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COMPLETE MONOTONICITY OF THE REMAINDER OF AN ASYMPTOTIC EXPANSION OF THE GENERALIZED GURLAND'S RATIO

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Abstract. Let $a,b,c,d \in \mathbb{R}$ with $a+b=c+d=2r+1$. Then

$$\ln \frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} \sim \sum_{k=1}^{\infty} \frac{B_{2k}(\theta_1)-B_{2k}(\theta_2)}{k(2k-1)(x+r)^{2k-1}} \text{ as } x\to\infty,$$

where $(\delta_1,\delta_2)=(|a-b|,|c-d|)=(1-2\theta_1,1-2\theta_2)$. When $0\leqslant\delta_2<\delta_1\leqslant 1$, the function

$$x\mapsto(-1)^m\left[\ln\frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)}-\sum_{k=1}^{m}\frac{B_{2k}(\theta_1)-B_{2k}(\theta_2)}{k(2k-1)(x+r)^{2k-1}}\right]$$

for $m\in\mathbb{N}$ is completely monotonic on $(-r,\infty)$. This yields some known and new results.

1. Introduction

The ratio of gamma functions

$$T(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma((x+y)/2)^2} \quad x,y>0,$$

is called Gurland's ratio by Merkle in [16] due to Gurland's paper [12]. In probability theory and their applications, the ratio $T(x,x+2v)$ for $x,x+2v>0$ is in connection with the variance of an estimator involving gamma distribution; while the ratios

$$T\left(\frac{1}{p},\frac{3}{p}\right)=\frac{\Gamma(1/p)\Gamma(3/p)}{\Gamma(2/p)^2} \text{ and } T\left(\frac{1}{p},\frac{5}{p}\right)=\frac{\Gamma(5/p)\Gamma(1/p)}{\Gamma(3/p)^2} \text{ for } p>0,$$

called Mallat ratio [14] and Kurtosis ratio [35], respectively, are used to estimate the shape parameter $\hat{p}$ in a generalized Gaussian density. Gurland's ratio has attracted the attention of some scholars on this account, and some of interesting results were found, including inequalities [4], [10], [13], [15], [16], [19], [22], [25], (complete) monotonicity [6], [16], [20], [29], [30], [32], [33], [34], asymptotic expansions [4], [8], [24], [32].

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For $a,b,c,d \in \mathbb{R}$, let us consider the following ratio of gamma functions

$$x \mapsto Q_{a,b;c,d}(x) = \frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)}, \quad x > -\min\{a,b,c,d\}.$$

Clearly, $Q_{a,b;c,d}(x)$ is a generalization of Gurland's ratio $T(x+a,x+b)$, and we call it as generalized Gurland's ratio. In 1986, Bustoz and Ismail [6, Theorem 6] showed that the function

$$x \mapsto p(x;a,b) = \frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)} = Q_{0,a+b;a,b}(x)$$

for $a,b \geqslant 0$ is logarithmically completely monotonic on $(0,\infty)$ (see also [16, Lemma 1], [33, Corollary 3.6]). In 2017, Yang and Zheng [33, Corollary 4.9] proved that the function $Q_{a,b;c,d}(x)$ is logarithmically completely monotonic on $(-\min\{a,b,c,d\},\infty)$ if and only if $a+b \leqslant c+d$ and $\min\{a,b\} \leqslant \min\{c,d\}$, and $\ln Q_{a,b;c,d}(x)$ is completely monotonic on $(-\min\{a,b,c,d\},\infty)$ if and only if $a+b=c+d$ and $\min\{a,b\} \leqslant \min\{c,d\}$. In 2019, the authors further proved in [34, Theorem 1] that, for fixed $p,q,r,s,u,v \in \mathbb{R}$ with $(p-q)(r-s)(u-v) \neq 0$ and $\rho = \min(p,q,r,s) + \min(u,v)$, the function

$$x \mapsto \frac{\ln Q_{p+u,q+v;p+v,q+u}(x)}{(p-q)(u-v)} - \frac{\ln Q_{r+u,s+v;r+v,s+u}(x)}{(r-s)(u-v)}$$

is completely monotonic on $(-\rho,\infty)$ if and only if $p+q \leqslant r+s$ and $\min(p,q) \leqslant \min(r,s)$.

The aim of this paper is to further investigate the asymptotic expansion of function $Q_{a,b;c,d}(x)$, and the complete monotonicity of the remainder of the asymptotic expansion of $\ln Q_{a,b;c,d}(x)$. To state our results, we need two basic knowledge. The first is the Bernoulli polynomials $B_n(x)$ defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi, \tag{1}$$

which satisfy the following properties listed in [1, (23.1.8), (23.1.6), (23.2.5), (23.1.14), (23.1.21)]:

PROPERTY 1. $B_n(1-x) = (-1)^n B_n(x)$;

PROPERTY 2. $B_n(x+1) - B_n(x) = nx^{n-1}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$;

PROPERTY 3. $B_n'(x) = nB_{n-1}(x)$ and $n\int_a^x B_{n-1}(t)\,dt = B_n(x) - B_n(a)$, $n \in \mathbb{N}$;

PROPERTY 4. $(-1)^{n+1} B_{2n+1}(x) > 0$, $x \in (0,1/2)$, $n \in \mathbb{N}$;

PROPERTY 5. $B_n(1/2) = -\left(1-2^{1-n}\right)B_n$, $n \in \mathbb{N}_0$.

The second is (logarithmically) completely monotonic functions. A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies

$$(-1)^k f^{(k)}(x) \geqslant 0$$

for all $k \in \mathbb{N}_0$ on I (see [3], [26]). A positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geqslant 0$$

for all $k \in \mathbb{N}$ on I (see [2], [21]). It was pointed out in [21] that if f is logarithmically completely monotonic on I then f is completely monotonic on I, and not vice versa.

