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LIOUVILLE TYPE THEOREMS FOR FRACTIONAL ELLIPTIC SYSTEMS WITH COUPLED TERMS

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Abstract. In this paper, we study the fractional elliptic system with coupled terms

$$\begin{cases}(-\Delta)^s u = (q+1)u^q v^{p+1} & \text{in } \mathbb{R}^N\\ (-\Delta)^s v = (p+1)v^p u^{q+1} & \text{in } \mathbb{R}^N,\end{cases}$$

where $0 < s < 1$ and $N > 2s$. We first prove that if $p > -1$, $q > -1$ and $p+q+1 \leqslant \frac{N}{N-2s}$, then the system has no positive supersolution. In the case $p,q > 0$ we establish the nonexistence result of stable positive solutions. Our results generalize some results in *[Li, Yayun; Lei, Yutian; Commun. Pure Appl. Anal. 17 (2018), no. 5, 1749–1764.]* to the system involving the fractional Laplacian.

1. Introduction

In the last decades, the fractional Laplacian has been widely used to model various physical phenomena, such as the turbulence, water waves, anomalous diffusion, phase transitions, flame propagation and quasi-geostrophic flows, see [3, 5, 6, 26] and references given there. Further applications of the fractional Laplacian in probability, optimization and finance can be found in [1, 22]. In particular, the fractional Laplacian can be seen as the infinitesimal generator of a stable Lévy process [22].

The fractional Laplacian $(-\Delta)^s$, $0 < s < 1$, is defined on the space of rapidly decreasing functions as a nonlocal pseudo-differential operator

$$(-\Delta)^s u(x) = c_{N,s} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{u(x)-u(\xi)}{|x-\xi|^{N+2s}} d\xi,$$

where $c_{N,s}$ is the normalization constant and

$$B(x,\varepsilon) = \{\xi \in \mathbb{R}^N ; |\xi - x| \leqslant \varepsilon\}.$$

Equivalently, the fractional Laplacian is also defined by the Fourier transform

$$\mathscr{F}\left((-\Delta)^s u\right)(\xi) = |\xi|^s \mathscr{F}(u)(\xi),$$

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where $\mathscr{F}(u)$ is the Fourier transform of u. In the distribution sense, the fractional Laplacian can be defined on the space

$$\mathscr{L}_s(\mathbb{R}^N) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^N);\ \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}} dx < \infty \right\}$$

by

$$\langle (-\Delta)^s u, \varphi \rangle = \langle u, (-\Delta)^s \varphi \rangle .$$

In this paper, we are concerned with the fractional elliptic system with coupled terms

$$\begin{cases} (-\Delta)^s u = (q+1) u^q v^{p+1} \text{ in } \mathbb{R}^N \\ (-\Delta)^s v = (p+1) v^p u^{q+1} \text{ in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $0 < s < 1$, $p > -1$ and $q > -1$.

The systems of type (1) appear as limit problems in many phenomena, such as Bose-Einstein condensates, chemical reaction, population evolution. Some of these systems are known as the fractional Gross-Pitaevskii system, the fractional Lotka-Volterra system or the fractional Schrödinger system [27–30, 32]. When $p, q > -1$, under some simple scaling argument $u_1 = c_1 u, v_1 = c_2 v$, the system (1) is equivalent to

$$\begin{cases} (-\Delta)^s u_1 = u_1^q v_1^{p+1} \text{ in } \mathbb{R}^N \\ (-\Delta)^s v_1 = v_1^p u_1^{q+1} \text{ in } \mathbb{R}^N. \end{cases} \tag{2}$$

In [30], by developing the technique in [24], the authors studied the symmetry of components of the system containing (2) as a special case. In particular, it was shown that the components u_1, v_1 satisfy $u_1 \leqslant v_1$ or $v_1 \leqslant u_1$ with $p, q > 1$. In addition, if $1 < q \leqslant \frac{N}{N-2s}$ or $1 < p \leqslant \frac{N}{N-2s}$, then the result in [30] implies the nonexistence of positive solutions of the system (2).

