Mathematical
Inequalities
& Applications
Volume 27, Number 1 (2024), 249–259 doi:10.7153/mia-2024-27-19

SOME RESULTS RELATED TO THE HEINZ INEQUALITY IN C*–ALGEBRA

AMEUR SEDDIK

(*Communicated by J.-C. Bourin*)

Abstract. Let f, g be two continuous non-negative real-valued functions defined on the non-negative half-line $[0,\infty)$ that satisfy the condition $f(t)g(t)=t$, for all $t\geqslant 0$, and let P and Q denote two positive elements in an unital C*-algebra $\mathscr{A}$. We shall show that the following model of inequality holds:

$$\forall X\in\mathscr{A},\ \ \|f(P)Xg(Q)+g(P)Xf(Q)\|\geqslant 2\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|.$$

Through this model, we shall establish the universality of the Heinz operator norm inequality and related inequalities within the broad spectrum of any abstract unital C*-algebra.

1. Introduction

Let $\mathfrak{B}(H)$ be the C^*-algebra of all bounded linear operators on a complex separable Hilbert space H. We consider an abstract unital C*-algebra $\mathscr{A}$ with unit I of norm one.

In operator theory, there exist several remarkable operator norm inequalities. In this work, we concentrate to a family of them related to the Heinz inequality, where each of which can follow from the others:

For every two positive operators P and Q in $\mathfrak{B}(H)$, and for every $\lambda\in[0,1]$, the following double operator inequality holds:

$$\forall X\in\mathfrak{B}(H),\ \ \|PX+XQ\|\geqslant\left\|P^{\lambda}XQ^{1-\lambda}+P^{1-\lambda}XQ^{\lambda}\right\|\geqslant 2\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|. \tag{1}$$

For every two positive operators P and Q in $\mathfrak{B}(H)$, the following operator inequality holds:

$$\forall X\in\mathfrak{B}(H),\ \ \|PX+XQ\|\geqslant 2\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|. \tag{2}$$

For every operators A and B in $\mathfrak{B}(H)$, the following operator inequality holds:

$$\forall X\in\mathfrak{B}(H),\ \ \|A^*AX+XBB^*\|\geqslant 2\,\|AXB\|. \tag{3}$$

Mathematics subject classification (2020): 46L05, 47A05, 47A30, 47B15.

Keywords and phrases: Arithmetic-geometric mean inequality, Heinz inequality, C*-algebra, operator norm inequality.

Paper MIA-27-19

Note that the first inequality (1) was introduced by Heinz [5], but its proof is somewhat complicated. So, McIntosh [9] proved the inequality (2) (where (2) follows immediately from (1) with $\lambda = 0, 1, \frac{1}{2}$) and deduced from it the inequality (1) using an iteration method. It is is easy to see the equivalence between (2) and (3) and where each of them is called the arithmetic-geometric mean inequality. In [2, 4, 10], we find a largest family of inequalities including the three given above inequalities and follows from each other. A fourth inequality of this family is given by: for every invertible selfadjoint operator S in $\mathfrak{B}(H)$, the following inequality holds:

$$\forall X \in \mathfrak{B}(H), \quad \left\|SXS^{-1} + S^{-1}XS\right\| \geqslant 2\left\|X\right\|. \tag{4}$$

Independently of the work of Heinz and McIntosh and with another motivation, Corach et al. [3] have proved this last inequality.

From matrix theory to operator theory, the four distinguished inequalities presented before were reformulated in a general situation with any unitarily invariant norm instead of the operator norm, see [1, 6, 7, 8].