The famous Bernstein Theorem [26, p. 161, Theorem 12b] tells us that the function $f(x)$ is completely monotonic on $(0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where $\mu(t)$ is nondecreasing and the integral converges for $0 < x < \infty$.

Now we state our main result as follows.

THEOREM 1. *Let $a,b,c,d \in \mathbb{R}$ with $a+b=c+d=2r+1$ and let $\delta_1 = |a-b|$, $\delta_2 = |c-d|$. The following statements are valid.*

(i) It holds that

$$\ln \frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} = \int_0^\infty \frac{\cosh(\delta_1 t/2) - \cosh(\delta_2 t/2)}{t \sinh(t/2)} e^{-(x+r)t} dt \tag{2}$$

$$\sim \sum_{k=1}^{\infty} \frac{B_{2k}(\theta_1) - B_{2k}(\theta_2)}{k(2k-1)(x+r)^{2k-1}} \text{ as } x \to \infty, \tag{3}$$

where $\theta_k = (1-\delta_k)/2$, $k = 1,2$.

(ii) Let

$$D_m(x) = \ln \frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} - \sum_{k=1}^{m} \frac{B_{2k}(\theta_1) - B_{2k}(\theta_2)}{k(2k-1)(x+r)^{2k-1}}.$$

If $0 \leqslant \delta_2 < \delta_1 \leqslant 1$, then for any integer $m \in \mathbb{N}$, the function $x \mapsto (-1)^m D_m(x)$ is completely monotonic on $(-r,\infty)$. Consequently, the inequality

$$|D_m(x)| < \frac{|B_{2m+2}(\theta_1) - B_{2m+2}(\theta_2)|}{(m+1)(2m+1)(x+r)^{2m+1}} \tag{4}$$

holds for $x > -r$, where the upper bound is sharp.

REMARK 1. Using Property 3 we see that

$$B_{2k}(\theta_1) - B_{2k}(\theta_2) = -2k \int_{\theta_1}^{\theta_2} B_{2k-1}(\theta) d\theta.$$

If $0 \leqslant \delta_2 < \delta_1 \leqslant 1$ then $0 \leqslant \theta_1 < \theta_2 \leqslant 1/2$. By Property 4 we find that

$$B_{2k}(\theta_1) - B_{2k}(\theta_2) < (>) 0 \text{ if } k \text{ is odd (even)},$$

which shows that the series given in (3) is alternate if $0 \leqslant \delta_2 < \delta_1 \leqslant 1$.

2. Consequences and remarks

Let $\delta_2 = 0$. Then $\theta_2 = 1/2$ and $c = d = (a+b)/2 = r + 1/2$. Using Theorem 1 and replacing (δ, θ) with (δ_1, θ_1) we have

COROLLARY 1. *Let $a, b \in \mathbb{R}$ with $\delta = |a-b| \neq 0$, $r = (a+b-1)/2$.*
(i) The following integral representation and asymptotic expansion

$$\ln \frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+(a+b)/2)^2} = \int_0^\infty \frac{\cosh(\delta t/2) - 1}{t \sinh(t/2)} e^{-(x+r)t} dt$$
$$\sim \sum_{k=1}^\infty \frac{B_{2k}(\theta) - B_{2k}(1/2)}{k(2k-1)(x+r)^{2k-1}} \text{ as } x \to \infty$$

holds, where $\theta = (1-\delta)/2$.
(ii) Let

$$D_m(x;a,b) = \ln \frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+(a+b)/2)^2} - \sum_{k=1}^m \frac{B_{2k}(\theta) - B_{2k}(1/2)}{k(2k-1)(x+r)^{2k-1}}.$$

If $0 < \delta \leqslant 1$, then the function $x \mapsto (-1)^m D_m(x;a,b)$ for $m \in \mathbb{N}$ is completely monotonic on $(-r, \infty)$.

REMARK 2. Corollary 1 was established in [24, Theorems 1 and 2]. This shows that the Theorem 1 is a generalization of [24, Theorems 1 and 2].

Assume that $b \geqslant a$ and $d \geqslant c$. From the conditions that $a+b = c+d$ and $\delta_2 < \delta_1$ it is deduced that $b > d \geqslant c > a$. Note that

$$\frac{1}{c-a} \ln \frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} = -\frac{\ln\Gamma(x+c) - \ln\Gamma(x+a)}{c-a} + \frac{\ln\Gamma(x+b) - \ln\Gamma(x+d)}{b-d}.$$

Taking $c \to a$ (which implies that $d \to b$) gives

$$\lim_{c \to a} \frac{1}{c-a} \ln \frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} = \psi(x+b) - \psi(x+a).$$

Since

$$\frac{\delta_1 - \delta_2}{c-a} = \frac{b-a-d+c}{c-a} = 2 \text{ and } \frac{\theta_1 - \theta_2}{c-a} = -\frac{1}{2} \frac{\delta_1 - \delta_2}{c-a} = -1,$$

we have

$$\lim_{c\to a}\frac{\cosh(\delta_1 t/2)-\cosh(\delta_2 t/2)}{c-a}=\frac{\delta_1-\delta_2}{c-a}\lim_{\delta_2\to\delta_1}\frac{\cosh(\delta_1 t/2)-\cosh(\delta_2 t/2)}{\delta_1-\delta_2}$$
$$=t\sinh(\delta_1 t/2),$$

$$\lim_{c\to a}\frac{B_{2k}(\theta_1)-B_{2k}(\theta_2)}{c-a}=\frac{\theta_1-\theta_2}{c-a}\lim_{\theta_2\to\theta_1}\frac{B_{2k}(\theta_1)-B_{2k}(\theta_2)}{\theta_1-\theta_2}$$
$$=-2kB_{2k-1}(\theta_1),$$

where the last equality holds due to Property 3. Using Theorem 1 and replacing (δ,θ) with (δ_1,θ_1) we have