In [32], the symmetry and nonexistence result of positive solutions to the fractional Schrödinger system

$$\begin{cases} (-\Delta)^s u_1 = u_1^{\beta_1} v_1^{\tau_1} \text{ in } \mathbb{R}^N \\ (-\Delta)^t v_1 = v_1^{\beta_2} u_1^{\tau_2} \text{ in } \mathbb{R}^N. \end{cases} \tag{3}$$

with $0 < s, t < 1$ and $\tau_i, \beta_i > 0$ was investigated. By using the method of moving planes, the authors proved the nonexistence of positive solutions to the system in the subcritical case and established the symmetry of solutions in the critical case. On the other hand, in [2], the author established some Liouville type theorems for supersolutions in exterior domains of the system (3) by using a probability approach.

In the recent paper [23], the author classified positive solutions to the system with coupled terms

$$\begin{cases} -\Delta u = (q+1) u^q v^{p+1} \\ -\Delta v = (p+1) v^p u^{q+1}, \end{cases} \tag{4}$$

where $p, q > 0$ and $\max(p, q) > 1$. It was shown in [23] that the system (4) has no positive solution when $1 < p+q+1 \leqslant \frac{N}{N-2}$. This type of result for the system (1) was also proved in [2] when $p,q>0$. However, the case $p\leqslant 0$ or $q\leqslant 0$ has not been treated in these papers. Therefore, our first purpose in this paper is to prove a complement result on the nonexistence of positive supersolutions (u,v) of the system (1), i.e.

$$\begin{cases}(-\Delta)^s u \geqslant (q+1)u^q v^{p+1} \text{ in } \mathbb{R}^N\\ (-\Delta)^s v \geqslant (p+1)v^p u^{q+1} \text{ in } \mathbb{R}^N,\end{cases}$$

in the case $p,q>-1$ and $p+q+1\leqslant \frac{N}{N-2s}$. Here and in what follows, u and v are considered in $C^{2\sigma}(\mathbb{R}^N)\cap\mathscr{L}_s(\mathbb{R}^N)$ with $\sigma>s$. More precisely, we prove the following theorem.

THEOREM 1. *Let $p>-1$ and $q>-1$. The system (1) has no positive supersolution when $p+q+1\leqslant \frac{N}{N-2s}$.*

To prove Theorem 1 we first use the equivalent between (1) and (2) and then shall exploit a type of reduction introduced in [24], see also [11,30]. This type of reduction allows one to reduce the system to an inequality.

We now consider the case $p,q>0$. We shall study the nonexistence of positive stable solutions to the system (1). Note that the energy functional of the system (1) is given by

$$\begin{aligned}E(u,v)=&\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}dxdy\\ &+\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v(x)-v(y))^2}{|x-y|^{N+2s}}dxdy-\int_{\mathbb{R}^N}u^{q+1}v^{p+1}dx.\end{aligned}$$

In this paper, we follow the definition of stability in [23].

DEFINITION 1. A positive solution (u,v) of (1) is called u-stable (resp. v-stable) if for all $\phi\in C_c^1(\mathbb{R}^N)$,

$$q(q+1)\int_{\mathbb{R}^N}u^{q-1}v^{p+1}\phi^2dx\leqslant\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\phi(x)-\phi(y))^2}{|x-y|^{N+2s}}dxdy \tag{5}$$

$$\left(\text{resp. } p(p+1)\int_{\mathbb{R}^N}u^{q+1}v^{p-1}\phi^2dx\leqslant\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\phi(x)-\phi(y))^2}{|x-y|^{N+2s}}dxdy.\right).$$

As mentioned in [23], this definition is a little "partial". Stability often comes from the fact that the second variation of energy functional is nonnegative. In fact, the energy functional is a two-variables function and then the nonnegative definite property of the hessian matrix is too strong to calculate easily "total" derivatives. The condition (5) is just like to calculate the "partial" derivative of $E(u,v)$.

In recent years, the existence and nonexistence of stable solutions to elliptic equations or systems has been attracted much attention of researchers. We next review some related results on this topic in literature. The stable solutions in the whole space $\mathbb{R}^N$ to the equation

$$-\Delta u=f(u) \text{ in } \mathbb{R}^N$$

was classified in [9, 14–16]. The critical exponents in some concrete cases of nonlinearity were explicitly computed and the sharpness was also proved.

Concerning the Lane-Emden system with or without weights

$$\begin{cases} -\Delta u = w_1(x)v^p \text{ in } \mathbb{R}^N \\ -\Delta v = w_2(x)u^q \text{ in } \mathbb{R}^N, \end{cases}$$

the nonexistence of stable solutions was studied in [7, 20, 21]. However, the sharpness of the nonexistence results for the system is left open.