Our focus in this paper is to extend the above results to an abstract unital C*-algebra where we shall present:

1. a model of inequality (the main theorem) given by: for f, g be two continuous non-negative (resp. positive) real-valued functions defined on the non-negative (resp. positive) half-line $[0,\infty)$ (resp. $(0,\infty)$) satisfying $f(t)g(t) = t$, for all $t \geqslant 0$ (resp. $t > 0$), and for two positive (resp. invertible positive) elements P, Q in $\mathscr{A}$, the following inequality holds:

$$\forall X \in \mathscr{A}, \quad \|f(P)Xg(Q) + g(P)Xf(Q)\| \geqslant 2\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|,$$

2. for two positive (resp. invertible positive) elements P, Q in $\mathscr{A}$, and for $\lambda \in [0,1]$ (resp. $\lambda \in \mathbb{R}$), the following inequality holds:

$$\forall X \in \mathscr{A}, \quad \left\|P^{\lambda}XQ^{1-\lambda} + P^{1-\lambda}XQ^{\lambda}\right\| \geqslant 2\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|,$$

3. the four above inequalities (1)–(4) remain true.

All these results are given in the section 3.

In section 2, we shall give some observations about the numerical version of the Heinz inequality, and we introduce four models of numerical inequalities and where one of them includes the numerical version of Heinz inequality.

2. Some observations about the numerical version of Heinz inequality.

In this section, we shall interest in the numerical version of Heinz inequality (1) that says: for every $\lambda \in [0,1]$, the following holds:

$$\forall s,t > 0, \quad \frac{s+t}{2} \geqslant \frac{t^{\lambda}s^{1-\lambda} + t^{1-\lambda}s^{\lambda}}{2} \geqslant \sqrt{st}. \tag{5}$$

A simple proof of (5) was previously accomplished given that the function: $[0,1] \to \mathbb{R}$, $\lambda \mapsto t^{\lambda}s^{1-\lambda} + t^{1-\lambda}s^{\lambda}$, is continuous, convex, and has symmetry about the vertical line $\lambda = \frac{1}{2}$ (where s,t are two positive parameters).

We propose a straightforward and fundamental proof of (5) that eschews reliance on the principles of continuity and convexity by considering the two positive real-valued functions f and g defined on the positive half-line $(0,\infty)$ by $f(t) = t^{\lambda}$ and $g(t) = t^{1-\lambda}$, for every $t > 0$ and where λ is a parameter in $[0,1]$. Indeed, for $s,t > 0$, we have:

$$\begin{aligned} \frac{s+t}{2} - \frac{t^{\lambda}s^{1-\lambda} + t^{1-\lambda}s^{\lambda}}{2} &= \frac{1}{2}\left(f(t) - f(s)\right)\left(g(t) - g(s)\right), \\ &\geqslant 0, \text{ since } f \text{ and } g \text{ are both non-decreasing,} \end{aligned}$$

and

$$\frac{t^{\lambda}s^{1-\lambda} + t^{1-\lambda}s^{\lambda}}{2} - \sqrt{st} = \frac{1}{2}\left(\sqrt{f(t)g(s)} - \sqrt{f(s)g(t)}\right)^2 \geqslant 0.$$

This proves easily the numerical version of Heinz inequality .

Extending the analysis, if λ lies outside the interval $[0,1]$, in the domain of $\mathbb{R}\setminus[0,1]$, one function among f and g is non-decreasing while the other is non-increasing. Consequently:

$$\frac{s+t}{2} - \frac{t^{\lambda}s^{1-\lambda} + t^{1-\lambda}s^{\lambda}}{2} = \frac{1}{2}\left(f(t) - f(s)\right)\left(g(t) - g(s)\right) \leqslant 0.$$

This leads to the conclusion that for every $\lambda \in \mathbb{R}\setminus[0,1]$, the inequality below is valid:

$$\forall s,t > 0, \quad \frac{t^{\lambda}s^{1-\lambda} + t^{1-\lambda}s^{\lambda}}{2} \geqslant \frac{s+t}{2} \geqslant \sqrt{st}. \tag{6}$$

It is important to note that the second part of (6) is recognized as the classical numerical arithmetic-geometric mean inequality. Inequality (6) can thus be regarded as a dual inequality to the Heinz inequality (5).