COROLLARY 2. *Let $a,b\in\mathbb{R}$ with $\delta=b-a>0$, $r=(a+b-1)/2$ and $\theta=(1-\delta)/2$. (i) It holds that*

$$\psi(x+b)-\psi(x+a)=\int_0^\infty\frac{\sinh(\delta t/2)}{\sinh(t/2)}e^{-(x+r)t}dt$$
$$\sim\sum_{k=1}^{\infty}\frac{-2B_{2k-1}(\theta)}{(2k-1)(x+r)^{2k-1}}\ \text{as } x\to\infty.$$

(ii) Let

$$D_m^*(x;b,a)=\psi(x+b)-\psi(x+a)+\sum_{k=1}^{m}\frac{2B_{2k-1}(\theta)}{(2k-1)(x+r)^{2k-1}}.$$

If $0<\delta\leqslant 1$, then the function $x\mapsto(-1)^m D_m^*(x;b,a)$ for $m\in\mathbb{N}$ is completely monotonic on $(-r,\infty)$.

REMARK 3. Let

$$R_m(x;b,a)=\frac{\ln\Gamma(x+b)-\ln\Gamma(x+a)}{b-a}-\ln(x+r)-\sum_{k=1}^{m}\frac{B_{2k+1}(\theta)}{\delta k(2k+1)(x+r)^{2k}}.$$

In 2020, Yang, Tian and Ha [31] proved that, under the conditions as in Corollary 2, the function $x\mapsto(-1)^m R_m(x;b,a)$ is completely monotonic on $(-r,\infty)$. Now we present a simple proof of Theorem 2 in [31] using Corollary 2. In fact, since $\lim_{x\to\infty}R_m(x;b,a)=0$, it suffices to prove that $(-1)^{m+1}R_m'(x;b,a)$ is completely monotonic on $(-r,\infty)$. Differentiation yields

$$\delta R_m'(x;b,a)=\psi(x+b)-\psi(x+a)-\frac{\delta}{x+r}+\sum_{k=1}^{m}\frac{2B_{2k+1}(\theta)}{(2k+1)(x+r)^{2k+1}}.$$

Since $2B_1(\theta)=2\theta-1=-\delta$, we have

$$-\frac{\delta}{x+r}+\sum_{k=1}^{m}\frac{2B_{2k+1}(\theta)}{(2k+1)(x+r)^{2k+1}}=\sum_{k=0}^{m}\frac{2B_{2k+1}(\theta)}{(2k+1)(x+r)^{2k+1}},$$

and then, $\delta R_m'(x;b,a)$ can be written as

$$\delta R_m'(x;b,a) = \psi(x+b) - \psi(x+a) + \sum_{k=0}^{m} \frac{2B_{2k+1}(\theta)}{(2k+1)(x+r)^{2k+1}} = D_{m+1}^*(x;b,a).$$

By Corollary 2 the required complete monotonicity follows.

We continue to observe Corollary 2. Evidently, $x \mapsto (-1)^m \lim_{b\to a}[D_m^*(x;b,a)/\delta]$ for $m \in \mathbb{N}$ is also completely monotonic on $(-r,\infty)$. Applying L'Hospital rule with Properties 1 and 3, we have

$$\lim_{b\to a} \frac{\psi(x+b) - \psi(x+a)}{b-a} = \psi'(x+a),$$

$$\lim_{b\to a} \frac{B_{2k-1}(\theta)}{\delta} = \lim_{\theta\to 1/2} \frac{B_{2k-1}(\theta)}{1-2\theta} = -\frac{1}{2}(2k-1)B_{2k-2}\left(\frac{1}{2}\right).$$

Then

$$\lim_{b\to a} \frac{D_m^*(x;b,a)}{b-a} = \psi'(x+a) - \sum_{k=1}^{m} \frac{B_{2k-2}(1/2)}{(x+a-1/2)^{2k-1}}.$$

Taking $a = 1/2$ gives the following corollary.

COROLLARY 3. *Let*

$$D_m^*(x) = \psi'\left(x+\frac{1}{2}\right) - \sum_{k=0}^{m-1} \frac{B_{2k}(1/2)}{x^{2k+1}}.$$

The function $x \mapsto (-1)^m D_m^*(x)$ for $m \in \mathbb{N}$ is completely monotonic on $(0,\infty)$.

REMARK 4. Let

$$g_m(x) = \ln\Gamma\left(x+\frac{1}{2}\right) - x\ln x + x - \frac{1}{2}\ln(2\pi) + \sum_{k=1}^{m} \frac{\left(1-2^{1-2k}\right)B_{2k}}{2k(2k-1)x^{2k-1}}.$$

Yang [28, Theorem 4] proved that the function $x \mapsto (-1)^{m+1} g_m(x)$ is completely monotonic on $(0,\infty)$. Now we give a concise proof of this assertion. In fact, differentiation yields

$$g_m'(x) = \psi\left(x+\frac{1}{2}\right) - \ln x - \sum_{k=1}^{m} \frac{\left(1-2^{1-2k}\right)B_{2k}}{2kx^{2k}},$$

$$g_m''(x) = \psi'\left(x+\frac{1}{2}\right) + \sum_{k=0}^{m} \frac{\left(1-2^{1-2k}\right)B_{2k}}{x^{2k+1}} = D_{m+1}^*(x),$$

where the last equality holds due to $B_{2k}(1/2) = -\left(1-2^{1-2k}\right)B_{2k}$ derived from Property 5. By Corollary 3 we see that $x \mapsto (-1)^{m+1} D_{m+1}^*(x)$ for $m \in \mathbb{N}_0$ is completely monotonic on $(0,\infty)$, and so is $(-1)^{m+1} g_m''(x)$ on $(0,\infty)$. In view of $\lim_{x\to\infty} g_m(x) = \lim_{x\to\infty} g_m'(x) = 0$, we find that $x \mapsto (-1)^{m+1} g_m(x)$ is also completely monotonic on $(0,\infty)$.