Very recently, the classification of stable solutions to elliptic problems involving the fractional Laplacian has been much studied. In [8], the authors classified stable solutions to the fractional Lane-Emden equation by using a combination of the monotonicity formula and some nonlinear integral estimates. This idea was then used in [17–19, 25, 31] to deal with higher order fractional elliptic equations with weights. Concerning the fractional Lane-Emden system, the authors in [12] have given a sufficient condition for the nonexistence of stable solutions by exploiting the technique in [13].

The second purpose of this paper is to generalize a nonexistence result of stable solutions in [23] to the fractional setting. Before presenting our second result, let us recall an extension result due to Caffarelli and Silvestre [4].

THEOREM A. *Let $0 < s < \sigma < 1$ and $u \in C^{2\sigma}(\mathbb{R}^N) \cap \mathscr{L}_s(\mathbb{R}^N)$. For $(x,t) \in \mathbb{R}^{N+1}_+$, we define*

$$U(x,t) = \int_{\mathbb{R}^N} P_s(x-z,t)u(z)dz,$$

where $P_s(x,t)$ is the Poisson kernel

$$P_s(x,t) = p(N,s)\frac{t^{2s}}{(|x|^2+t^2)^{\frac{N+2s}{2}}}$$

and $p(N,s)$ is the normalization constant. Then $U \in C^2(\mathbb{R}^{N+1}_+) \cap C(\overline{\mathbb{R}^{N+1}_+})$, $t^{1-2s}\partial_t U \in C(\overline{\mathbb{R}^{N+1}_+})$ and

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ U = u & \text{on } \partial\mathbb{R}^{N+1}_+ \\ -\lim\limits_{t\to 0} t^{1-2s}\partial_t U = \kappa_s(-\Delta)^s u & \text{on } \partial\mathbb{R}^{N+1}_+ \end{cases}. \tag{6}$$

Here $\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}$ and Γ is the usual Gamma function.

Given $W \in C(\overline{\mathbb{R}^{N+1}_+})$, define

$$\overline{W}(r) = \frac{1}{r^{N+1-2s}}\int_{\partial^+ B_r^+} t^{1-2s}W,$$

where $\partial^+ B_r^+ = \partial B_r \cap \{t > 0\}$ and B_r is the ball in $\mathbb{R}^{N+1}$ centered at $(0,0)$ with radius r. Let (u,v) be a positive solution of (1) and denote by U (resp. V) the extension of u (resp. v) in the sense of Theorem A. According to [30], $\overline{U}(r)$ and $\overline{V}(r)$ are non-increasing and there holds

$$a := \lim_{r \to \infty} \overline{U}(r) = 0$$

or

$$b := \lim_{r \to \infty} \overline{V}(r) = 0.$$

This assertion is true because if $\lim_{r\to\infty} \overline{U}(r) > 0$ and $\lim_{r\to\infty} \overline{V}(r) > 0$, then there is $\varepsilon > 0$ such that $u(x) > \varepsilon$ and $v(x) > \varepsilon$ which implies $(-\Delta)^s u \geqslant C > 0$. This is a contradiction since $(-\Delta)^s u \geqslant C > 0$ has no positive solution [11].

The second result in this paper reads as follows.

THEOREM 2. *Assume that* $p, q > 0$, $0 < s < 1$ *and* $N > 2s$.

1. *If* $a = 0$ *and* $q > 1$ *then* (1) *has no u-stable solution provided that*

$$N < 2s + \frac{4s}{p+q}\left(q + \sqrt{q^2 - q}\right).$$

2. *If* $b = 0$ *and* $p > 1$ *then* (1) *has no v-stable solution provided that*

$$N < 2s + \frac{4s}{p+q}\left(p + \sqrt{p^2 - p}\right).$$

Remark that Theorem 2 with $s = 1$ was established in [23] by using the technique developed by Farina [15]. Nevertheless, in the nonlocal situation $0 < s < 1$, it requires another approach to deal with the problem since $(-\Delta)^s \phi$ has no compact support in general for $\phi \in C_c^\infty(\mathbb{R}^N)$. In this paper, we exploit the technique developed by the first author and V. H. Nguyen in [13] which consists of three main steps:

- Prove an integrability of the right hand side of the system.
- Establish uniform upper bound of some nonlinear integrals.
- Use a scaling argument to arrive at the nonexistence result.

The rest of this paper is devoted to the proof of our main results.

2. Proof of Theorems 1 and 2

In which follows, we denote by C a positive constant which may change from line to line and independent of solutions of (1).

