies the next statement:

LEMMA 12. *Let* $q(\cdot)$ *be an exponent such that* $q(\cdot) \in \mathscr{P}(\Omega)$. *Then there is a positive constant* $\overline{C}_{q(\cdot)}$ *such that for all* $x \in \Omega$ *and* $r > 0$,

$$\left\| \chi_{D(x,r)} \right\|_{L^{q'(\cdot)}} \leqslant \overline{C}_{q(\cdot)} \mu(D(x,r))^{\frac{1}{q'(x)}}.$$

Moreover, the constant $\overline{C}_{q(\cdot)}$ *is such that*

$$\sup_{0<\varepsilon<\eta} \overline{C}_{q(\cdot)-\varepsilon} < \infty$$

for some small positive constant η.

The following statement is a quantitative version of Theorem 4.6 in [20].

PROPOSITION 2. *Let* $\mu(\Omega) < \infty$ *and let* μ *satisfy* (10). *Let* k_γ *satisfy (a) and (b) of Definition 3. Let* γ *and* ε *be constants such that* $0 < \varepsilon \leqslant \gamma < n$. *Assume that* $\left(p(\cdot), q(\cdot)\right) \in \widetilde{\boldsymbol{P}}(\Omega)$, $\frac{n}{\gamma} < p_- \leqslant p_+ < \frac{n}{\gamma - \varepsilon}$. *If* $q(\cdot) \in \mathscr{P}(\Omega)$ *and* $p(\cdot) \in \mathscr{P}^{\log}(\Omega)$ *then there exists a constant* $C = C_{k_\gamma, n, q(\cdot), p(\cdot), \varepsilon}$ *such that*

$$[K^\gamma f]_{\widetilde{H}^{p(\cdot)}_{\gamma,\eta}(\Omega)} \leqslant C \|f\|_{L^{p(\cdot)}_{q(\cdot)}(\Omega)},$$

where C *satisfies the condition*

$$\sup_{0<\lambda<\eta} C_{k_\gamma, n, q(\cdot)-\lambda, p(\cdot), \varepsilon} < \infty$$

with a small positive constant η.

Proof. By conditions (a) and (b) of Definition 3 we find that

$$
\begin{aligned}
|K^\gamma f(x+h) - K^\gamma f(x)| &\leqslant \int_\Omega \left|k_\gamma(x+h,y) - k_\gamma(x,y)\right| |f(y)| d\mu(y) \\
&\leqslant \int_{D(x,2|h|)} \left|k_\gamma(x+h,y)\right| |f(y)| d\mu(y) \\
&\quad + \int_{D(x,2|h|)} \left|k_\gamma(x,y)\right| |f(y)| d\mu(y) \\
&\quad + \int_{\Omega\setminus D(x,2|h|)} \left|k_\gamma(x+h,y) - k_\gamma(x,y)\right| |f(y)| d\mu(y) \\
&\leqslant C_{k_\gamma} \int_{D(x,2|h|)} \frac{|f(y)|}{|x+h-y|^{n-\gamma}} d\mu(y) \\
&\quad + C_{k_\gamma} \int_{D(x,2|h|)} \frac{|f(y)|}{|x-y|^{n-\gamma}} d\mu(y) \\
&\quad + C_{k_\gamma} |h|^\varepsilon \int_{\Omega\setminus D(x,2|h|)} \frac{|f(y)|}{|x+h-y|^{n-\gamma+\varepsilon}} d\mu(y) \\
&=: I_1 + I_2 + I_3,
\end{aligned}
$$

where $|h|$ is small and $x+h \in \Omega$.

Further, by using the representation $|x+h-y|^{\gamma-n} = \frac{n-\gamma}{1-2^{\gamma-n}} \int\limits_{|x+h-y|}^{2|x+h-y|} t^{\gamma-n-1} dt$, and Fubini's theorem, we see that

$$
I_1 \leqslant C_{k_\gamma,n} \int_0^{6|h|} f_t(x+h) t^{\gamma-1} dt,
$$

where $f_t(x) := \frac{1}{t^n} \int_{D(x,t)} |f(y)| d\mu(y)$ and the positive constant $C_{k_\gamma,n}$ depends only k_γ, n.