Let $(a,b) = (p, 1-p)$ and $(c,d) = (q, 1-q)$ with $p \neq q$ in Theorem 1. Then $r = 0$. By Theorem 1 we obtain the following corollary.

COROLLARY 4. *Let $p, q \in \mathbb{R}$ with $p \neq q$ and let $\delta_1 = |1-2p|$, $\delta_2 = |1-2q|$. The following statements are valid.*

(i) It holds that

$$\ln \frac{\Gamma(x+p)\Gamma(x+1-p)}{\Gamma(x+q)\Gamma(x+1-q)} = \int_0^\infty \frac{\cosh(\delta_1 t/2) - \cosh(\delta_2 t/2)}{t \sinh(t/2)} e^{-xt} dt$$
$$\sim \sum_{k=1}^\infty \frac{B_{2k}(p) - B_{2k}(q)}{k(2k-1)x^{2k-1}} \text{ as } x \to \infty.$$

(ii) Let

$$\Delta_m(x) = \ln \frac{\Gamma(x+p)\Gamma(x+1-p)}{\Gamma(x+q)\Gamma(x+1-q)} - \sum_{k=1}^m \frac{B_{2k}(p) - B_{2k}(q)}{k(2k-1)x^{2k-1}}.$$

If $0 \leqslant p < q \leqslant 1/2$, then for any integer $m \in \mathbb{N}$, the function $x \mapsto (-1)^m \Delta_m(x)$ is completely monotonic on $(0, \infty)$. Consequently, the inequality

$$|\Delta_m(x)| < \frac{|B_{2m+2}(p) - B_{2m+2}(q)|}{(m+1)(2m+1)x^{2m+1}}$$

holds for $x > 0$, where the upper bound is sharp.

A transformation formula of asymptotic expansions was established in [7, Lemma 3] (see also [8, Lemma 3.5]), which states that

$$\exp\left(\sum_{n=1}^\infty \frac{u_n}{x^n}\right) \sim \sum_{n=0}^\infty \frac{v_n}{x^n} \quad \text{as } x \to \infty,$$

with $v_0 = 1$ and

$$v_n = \frac{1}{n} \sum_{k=1}^n k u_k v_{n-k} \quad \text{for } n \geqslant 1.$$

Writing the asymptotic expansion (3) as

$$\frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} \sim \exp\left[\sum_{n=1}^\infty \frac{u_n}{(x+r)^n}\right],$$

where

$$u_{2n-1} = \frac{B_{2n}(\theta_1) - B_{2n}(\theta_2)}{n(2n-1)(x+r)^{2n-1}} \quad \text{and} \quad u_{2n} = 0,$$

then employing the above transformation formula of asymptotic expansions, we obtain another asymptotic expansion of $Q_{a,b;c,d}(x)$.

COROLLARY 5. *Let $a,b,c,d \in \mathbb{R}$ with $a+b=c+d=2r+1$ and let $\delta_1=|a-b|$, $\delta_2=|c-d|$. Then as $x\to\infty$,*

$$Q_{a,b;c,d}(x)=\frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)}\sim\sum_{n=0}^{\infty}\frac{v_n}{(x+r)^n}$$

with $v_0=1$ and

$$v_n=\frac{1}{n}\sum_{j=1}^{\lfloor (n+1)/2\rfloor}\frac{1}{j}\left[B_{2j}(\theta_1)-B_{2j}(\theta_2)\right]v_{n-2j+1},$$

where $\theta_k=(1-\delta_k)/2,\ k=1,2.$

We close this section with two examples.

EXAMPLE 1. In Corollary 4, taking $(p,q)=(0,1/4)$ gives $(\delta_1,\delta_2)=(1,1/2)$. Then

$$\ln\frac{\Gamma(x)\Gamma(x+1)}{\Gamma(x+1/4)\Gamma(x+3/4)}\sim\sum_{k=1}^{\infty}\frac{B_{2k}(0)-B_{2k}(1/4)}{k(2k-1)x^{2k-1}}\ \text{ as } x\to\infty,$$

and the function

$$x\mapsto(-1)^m\left[\ln\frac{\Gamma(x)\Gamma(x+1)}{\Gamma(x+1/4)\Gamma(x+3/4)}-\sum_{k=1}^{m}\frac{B_{2k}(0)-B_{2k}(1/4)}{k(2k-1)x^{2k-1}}\right]$$

is completely monotonic on $(0,\infty)$. Hence, the double inequality

$$\sum_{k=1}^{2m}\frac{B_{2k}(0)-B_{2k}(1/4)}{k(2k-1)x^{2k-1}}<\ln\frac{\Gamma(x)\Gamma(x+1)}{\Gamma(x+1/4)\Gamma(x+3/4)}<\sum_{k=1}^{2n-1}\frac{B_{2k}(0)-B_{2k}(1/4)}{k(2k-1)x^{2k-1}}$$

holds for $x>0$ and $m,n\in\mathbb{N}$. In particular, when $m=1$, $n=2$ we have

$$\frac{3}{16x}-\frac{3}{512x^3}<\ln\frac{\Gamma(x)\Gamma(x+1)}{\Gamma(x+1/4)\Gamma(x+3/4)}<\frac{3}{16x}-\frac{3}{512x^3}+\frac{33}{20480x^5}$$

for $x>0$.