Furthermore, the proof outlined above maintains the validity of the second part of inequality (5) for any real-valued λ. Therefore, it holds true that for any $\lambda \in \mathbb{R}$:

$$\forall s,t > 0, \quad \frac{t^{\lambda}s^{1-\lambda} + t^{1-\lambda}s^{\lambda}}{2} \geqslant \sqrt{st}, \tag{7}$$

Combining the two inequalities (5) and (6), it is derived that for every $\lambda \in \mathbb{R}\setminus[0,1]$ and for every $\mu \in [0,1]$, the following inequality holds:

$$\forall s,t > 0, \quad t^{\lambda}s^{1-\lambda} + t^{1-\lambda}s^{\lambda} \geqslant t^{\mu}s^{1-\mu} + t^{1-\mu}s^{\mu}. \tag{8}$$

Our last observation is to see that the condition $\lambda \in [0,1]$ is necessary and sufficient to have the first part of the Heinz inequality (5) holds.

Indeed, assume that the first part of (5) holds for some $\lambda \in \mathbb{R}\setminus[0,1]$. So, from (5) and (6), we have $t^{\lambda}s^{1-\lambda} + t^{1-\lambda}s^{\lambda} = s+t$, for all $s,t > 0$. This gives use that $\left(t^{\lambda} - s^{\lambda}\right)\left(t^{1-\lambda} - s^{1-\lambda}\right) = 0$, for all $s,t > 0$. So, we have $\left(2^{\lambda} - 1\right)\left(2^{1-\lambda} - 1\right) = 0$.

Hence, $\lambda = 0$ or $\lambda = 1$, contradiction with $\lambda \in \mathbb{R}\setminus[0,1]$. This proves that the condition $\lambda \in [0,1]$ is necessary to have the first part of (5) holds. This proves that the condition $\lambda \in [0,1]$ is necessary and sufficient for the first part of (5).

In this section, we shall show that each of the four above inequalities is a part of a model including infinitely numerical inequalities. In particular, the numerical version of the Heinz inequality becomes a simple example of a model including infinitely of inequalities.

In the following definition, we introduce the notion of the conjugate of a numerical function which is the basis of all results of this paper.

DEFINITION 1. If f, g be two continuous non-negative (resp. positive) real-valued functions defined on the non-negative (resp. positive) half-line $[0,\infty)$ (resp. $(0,\infty)$), we say that g is a conjugate of f, if $f(t)g(t) = t$, for all $t \geqslant 0$ (resp. $t > 0$).

Note that each continuous function $f : (0,\infty) \to (0,\infty)$ has a unique continuous conjugate function g defined on $(0,\infty)$ given by $g(t) = \frac{t}{f(t)}$, for all $t > 0$, and every continuous function $f : [0,\infty) \to [0,\infty)$ has at most one continuous conjugate function $g : [0,\infty) \to [0,\infty)$.

We denote by:

(i). C_+^* (resp. C_+), the class of all continuous function $f : (0,\infty) \to (0,\infty)$ (resp. $f : [0,\infty) \to [0,\infty)$ having a continuous conjugate function $g : [0,\infty) \to [0,\infty)$),

(ii). $\widetilde{f}$, the conjugate of f, where $f \in C_+^*$ (resp. C_+),

(iii). H_λ, the element in C_+^* (resp. C_+) given by $H_\lambda(t) = t^\lambda$, for all $t > 0$ (resp. $t \geqslant 0$), where $\lambda \in \mathbb{R}$ (resp. $\lambda \in [0,1]$), and where the conjugate of H_λ is $H_{1-\lambda}$,

(iv). $(C1)$, the class of all $f \in C_+^*$ (resp. C_+) such that f and $\widetilde{f}$ are both non-decreasing,

(v). $(C2)$, the class of all $f \in C_+^*$ (resp. C_+) such that one function among f and $\widetilde{f}$ is non-decreasing while the other is non-increasing.

Note that H_λ belongs to the subclass $(C1)$ (resp. $(C2)$) of C_+^*, if $\lambda \in [0,1]$ (resp. $\lambda \in \mathbb{R}\setminus[0,1]$); and H_λ belongs to the subclass $(C1)$ of C_+, for all $\lambda \in [0,1]$.