Applying now the Hölder inequality in the space $L^{q(\cdot)}(\Omega)$, the growth condition for μ, the assumptions $1/q'(\cdot) \in \mathscr{P}(\Omega)$, $p(\cdot) \in \mathscr{P}^{\log}(\Omega)$, and observing that

$$
\frac{1}{c} |h|^{\gamma-n/p(x+h)} \leqslant |h|^{\gamma-n/p(x)} \leqslant c|h|^{\gamma-n/p(x+h)}
$$

for some constant $c > 1$, we find that

$$
\begin{aligned}
f_t(x+h) &\leqslant C_{q(\cdot)} t^{-n} \left\|\chi_{D(x+h,t)} f\right\|_{L^{q(\cdot)}(\Omega)} \left\|\chi_{D(x+h,t)}\right\|_{L^{q'(\cdot)}(\Omega)} \\
&\leqslant C_{q(\cdot)} \overline{C}_{q(\cdot)} t^{-n} \mu(D(x+h,t))^{1/q'(x+h)} \left\|\chi_{D(x+h,t)} f\right\|_{L^{q(\cdot)}(\Omega)} \\
&\leqslant C_{q(\cdot)} \overline{C}_{q(\cdot)} C_0 t^{-n/q(x+h)} \left\|\chi_{D(x+h,t)} f\right\|_{L^{q(\cdot)}(\Omega)} \\
&= C_{q(\cdot)} \overline{C}_{q(\cdot)} C_0 t^{-n/p(x+h)} t^{n/p(x+h)-n/q(x+h)} \left\|\chi_{D(x+h,t)} f\right\|_{L^{q(\cdot)}(\Omega)} \\
&\leqslant C_{q(\cdot)} \overline{C}_{q(\cdot)} C_0^2 t^{-n/p(x+h)} \mu(D(x+h,t))^{1/p(x+h)-1/q(x+h)} \left\|\chi_{D(x+h,t)} f\right\|_{L^{q(\cdot)}(\Omega)} \\
&\leqslant D \|f\|_{L^{p(\cdot)}_{q(\cdot)}(\Omega)} t^{-n/p(x+h)},
\end{aligned}
$$

where

$$D = C_0^2\, C_{q(\cdot)}\overline{C}_{q(\cdot)}. \tag{13}$$

Consequently, since $1/p(\cdot) \in \mathscr{P}^{\log}(\Omega)$ and $\gamma > \frac{n}{p_-}$, we have

$$\begin{aligned} I_1 &\leqslant C_{k_\gamma} D \frac{6^n}{\gamma - \frac{n}{p_-}} \|f\|_{L^{p(\cdot)}_{q(\cdot)}(\Omega)} |h|^{\gamma - n/p(x+h)} \\ &\leqslant C \|f\|_{L^{p(\cdot)}_{q(\cdot)}(\Omega)} |h|^{\gamma - n/p(x)}, \end{aligned}$$

where

$$C = c\, C_{k_\gamma} D\, \frac{6^n}{\gamma - \frac{n}{p_-}} \tag{14}$$

Further, similar arguments yield

$$I_2 \leqslant C\|f\|_{L^{p(\cdot)}_{q(\cdot)}(\Omega)} |h|^{\gamma - n/p(x)}.$$

To estimate I_3 we observe that if $|x-y| \geqslant 2|h|$, then $|x-y+h| \geqslant |x-y| - |h| \geqslant |h|$. Therefore

$$\begin{aligned} I_3 &\leqslant C_{k_\gamma} |h|^{\varepsilon} \int_{\Omega \setminus D(x,2|h|)} \frac{|f(y)|}{|x-y+h|^{n-\gamma+\varepsilon}} d\mu(y) \\ &\leqslant C_{k_\gamma} |h|^{\varepsilon} \int_{\Omega \setminus D(x,2|h|)} |f(y)| \left(\int_{|x-y+h|}^{2|x-y+h|} t^{\gamma-n-\varepsilon-1} dt \right) d\mu(y) \\ &\leqslant C_{k_\gamma} |h|^{\varepsilon} \int_{|h|}^{\infty} t^{\gamma-n-\varepsilon-1} \left(\int_{D(x+h,t)} |f(y)| d\mu(y) \right) dt \\ &= C_{k_\gamma} |h|^{\varepsilon} \int_{|h|}^{\infty} t^{\gamma-\varepsilon-1} f_t(x+h) dt. \end{aligned}$$

Repeating the arguments used to estimate I_1 we see that

$$f_t(x+h) \leqslant D t^{-n/p(x+h)} \|f\|_{L^{p(\cdot)}_{q(\cdot)}(\Omega)}.$$

Since we assumed that $\gamma < \varepsilon + n/p_+$ and $1/p(\cdot) \in \mathscr{P}^{\log}(\Omega)$, we obtain

$$\begin{aligned} I_3 &\leqslant C\|f\|_{L^{p(\cdot)}_{q(\cdot)}(\Omega)} |h|^{\varepsilon} \int_{|h|}^{\infty} t^{\gamma - n/p(x+h) - \varepsilon - 1} dt \\ &\leqslant C\|f\|_{L^{p(\cdot)}_{q(\cdot)}(\Omega)} |h|^{\gamma - n/p(x)}, \end{aligned}$$

where $C \equiv C_{k_\gamma, n, q(\cdot), p(\cdot), \varepsilon}$ is a positive constant.