EXAMPLE 2. In Corollary 5, Taking $(a,b)=(0,1)$ and $(c,d)=(1/2,1/2)$ gives $r=0$, $(\delta_1,\delta_2)=(1,0)$ and $(\theta_1,\theta_2)=(0,1/2)$. Then as $x\to\infty$,

$$\frac{\Gamma(x)\Gamma(x+1)}{\Gamma(x+1/2)^2}\sim\sum_{n=0}^{\infty}\frac{v_n}{x^n}$$

with $v_0=1$ and

$$v_n=\frac{1}{n}\sum_{j=1}^{\lfloor (n+1)/2\rfloor}\frac{1}{j}\left[B_{2j}(0)-B_{2j}(1/2)\right]v_{n-2j+1}$$
$$=\frac{1}{n}\sum_{j=1}^{\lfloor (n+1)/2\rfloor}\frac{1}{j}\left(2-2^{1-2j}\right)B_{2j}v_{n-2j+1},$$

where the last equality holds due to $B_{2j}(0) = B_{2j}$ and $B_{2j}(1/2) = -\left(1-2^{1-2j}\right)B_{2j}$. A direct computation leads to

$$v_1 = \frac{1}{4},\ v_2 = \frac{1}{32},\ v_3 = -\frac{1}{128},\ v_4 = -\frac{5}{2048},\ v_5 = \frac{23}{8192},\ v_6 = \frac{53}{65\,536}.$$

Noting that $\Gamma(x) = \Gamma(x+1)/x$, we arrive at

$$\left[\frac{\Gamma(x+1)}{\Gamma(x+1/2)}\right]^2 \sim x + \frac{1}{4} + \frac{1}{32x} - \frac{1}{128x^2} - \frac{5}{2048x^3} + \frac{23}{8192x^4} + \frac{53}{65\,536x^5} + \ldots$$

as $x \to \infty$.

REMARK 5. The ratio $W(x) = \Gamma(x+1)/\Gamma(x+1/2)$ is called Wallis' fraction (see [8]). The asymptotic expansion was derived in [5] (see also [17]). Two nice asymptotic expansions of Wallis' fraction were presented in [11], [31]. More asymptotic expansions of $W(x)$ can be found in [9], [23], [27].

3. Lemmas

To prove the first part of Theorem 1, we need the following special case of Watson's lemma.

LEMMA 1. ([18, Section 2.3]) *Assume that the Laplace transform* $\int_0^\infty f(t)e^{-xt}dt$ *converges for all sufficiently large* x, *and* $f(t)$ *is infinitely differentiable in a neighborhood of the origin. Then*

$$\int_0^\infty f(t)e^{-xt}dt \sim \sum_{n=0}^\infty \frac{f^{(n)}(0)}{x^{n+1}}, \qquad x \to \infty.$$

LEMMA 2. *Let* $0 < v < u$. *The function*

$$t \mapsto \phi_{u,v}(t) = \frac{\sinh\left(v\sqrt{t}\right)}{\sinh\left(u\sqrt{t}\right)}$$

is logarithmically completely monotonic on $(0,\infty)$. *Therefore,* $\phi_{u,v}(t)$ *is completely monotonic on* $(0,\infty)$

Proof. To prove the required logarithmically complete monotonicity of $\phi_{u,v}(t)$, it suffices to prove that $-[\ln\phi_{u,v}(t)]'$ is completely monotonic on $(0,\infty)$. It was listed in [1, Eq. (4.5.68)] that

$$\frac{\sinh z}{z} = \prod_{n=1}^\infty \left(1 + \frac{z^2}{n^2\pi^2}\right)$$

for $z \in \mathbb{C}$. Logarithmic differentiation yields

$$\left[\ln\frac{\sinh\left(u\sqrt{t}\right)}{u\sqrt{t}}\right]' = \frac{d}{dt}\sum_{n=1}^\infty \ln\left(1 + \frac{u^2t}{n^2\pi^2}\right) = \sum_{n=1}^\infty \frac{1}{\pi^2n^2/u^2 + t}.$$

We thus obtain that

$$-\left[\ln\phi_{u,v}(t)\right]' = -\left[\ln\frac{v}{u} + \ln\frac{\sinh\left(v\sqrt{t}\right)}{v\sqrt{t}} - \ln\frac{\sinh\left(u\sqrt{t}\right)}{u\sqrt{t}}\right]'$$
$$= -\sum_{n=1}^{\infty}\frac{1}{t+\pi^2n^2/v^2} + \sum_{n=1}^{\infty}\frac{1}{t+\pi^2n^2/u^2}$$
$$= \frac{\pi^2\left(u^2-v^2\right)}{u^2v^2}\sum_{n=1}^{\infty}\frac{n^2}{\left(t+\pi^2n^2/u^2\right)\left(t+\pi^2n^2/v^2\right)}.$$

Since $t \mapsto 1/(t+\alpha)$ $(\alpha > 0)$ is completely monotonic on $(0,\infty)$, so is $-\left[\ln\phi_{u,v}(t)\right]'$ on $(0,\infty)$.