For $f \in C_+^*$ (resp. C_+), it is clear that for every s, $t > 0$ (resp. s, $t \geqslant 0$), the two following equalities hold:

$$\begin{cases} \left(f(t)\widetilde{f}(s) + f(s)\widetilde{f}(t)\right) - 2\sqrt{st} = \left(\sqrt{f(t)\widetilde{f}(s)} - \sqrt{f(s)\widetilde{f}(t)}\right)^2, \\ (s+t) - \left(f(t)\widetilde{f}(s) + f(s)\widetilde{f}(t)\right) = (f(t) - f(s))\left(\widetilde{f}(t) - \widetilde{f}(s)\right). \end{cases}$$

The following proposition follows immediately from the two last identities.

PROPOSITION 1. *Let* $f,\ g \in \mathrm{C}_{+}^{*}$ *(resp.* C_{+}*).*

(1). Assume that f *belongs to the class* $(C1)$*. Then, the following double inequality holds:*

$$\forall s,t>0\,(\textit{resp. } s,t\geqslant 0),\ \ \frac{s+t}{2}\geqslant\frac{1}{2}\left(f(t)\widetilde{f}(s)+f(s)\widetilde{f}(t)\right)\geqslant\sqrt{st}. \qquad (MI1)$$

(2). Assume that f *belongs to the class* $(C2)$*. Then, the following double inequality holds:*

$$\forall s,t>0\,(\textit{resp. } s,t\geqslant 0),\ \ \frac{1}{2}\left(f(t)\widetilde{f}(s)+f(s)\widetilde{f}(t)\right)\geqslant\frac{s+t}{2}\geqslant\sqrt{st}. \qquad (MI2)$$

(3). The following inequality holds:

$$\forall s,t>0\,(\textit{resp. } s,t\geqslant 0),\ \ \frac{1}{2}\left(f(t)\widetilde{f}(s)+f(s)\widetilde{f}(t)\right)\geqslant\sqrt{st}. \qquad (MI3)$$

(4). Assume that f *belongs to the class* $(C1)$ *and* g *belongs to the class* $(C2)$*. Then, the following inequality holds:*

$$\forall s,t>0\,(\textit{resp. } s,t\geqslant 0),\left(g(t)\widetilde{g}(s)+g(s)\widetilde{g}(t)\right)\geqslant\left(f(t)\widetilde{f}(s)+f(s)\widetilde{f}(t)\right). \qquad (MI4)$$

It is clear that with the class C_{+}^{*}:

(i). the model of inequality ($MI1$) includes the numerical version of Heinz inequality (5) with $f=H_{\lambda}$ and where $\lambda\in[0,1]$,

(ii). the model of inequality ($MI2$) includes the inequality dual of Heinz (6) with $f=H_{\lambda}$ and where $\lambda\in\mathbb{R}\backslash[0,1]$,

(iii). the model ($MI3$) includes the inequality (7) with $f=H_{\lambda}$ and $\lambda\in\mathbb{R}$,

(iv). the model ($MI4$) includes the inequality (8) with $f=H_{\mu}$, where $\mu\in[0,1]$, and $g=H_{\lambda}$, where $\lambda\in\mathbb{R}\backslash[0,1]$.

3. Model of norm inequality in unital C*-algebra and applications

In this section, we elaborate on a model of norm inequalities within the framework of an abstract unital C*-algebra. Through this model, we shall establish the universality of the Heinz inequality, denoted herein as (1), along with the four other known operator norm inequalities indexed as (2), (3), (4) within the broad spectrum of any abstract unital C*-algebra.

We introduce the following notations needed for this section:

(i). $\mathscr{P}(\mathscr{A})=\left\{\varphi\in\mathscr{A}':\varphi(I)=1=\|\varphi\|\right\}$, the set of all states on $\mathscr{A}$, (ii). $V(A) = \{\varphi(A) : \varphi \in \mathscr{P}(\mathscr{A})\}$, the algebraic numerical range of an element $A \in \mathscr{A}$,

(iii) $|A|$, the positive square root of the positive element A^*A of $\mathscr{A}$ (it is called the modulus of A), where $A \in \mathscr{A}$.