Combining the estimates for I_1, I_2 and I_3 we finally obtain that

$$\|K^{\gamma} f\|_{H^{\gamma - \frac{n}{p(\cdot)}}(\Omega)} \leqslant C_{k_\gamma, n, q(\cdot), p(\cdot), \varepsilon} \|f\|_{L^{p(\cdot)}_{q(\cdot)}(\Omega)},$$

where the constant $C_{k_\gamma,n,q(\cdot),p(\cdot),\varepsilon}$ satisfies

$$\sup_{0<c<\sigma} C_{k_\gamma,n,q(\cdot)-c,p(\cdot),\varepsilon} < \infty$$

for some small positive constant σ. $\square$

Finally, taking into account Proposition 2 and Lemma 9 we have

THEOREM 2. *Let Ω be an open set in $\mathbb{R}^d$ and let μ be a Borel measure on Ω satisfying condition (10). Suppose that $\mu(\Omega) < \infty$. Further, let $p(\cdot)$ and $q(\cdot)$ be variable exponents on Ω such that $\Big(p(\cdot), q(\cdot)\Big) \in \widetilde{\boldsymbol{P}}(\Omega)$. Assume that $\varphi(\cdot) \in A_{q(\cdot)}$. Suppose that γ and ε are positive constants such that $\frac{n}{\gamma} < p_- \leqslant p_+ < \frac{n}{\gamma-\varepsilon}$. Let $q(\cdot) \in \mathscr{P}(\Omega)$ and let $p(\cdot) \in \mathscr{P}^{log}(\Omega)$. Then the operator J_Ω^γ is bounded from $L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)$ to $\widetilde{H}^{p(\cdot)}_{\gamma,n}(\Omega)$, i.e. there is a positive constant c_0 such that for all $f \in L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)$,*

$$[J_\Omega^\gamma f]_{\widetilde{H}^{p(\cdot)}_{\gamma,n}(\Omega)} \leqslant c_0 \|f\|_{L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)}. \tag{15}$$

DEFINITION 4. Let $\Omega \subset \mathbb{R}^d$ be an open set and let n be a positive constant. We say that a Borel measure μ defined on Ω satisfies condition $A(n,\Omega)$ (or $\mu \in A(n,\Omega)$) if there is a positive constant c_0 such that for all $x \in \Omega$ and $R \in (0, diam\Omega)$,

$$\frac{1}{c_0} R^n \leqslant \mu\big(D(x,R)\big) \leqslant c_0 R^n. \tag{16}$$

For example, the induced Lebesgue measure $|\cdot|_d$ on $\Omega \subset \mathbb{R}^d$ satisfying the condition $|D(x,R)|_d \geqslant C_0 R^d$ belongs to the class $A(d,\Omega)$; arc-length measure ν on a regular curve Γ belongs to $A(1,\Gamma)$; the Haar measure on a nilpotent Lie groups G belongs to $A(Q,G)$, where Q is the homogeneous dimension of G.

It is easy to see that any measure μ satisfying condition (16) is doubling.

COROLLARY 1. *Let μ be a finite measure on a bounded open set $\Omega \subset \mathbb{R}^d$ satisfying the condition $A(n,\Omega)$. Suppose that $p(\cdot)$ and $q(\cdot)$ are variable exponents on Ω such that $\Big(p(\cdot), q(\cdot)\Big) \in \widetilde{\boldsymbol{P}}(\Omega)$. Assume that $\varphi(\cdot) \in \mathscr{A}_{q(\cdot)}$. Let γ and ε be positive constants such that $\frac{n}{\gamma} < p_- \leqslant p_+ < \frac{n}{\gamma-\varepsilon}$. Then the operator J_Ω^γ is bounded from $L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)$ to $\widetilde{H}^{p(\cdot)}_{\gamma,n}(\Omega)$, i.e. there is a positive constant c_0 such that for all $f \in L^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)$, (15) holds.*

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