As shown in [21], a (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic. Therefore, the function $\phi_{u,v}(t)$ is completely monotonic on $(0,\infty)$. □

LEMMA 3. *Let*

$$f(t) = \frac{\cosh(\delta_1 t/2) - \cosh(\delta_2 t/2)}{t\sinh(t/2)}.$$

If $0 \leqslant \delta_2 < \delta_1 \leqslant 1$, *then the function* $t \mapsto f\left(\sqrt{t}\right)$ *is completely monotonic on* $(0,\infty)$.

Proof. Since

$$\frac{\cosh\left(\delta_1\sqrt{t}/2\right) - \cosh\left(\delta_2\sqrt{t}/2\right)}{(\delta_1-\delta_2)\sqrt{t}/2} = \int_0^1 \sinh\left(v\sqrt{t}\right)dx,$$

where

$$v = v(x) = x\frac{\delta_1}{2} + (1-x)\frac{\delta_2}{2} \in \left(0,\frac{1}{2}\right), \tag{5}$$

due to $0 \leqslant \delta_2 < \delta_1 \leqslant 1$ and $x \in [0,1]$, $f\left(\sqrt{t}\right)$ can be represented as

$$f\left(\sqrt{t}\right) = (\delta_1-\delta_2)\frac{\cosh\left(\delta_1\sqrt{t}/2\right) - \cosh\left(\delta_2\sqrt{t}/2\right)}{(\delta_1-\delta_2)\sqrt{t}\sinh\left(\sqrt{t}/2\right)} = (\delta_1-\delta_2)\int_0^1 \frac{\sinh\left(v\sqrt{t}\right)}{\sinh\left(u\sqrt{t}\right)}dx,$$

where $u = 1/2$ and $v \in (0,1/2)$ is defined by (5). It follows from Lemma 2 that

$$(-1)^n\frac{d^n}{dt^n}f\left(\sqrt{t}\right) = (\delta_1-\delta_2)\int_0^1 (-1)^n\frac{d^n}{dt^n}\left[\frac{\sinh\left(v\sqrt{t}\right)}{\sinh\left(u\sqrt{t}\right)}\right]dx > 0$$

for $t > 0$. This completes the proof. □

LEMMA 4. *If* $g(x)$ *is completely monotonic on the interval* I *and* $x_0 \in I$, *then*

$$(-1)^{m+1}\left[g(x) - \sum_{k=0}^{m}\frac{g^{(k)}(x_0)}{k!}(x-x_0)^k\right] > 0$$

for all $x \in I$ *and* $m \in \mathbb{N}_0$.

Proof. It is known that

$$g(x)=\sum_{k=0}^{m}\frac{g^{(k)}(x_0)}{k!}(x-x_0)^k+\int_{x_0}^{x}g^{(m+1)}(t)\frac{(x-t)^m}{m!}dt.$$

Then

$$\begin{aligned}&(-1)^{m+1}\left[g(x)-\sum_{k=0}^{m}\frac{g^{(k)}(x_0)}{k!}(x-x_0)^k\right]\\&=\int_{x_0}^{x}(-1)^{m+1}g^{(m+1)}(t)\frac{(x-t)^m}{m!}dt>0\end{aligned}$$

for all $x\in I$, which completes the proof. □

The following lemma is crucial to prove the second part of Theorem 1.

LEMMA 5. *For* $m\in\mathbb{N}$, *let*

$$J_m(t)=\frac{\cosh(\delta_1 t/2)-\cosh(\delta_2 t/2)}{t\sinh(t/2)}-2\sum_{k=1}^{m}\frac{B_{2k}(\theta_1)-B_{2k}(\theta_2)}{(2k)!}t^{2k-2},\tag{6}$$

where $\theta_k=(1-\delta_k)/2$, $k=1,2$. *If* $0\leqslant\delta_2<\delta_1\leqslant 1$, *then* $(-1)^m J_m(t)>0$ *for* $t>0$.

Proof. We first show that

$$J_m\left(\sqrt{t}\right)=g(t)-\sum_{k=0}^{m-1}\frac{g^{(k)}(0)}{k!}t^k,$$

where $g(t)=f\left(\sqrt{t}\right)$. Using the definition of Bernoulli polynomials (1) yields

$$\begin{aligned}\frac{\cosh(\delta t/2)}{t\sinh(t/2)}&=\frac{1}{t}\frac{e^{\delta t/2}+e^{-\delta t/2}}{e^{t/2}-e^{-t/2}}=\frac{1}{t^2}\frac{te^{(\delta+1)t/2}+te^{(1-\delta)t/2}}{e^t-1}\\&=\sum_{n=0}^{\infty}\frac{B_n((1+\delta)/2)+B_n((1-\delta)/2)}{n!}t^{n-2}.\end{aligned}$$

By Property 1 it is easy to see that

$$B_n\left(\frac{1+\delta}{2}\right)+B_n\left(\frac{1-\delta}{2}\right)=\begin{cases}0 & \text{if } n=2m+1,\\ 2B_{2m}\left(\frac{1-\delta}{2}\right) & \text{if } n=2m,\end{cases}$$

which yields

$$\frac{\cosh(\delta t/2)}{t\sinh(t/2)}=\sum_{m=0}^{\infty}\frac{2B_{2m}((1-\delta)/2)}{(2m)!}t^{2m-2}.$$