We present the model cited before in the following main result:

PROPOSITION 2. *Consider f belonging to C_+ (resp. C_+^*), and let P and Q represent two elements of $\mathscr{A}$ that are positive (resp. invertible positive). Under these conditions, the following inequality holds:*

$$\forall X \in \mathscr{A}, \quad \left\| f(P)X\widetilde{f}(Q) + \widetilde{f}(P)Xf(Q) \right\| \geqslant 2\left\| P^{\frac{1}{2}}XQ^{\frac{1}{2}} \right\|. \tag{MI5}$$

Proof. Case 1. Suppose $f \in C_+$, with P and Q being positive elements.

Let $X \in \mathscr{A}$ and let $\varphi \in \mathscr{P}(\mathscr{A})$. The validity of inequality (*MI*5) is immediate in the case where $P^{\frac{1}{2}}XQ^{\frac{1}{2}} = 0$.

Assuming the contrary, that $P^{\frac{1}{2}}XQ^{\frac{1}{2}} \neq 0$ and put $A = f(P)X\widetilde{f}(Q)$ and $B = \widetilde{f}(P)Xf(Q)$.

Since $(A-B)^*(A-B) \geqslant 0$, and $(A-B)^*(A-B) = (A^*A + B^*B) - 2\mathrm{Re}(A^*B)$, so it follows immediately that

$$\varphi(A^*A + B^*B) \geqslant 2\mathrm{Re}\,\varphi(A^*B). \tag{A}$$

So, we obtain:

$$\begin{aligned}
\left\| f(P)X\widetilde{f}(Q) + \widetilde{f}(P)Xf(Q) \right\|^2 &= \|A+B\|^2, \\
&= \|A^*A + B^*B + 2\mathrm{Re}(A^*B)\|, \\
&\geqslant \varphi(A^*A + B^*B) + 2\mathrm{Re}\,\varphi(A^*B), \\
&\geqslant 4\mathrm{Re}\,\varphi(A^*B), \quad \text{using (A).} \\
&= 4\mathrm{Re}\,\varphi\left(\widetilde{f}(Q)X^*f(P)\widetilde{f}(P)Xf(Q)\right), \\
&= 4\mathrm{Re}\,\varphi\left(\widetilde{f}(Q)X^*PXf(Q)\right), \text{ since } f(P)\widetilde{f}(P) = P.
\end{aligned} \tag{B}$$

On the other hand, we have

$$\begin{aligned}
\sigma\left(\widetilde{f}(Q)X^*PXf(Q)\right) - \{0\} &= \sigma\left(X^*PXf(Q)\widetilde{f}(Q)\right) - \{0\}, \\
&= \sigma(X^*PXQ) - \{0\}, \quad \text{since } f(Q)\widetilde{f}(Q) = Q, \\
&= \sigma\left(Q^{\frac{1}{2}}X^*PXQ^{\frac{1}{2}}\right) - \{0\}.
\end{aligned}$$

Since $Q^{\frac{1}{2}}X^*PXQ^{\frac{1}{2}} = \left(P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right)^*\left(P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right)$ is a nonzero positive element of $\mathscr{A}$, then its norm $\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|^2 \in \sigma\left(Q^{\frac{1}{2}}X^*PXQ^{\frac{1}{2}}\right) - \{0\}$. So, $\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|^2 \in \sigma\left(\widetilde{f}(Q)X^*PXf(Q)\right) - \{0\}$. Since $\sigma\left(\widetilde{f}(Q)X^*PXf(Q)\right) \subset V\left(\widetilde{f}(Q)X^*PXf(Q)\right)$, thus, $\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|^2 \in V\left(\widetilde{f}(Q)X^*PXf(Q)\right)$. Then, we may choose the state φ such that $\varphi\left(\widetilde{f}(Q)X^*PXf(Q)\right) = \left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|^2$. Using the above inequality (B), the model ($MI5$) follows immediately.

Case 2. When $f \in C_+^*$, and P, Q are invertible positive, the argument proceeds along similar lines than the case 1. □

COROLLARY 1. *Let $\lambda \in [0,1]$ (resp. $\lambda \in \mathbb{R}$), and let P and Q be two positive (resp. invertible positive) elements in $\mathscr{A}$. Then, the following inequality holds:*

$$\forall X \in \mathscr{A}, \quad \left\|P^{\lambda}XQ^{1-\lambda} + P^{1-\lambda}XQ^{\lambda}\right\| \geqslant 2\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|. \tag{9}$$

Proof. Applying the model ($MI5$) with the map $f = H_{\lambda}$, where $\lambda \in [0,1]$ (resp. $\lambda \in \mathbb{R}$), the inequality (9) follows immediately. □

REMARK 1. In this context of an abstract unital C*-algebra, the above corollary shows us that:

1. the second part of the Heinz inequality (1) remains true,
2. in the case where P, Q are invertible positive, the second part of the Heinz inequality (1) remains true, but for every $\lambda \in \mathbb{R}$.

PROPOSITION 3. *Each of the three following inequalities holds and follows from each other:*

(i). For every $\lambda \in [0,1]$ and for every two positive elements P, Q in $\mathscr{A}$, the following inequality holds:

$$\forall X \in \mathscr{A}, \quad \|PX + XQ\| \geqslant \left\|P^{\lambda}XQ^{1-\lambda} + P^{1-\lambda}XQ^{\lambda}\right\|. \tag{10}$$

(ii). For every two positive elements P, $Q \in \mathscr{A}$, the following inequality holds:

$$\forall X \in \mathscr{A}, \quad \|PX + XQ\| \geqslant 2\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|. \tag{11}$$

(iii). For every two elements A, $B \in \mathscr{A}$, the following inequality holds:

$$\forall X \in \mathscr{A}, \quad \|A^*AX + XBB^*\| \geqslant 2\|AXB\|. \tag{12}$$

Proof. $(i) \Rightarrow (ii)$. Setting $\lambda = \frac{1}{2}$ in inequality (10) directly yields (11).

$(ii) \Rightarrow (iii)$. Given (ii), to establish (12), let $A,\ B,\ X \in \mathscr{A}$ and denote $R = |A|$ and $S = |B^*|$. It is trivial to verify that $\||A|X|B^*|\| = \|AXB\|$, $R^2 \geqslant 0$, $R = \left(R^2\right)^{\frac{1}{2}}$, $S^2 \geqslant 0$, $S = \left(S^2\right)^{\frac{1}{2}}$. From these and applying inequality (11), we obtain:

$$\begin{aligned}\|A^*AX + XBB^*\| &= \left\|R^2X + XS^2\right\|,\\ &\geqslant 2\|RXS\|,\\ &= 2\||A|X|B^*|\|,\\ &= 2\|AXB\|.\end{aligned}$$

The implication $(iii) \Rightarrow (ii)$ is easy to prove.

$(ii) \Rightarrow (i)$. Assuming (ii), for any $X \in \mathscr{A}$ and positive elements $P,\ Q \in \mathscr{A}$, consider the continuous function:

$$\varphi : [0,1] \to \mathbb{R},\ \ \lambda \mapsto \left\|P^{\lambda}XQ^{1-\lambda} + P^{1-\lambda}XQ^{\lambda}\right\|.$$

From the inequality (11) and by using the same argument as used by McIntosh [9] for proving the implication $(ii) \Rightarrow (i)$ in the case of the unital C*-algebra $\mathfrak{B}(H)$ (it remains true in our general situation), the following condition holds:

$$\forall \alpha, \beta \in [0,1],\ \ \varphi\left(\frac{\alpha+\beta}{2}\right) \leqslant \frac{1}{2}\left(\varphi(\alpha) + \varphi(\beta)\right). \tag{C}$$

Put $\Gamma = \{\lambda \in [0,1] : \varphi(\lambda) \leqslant \|PX + XQ\|\}$, $\Delta_n = \left\{\frac{k}{2^n} : k = 0,1,\ldots,n\right\}$ $(n \geqslant 1)$, and $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$.

With $0, 1 \in \Gamma$ and the inequality (C), employing an iterative technique, it is deduced that, $\Delta \subset \Gamma$. In addition, since Δ is dense in $[0,1]$ and Γ is closed, then $\Gamma = [0,1]$.