This together with $B_0(x) = 1$ gives

$$f(t) = \frac{\cosh(\delta_1 t/2) - \cosh(\delta_2 t/2)}{t \sinh(t/2)} \\ = 2\sum_{m=0}^{\infty} \frac{B_{2m}((1-\delta_1)/2) - B_{2m}((1-\delta_2)/2)}{(2m)!} t^{2m-2} \\ = 2\sum_{k=0}^{\infty} \frac{B_{2k+2}(\theta_1) - B_{2k+2}(\theta_2)}{(2k+2)!} t^{2k} \tag{7}$$

for $|t| < 2\pi$. We thus obtain the Taylor series of the function $g(t) = f(\sqrt{t})$ about $t = 0$:

$$g(t) = f(\sqrt{t}) = 2\sum_{k=0}^{\infty} \frac{B_{2k+2}(\theta_1) - B_{2k+2}(\theta_2)}{(2k+2)!} t^k,$$

which converges for $0 \leqslant t < 4\pi^2$. Noting that

$$2\sum_{k=1}^{m} \frac{B_{2k}(\theta_1) - B_{2k}(\theta_2)}{(2k)!} t^{k-1} = 2\sum_{k=1}^{m-1} \frac{B_{2k+2}(\theta_1) - B_{2k+2}(\theta_2)}{(2k+2)!} t^k,$$

we have

$$J_m(\sqrt{t}) = g(t) - \sum_{k=0}^{m-1} \frac{g^{(k)}(0)}{k!} t^k.$$

By using Lemma 3, we find that $g(t) = f(\sqrt{t})$ is completely monotonic on $(0,\infty)$. It then follows from Lemma 4 that $(-1)^m J_m(\sqrt{t}) > 0$ for $t > 0$, which implies that $(-1)^m J_m(t) > 0$ for $t > 0$. This completes the proof. □

4. Proof of Theorem 1

We are in a position to prove Theorem 1.

Proof. (i) Using the integral representation of $\ln\Gamma(x)$ [1, p.258, (6.1.50)]

$$\ln\Gamma(x) = \int_0^\infty \left((x-1)e^{-t} - \frac{e^{-t} - e^{-xt}}{1-e^{-t}}\right)\frac{dt}{t} := \int_0^\infty \xi(x,t)\frac{dt}{t} \quad (x > 0),$$

we get

$$\ln\Gamma(x+a) + \ln\Gamma(x+b) - \ln\Gamma(x+c) - \ln\Gamma(x+d) = \int_0^\infty \eta(x,t)\,dt,$$

where

$$\eta(x,t) = \frac{1}{t}\left[\xi(x+a,t) + \xi(x+b,t) - \xi(x+c,t) - \xi(x+d,t)\right].$$

An easy verification gives

$$\eta(x,t) = \frac{e^{-at} + e^{-bt} - e^{-ct} - e^{-dt}}{t(1-e^{-t})} e^{-tx},$$

and then,

$$\ln \frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} = \int_0^\infty f(t)\, e^{-(x+r)t}\, dt,$$

where

$$f(t) = e^{rt} \frac{e^{-at} + e^{-bt} - e^{-ct} - e^{-dt}}{t(1-e^{-t})}.$$

Clearly, $f(t)$ can be written as

$$\begin{aligned} f(t) &= \frac{e^{(b-a)t/2} + e^{(a-b)t/2} - e^{(d-c)t/2} - e^{(c-d)t}/2}{t\left(e^{t/2} - e^{-t/2}\right)} \\ &= \frac{\cosh(\delta_1 t/2) - \cosh(\delta_2 t/2)}{t \sinh(t/2)}. \end{aligned}$$

On the other hand, from the Taylor series of $f(t)$ at $t=0$ proved in (7) we find that $f^{(2n+1)}(0) = 0$ and

$$\begin{aligned} f^{(2n)}(0) &= (2n)! \frac{f^{(2n)}(0)}{(2n)!} = (2n)! 2 \frac{B_{2n+2}(\theta_1) - B_{2n+2}(\theta_2)}{(2n+2)!} \\ &= \frac{B_{2n+2}(\theta_1) - B_{2n+2}(\theta_2)}{(n+1)(2n+1)}. \end{aligned}$$

By Lemma 1, we get that

$$\int_0^\infty f(t)\, e^{-(x+r)t}\, dt \sim \sum_{n=0}^\infty \frac{f^{(2n)}(0)}{(x+r)^{2n+1}} = \sum_{n=0}^\infty \frac{B_{2n+2}(\theta_1) - B_{2n+2}(\theta_2)}{(n+1)(2n+1)(x+r)^{2n+1}}$$

as $x \to \infty$, which proves part one of this theorem.

(ii) Firstly, we establish the integral representation of $D_m(x)$. By the integral representation (2) and

$$\frac{1}{x^n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-xt}\, dt,$$

we immediately get

$$\begin{aligned} D_m(x) &= \int_0^\infty \frac{\cosh(\delta_1 t/2) - \cosh(\delta_2 t/2)}{t \sinh(t/2)} e^{-(x+r)t}\, dt \\ &\quad - \sum_{k=1}^m \frac{B_{2k}(\theta_1) - B_{2k}(\theta_2)}{k(2k-1)} \frac{1}{(2k-2)!} \int_0^\infty t^{2k-2} e^{-(x+r)t}\, dt \\ &= \int_0^\infty J_m(t)\, e^{-(x+r)t}\, dt, \end{aligned}$$

where $J_m(t)$ is defied by (6). Then

$$(-1)^m D_m(x) = \int_0^\infty (-1)^m J_m(t)\, e^{-(x+r)t}\, dt.$$

Secondly, from Lemma 5 and Bernstein Theorem it follows that $x \mapsto (-1)^m D_m(x)$ is completely monotonic on $(-r,\infty)$.