This proves (i).

On the other hand, the inequality (11) holds by taking $\lambda = 1/2$ in the inequality (9) given in the above corollary. This proves the Proposition. □

REMARK 2. (1). Let $\lambda \in [0,1]$ and let two positive elements $P,\ Q$ in $\mathscr{A}$. Using the last corollary and the last proposition, the following double inequality holds:

$$\forall X \in \mathscr{A},\ \ \|PX + XQ\| \geqslant \left\|P^{\lambda}XQ^{1-\lambda} + P^{1-\lambda}XQ^{\lambda}\right\| \geqslant 2\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|. \tag{13}$$

The double inequality (13) (resp. the two inequalities (11) and (12)) asserts that the Heinz inequality (1) (resp. the two arithmetic-geometric mean inequalities (2) and (3)) remains (resp. remain) true in the general setting of an abstract unital C*-algebras.

(2). Note that the complete proof of the inequality (C) given by McIntosh is not in any journal, but it is presented in [4, Theorem 1].

(3). The model (*MI5*) yields infinitely of inequalities and one of them is the second part of the Heinz inequality (13) that contains the arithmetic-geometric mean inequality (11).

COROLLARY 2. *Let $R,\ S \in \mathscr{A}$.*

1. *Assume that $R,\ S$ are selfadjoint. The following inequality holds:*

$$\forall X \in \mathscr{A}, \quad \left\|R^2X + XS^2\right\| \geqslant 2\left\|RXS\right\|. \tag{14}$$

2. *Assume that S is invertible selfadjoint. The following inequality holds:*

$$\forall X \in \mathscr{A}, \quad \left\|SXS^{-1} + S^{-1}XS\right\| \geqslant 2\left\|X\right\|. \tag{15}$$

3. *Assume that $R,\ S$ are normal. The following inequality holds:*

$$\forall X \in \mathscr{A}, \quad \left\|R^2X\right\| + \left\|XS^2\right\| \geqslant 2\left\|RXS\right\|. \tag{16}$$

4. *Assume that S is invertible normal. The following inequality holds:*

$$\forall X \in \mathscr{A}, \quad \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geqslant 2\left\|X\right\|. \tag{17}$$

Proof. Consider an element $X \in \mathscr{A}$.

1. Given that R and S are self-adjoint, it follows that $\left(R^2\right)^{\frac{1}{2}} = |R|$ and $\left(S^2\right)^{\frac{1}{2}} = |S|$. Using (11) with $P = S^2$ and $Q = S^2$, and since $\|RXS\| = \||R|X|S|\|$, the inequality (14) follows immediately.
2. The inequality (15) follows immediately from the inequality (14).
3. In light of the normality of R and S, we deduce $\left\|R^2X\right\| = \|R^*RX\|$ and $\left\|XS^2\right\| = \|XSS^*\|$. Thus:

$$\begin{aligned}\left\|R^2X\right\| + \left\|XS^2\right\| &= \|R^*RX\| + \|XSS^*\|,\\ &\geqslant \|R^*RX + XSS^*\|,\\ &\geqslant 2\|RXS\|, \text{ using (12).}\end{aligned}$$

   This validates inequality (16).
4. Inequality (17) is immediately inferred from inequality (16). □

REMARK 3. Inequality (15) asserts that the Corach-Porta-Recht inequality (4) persists to hold within the framework of any abstract unital C*-algebra.

In the next proposition, using the model ($MI5$), we shall present an example of inequality that is not related to the Heinz inequality in this context of unital C^*-algebra.