Finally, we prove inequality (4). If m is even, then from the inequalities $D_m(x) > 0$, $D_{m+1}(x) < 0$ for $x > -r$ and the relation

$$D_{m+1}(x) = D_m(x) - \frac{B_{2m+2}(\theta_1) - B_{2m+2}(\theta_2)}{(m+1)(2m+1)(x+r)^{2m+1}}$$

it is deduced that

$$0 < D_m(x) < \frac{B_{2m+2}(\theta_1) - B_{2m+2}(\theta_2)}{(m+1)(2m+1)(x+r)^{2m+1}} \text{ for } x > -r. \tag{8}$$

If m is odd, then from the inequalities $D_m(x) < 0$, $D_{m+1}(x) > 0$ it is obtained that

$$\frac{B_{2m+2}(\theta_1) - B_{2m+2}(\theta_2)}{(m+1)(2m+1)(x+r)^{2m+1}} < D_m(x) < 0 \text{ for } x > -r. \tag{9}$$

Inequalities (8) and (9) imply (4). The limit relation

$$\lim_{x\to\infty} \left[(x+r)^{2m+1} |D_m(x)|\right] = \frac{|B_{2m+2}(\theta_1) - B_{2m+2}(\theta_2)|}{(m+1)(2m+1)},$$

implies that the upper bound given in (4) is sharp, which completes the proof. □

5. Concluding remarks

In this paper, we established an asymptotic expansion of $\ln Q_{a,b;c,d}(x)$ and showed that the remainder of this expansion has complete monotonicity (Theorem 1). From Corollaries 1–3 and Remarks 2–4 listed in Section 2 we see that certain known results are consequences of Theorem 1. As far as method and technique are concerned, Lemma 2 is refreshing. By means of this lemma, the proof of Theorem 2 in [31] can be greatly simplified.

Moreover, it should be noted that, in addition to the asymptotic expansion described in (3), there is another class of asymptotic expansion of $Q_{a,b;c,d}(x)$ in the form of hypergeometric series, which first appeared in [25]. In fact, using the Gaussian formula for the hypergeometric function (see [1, p. 556, (15.1.20)])

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-a)} = {}_2F_1(a,b;c;1) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!\,(c)_k} \; (-c \notin \mathbb{N}_0, \operatorname{Re}(c-a-b) > 0),$$

where $(a)_0 = 1$ for $a \neq 0$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geqslant 1$, we obtain two new asymptotic expansions of $Q_{a,b;c,d}(x)$ that, for $a,b,c,d \in \mathbb{R}$ with $a+b = c+d$,

$$\frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} = {}_2F_1(b-c,b-d;x+b;1) = \sum_{k=0}^{\infty} \frac{(b-c)_k (b-d)_k}{k!\,(x+b)_k},$$

$$\frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} = {}_2F_1(a-c,a-d;x+a;1) = \sum_{k=0}^{\infty} \frac{(a-c)_k (a-d)_k}{k!\,(x+a)_k},$$

which converge only if $x > -a$ and $x > -b$, respectively.

Assume that $b \geqslant a$ and $d \geqslant c$. If $b-a > d-c$ then $a < c$, $b > d$, and then $b > d \geqslant c > a$; If $b-a < d-c$ then $a > c$, $b < d$, and then $d > b \geqslant a > c$. Since $1/(x+\alpha)$ is completely monotonic in x, so are $1/(x+\alpha)_k$ in x for $k \geqslant 1$. Then the following theorem is immediate.

THEOREM 2. *Let $a,b,c,d \in \mathbb{R}$ with $a+b=c+d$. If $b-a > d-c \geqslant 0$, then the function*

$$x \mapsto (x+b)_m \left[\frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} - \sum_{k=0}^{m-1} \frac{(b-c)_k (b-d)_k}{k!\,(x+b)_k} \right]$$

is completely monotonic on $(-a,\infty)$.

Finally, $Q_{a,b;c,d}(x)$ can also be represented in the form of infinite product. Using Euler's formula for the gamma function [1, p. 255, (6.1.2)]

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{(z)_{n+1}} \quad (z \neq 0,-1,-2,\ldots),$$

we have that, for $a,b,c,d \in \mathbb{R}$ with $a+b=c+d$,

$$\frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} = \lim_{k\to\infty} \frac{(x+c)_{k+1}(x+d)_{k+1}}{(x+a)_{k+1}(x+b)_{k+1}} = \prod_{k=0}^{\infty} \frac{(k+x+c)(k+x+d)}{(k+x+a)(k+x+b)}. \tag{10}$$

THEOREM 3. *Let $a,b,c,d \in \mathbb{R}$ with $a+b=c+d$. If $b-a > d-c \geqslant 0$, then the function*

$$x \mapsto E_m(x) = \ln \frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)} - \sum_{k=0}^{m-1} \ln \frac{(k+x+c)(k+x+d)}{(k+x+a)(k+x+b)}$$

is completely monotonic on $(-a,\infty)$.

Proof. By (10) we see that

$$E_m(x) = \sum_{k=m}^{\infty} \ln \frac{(k+x+c)(k+x+d)}{(k+x+a)(k+x+b)}.$$

Differentiation yields

$$-E_m'(x) = \sum_{k=m}^{\infty} \phi(x+k),$$

where

$$\phi(y) = \frac{1}{y+a} + \frac{1}{y+b} - \frac{1}{y+c} - \frac{1}{y+d}.$$

Since $a+b=c+d$ and $b-a > d-c \geqslant 0$, we have $b > d \geqslant c > a$. Then $\phi(y)$ can be written as

$$\phi(y) = \frac{(b-c)(c-a)}{(y+a)(y+c)(y+d)} + \frac{(b-c)(c-a)}{(y+b)(y+c)(y+d)},$$

which is clearly completely monotonic in y, so is $\phi(x+k)$ in x. It then follows that $-E_m'(x)$ is completely monotonic on $(-a,\infty)$, and then, so is $E_m(x)$ due to $\lim_{x\to\infty} E_m(x)=0$. $\square$

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