PROPOSITION 4. *Let $a>0$, and let P and Q be two positive elements in $\mathscr{A}$. The following inequality holds:*

$$\forall X \in \mathscr{A}, \quad \left\|a^P X a^{-Q} Q + P a^{-P} X a^Q\right\| \geqslant 2 \left\|P^{\frac{1}{2}} X Q^{\frac{1}{2}}\right\|. \tag{18}$$

Proof. Define the map f on $[0,\infty)$ by $f(t)=a^t$, for $t \geqslant 0$. Clearly, f belongs to the class C_+. $f \in C_+$, and its conjugate is the map g defined on $[0,\infty)$ by $g(t)=ta^{-t}$, for $t \geqslant 0$. The inequality (18) is obtained promptly by applying the model ($MI5$) with the map f. $\square$

In operator theory:

(i). we have shown [11] that the class of all invertible operator $S \in \mathfrak{B}(H)$ satisfying the inequality of Corach-Porta-Recht (4) is the class of all operator of the form $S=\lambda M$, where λ is a non-zero scalar and M is an invertible selfadjoint operator of $\mathfrak{B}(H)$ (note that the inequality (15) is exactly the inequality of Corach-Porta-Recht (4) with $\mathscr{A}=\mathfrak{B}(H)$),

(ii). we have shown [12] that the class of all invertible operator $S \in \mathfrak{B}(H)$ satisfying the inequality (17) with the case $\mathscr{A}=\mathfrak{B}(H)$ is the class of all invertible normal operators in $\mathfrak{B}(H)$.

So, we may ask the two following problems:

PROBLEM 1. Is it true that the class of all invertible element $S \in \mathscr{A}$ satisfying the inequality (15) is the class of all element of the form $S=\lambda M$, where $\lambda \in \mathbb{C}^*$ and M is an invertible selfadjoint element of $\mathscr{A}$?

PROBLEM 2. Is it true that the class of all invertible element $S \in \mathscr{A}$ satisfying the inequality (17) is the class of all invertible normal elements in $\mathscr{A}$?

Acknowledgements. The author would like to thank the anonymous reviewers for their valuable comments and suggestions that improved the quality of this paper.

REFERENCES

[1] R. BHATIA AND C. DAVIS, *More matrix forms of the arithmetic-geometric mean inequality*, SIAM J. Matrix Anal. Appl. **14** (1993), 132–136.

[2] C. BOURAYA AND A. SEDDIK, *On the characterizations of some distinguished subclasses of Hilbert space operators*, Acta Sci. Math. (Szeged), **84** (2018), 611–627.

[3] G. CORACH, R. PORTA, AND L. RECHT, *An operator inequality*, Linear Algebra Appl., **142** (1990), 153–158.

[4] J. FUJII, M. FUJII, T. FURUTA, R. NAKAMOTO, *Norm inequalities equivalent to Heinz inequality*, Proc. Amer. Math. Soc. **118** (3) (1993), 827–830.

[5] E. HEINZ, *Beiträge zur Störungstheorie der Spectralzerlegung*, Math. Ann. **123** (1951), 415–438.
[6] R. A. HORN, *Norm bounds for Hadamard product and the arithmetic-geometric mean inequality for unitarily invariant norm*, Linear Algebra Appl. **223/224** (1995), 355–361.
[7] F. KITTANEH, *A note on the arithmetic-geometric mean inequality for matrices*, Linear Algebra Appl. **171** (1992), 1–8.
[8] R. MATHIAS, *A An arithmetic-geometric-harmonic mean inequality involving Hadamard product*, Linear Algebra Appl. **184** (1993), 71–78.
[9] A. MCINTOSH, *Heinz Inequalities and Perturbation of Spectral Families*, Macquarie Mathematical Reports, Macquarie Univ., 1979.
[10] A. SEDDIK, *Operator inequalities related to the arithmetic–geometric mean inequality and characterizations*, Adv. Oper. Theory **8**, 8 (2023), https://doi.org/10.1007/s43036.
[11] A. SEDDIK, *Some results related to Corach-Porta-Recht inequality*, Proc. Amer. Math. Soc. **129** (2001), 3009–3015.
[12] A. SEDDIK, *On the injective norm and characterization of some subclasses of normal operators by inequalities or equalities*, J. Math. Anal. Appl. **351** (2009), 277–284.

(Received September 16, 2023)

Ameur Seddik
Department of Mathematics
Faculty of Mathematics and Computer Science
University of Batna 2, Batna, Algeria
e-mail: a.seddik@univ-batna2.dz