2.1. Proof of Theorem 1

The proof is based on the reductions used in [11, 24, 30]. Since (1) and (2) are equivalent, it is enough to prove that the system (2) has no positive supersolution. Suppose on the contrary that the system (2) has a positive supersolution (u_1, v_1).

Set $w = u_1^a v_1^b$ where $a,b > 0$ and $a+b = 1$. Using [30, Lemma 2.4], we have

$$\begin{aligned}(-\Delta)^s w &\geqslant a u_1^{a-1} v_1^b (-\Delta)^s u_1 + b v_1^{b-1} u_1^a (-\Delta)^s v_1 \\ &\geqslant a u_1^{a+q-1} v_1^{b+p+1} + b v_1^{b+p-1} u_1^{a+q+1} \\ &= u_1^a v_1^b \left(a u_1^{q-1} v_1^{p+1} + b v_1^{p-1} u_1^{q+1} \right). \end{aligned} \tag{7}$$

Next, for any $m > 1$, the Young inequality implies

$$a u_1^{q-1} v_1^{p+1} + b v_1^{p-1} u_1^{q+1} \geqslant C \left(u_1^{q-1} v_1^{p+1} \right)^{\frac{1}{m}} \left(v_1^{p-1} u_1^{q+1} \right)^{\frac{m-1}{m}}. \tag{8}$$

Then, it results from (7) and (8) that

$$(-\Delta)^s w \geqslant C u_1^{a + \frac{q-1+(q+1)(m-1)}{m}} v_1^{b + \frac{p+1+(p-1)(m-1)}{m}} = C u_1^{a+q+1-\frac{2}{m}} v_1^{b+p-1+\frac{2}{m}}. \tag{9}$$

We now choose

$$m = \frac{2}{qb - pa + 1}$$

which verifies

$$q + 1 - \frac{2}{m} = a(p+q) \quad \text{and} \quad p - 1 + \frac{2}{m} = b(p+q).$$

It is necessary to show that for any $p,q > -1$, one can choose $a,b > 0$, $a+b = 1$ such that $m > 1$. Indeed,

1. If p,q are two positive number (or two negative numbers), then we can choose a,b such that $\frac{a}{b} = \frac{q}{p}$. Hence, $m = 2$.

2. If $-1 < q \leqslant 0 \leqslant p$ (resp. $-1 < p \leqslant 0 \leqslant q$), we choose a sufficiently small (resp. b sufficiently small), then we also have $m > 1$.

Therefore, (9) becomes

$$(-\Delta)^s w \geqslant C w^{1+p+q}. \tag{10}$$

According to [11, Corollary 1.2], this inequality has no positive solution. This is a contradiction. The proof is complete. □

2.2. Proof of Theorem 2

In this subsection, we prove the first assertion in Theorem 2 since the second one is proved by the same argument.

Suppose that (u,v), $u > 0, v > 0$, is a u-stable solution of (1). Assume that $q > 1$ and

$$a = \lim_{r \to \infty} \overline{U}(r) = 0.$$

This implies from [30, Lemma 3.1] that

$$u \leqslant \left(\frac{q+1}{p+1}\right)^{1/2} v. \tag{11}$$

In what follows, we use the function

$$\rho_{N+2s}(x) = (1+|x|^2)^{-\frac{N+2s}{2}}.$$

In the first step, we shall prove some integrability without the stability condition.

Step 1. Let (u,v) be a positive solution of (1). Then we have

$$\int_{\mathbb{R}^N} u^q(x)v^{p+1}(x)\rho_{N+2s}(x)dx < C\int_{\mathbb{R}^N} u(x)\rho_{N+2s}(x)dx < +\infty, \tag{12}$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R})$ be a test function, $0 \leqslant \phi \leqslant 1$ and

$$\phi(t) = \begin{cases} 1 & \text{if } |t| \leqslant 1 \\ 0 & \text{if } |t| > 2 \end{cases}.$$

Multiplying (1) by $\rho_{N+2s}\phi_R \in C_c^\infty(\mathbb{R}^N)$ and using an integration by parts, we have

$$(q+1)\int_{\mathbb{R}^N} u^q(x)v^{p+1}(x)\rho_{N+2s}(x)\phi_R(x)dx = \int_{\mathbb{R}^N} u(x)(-\Delta)^s(\rho_{N+2s}\phi_R)(x)dx, \tag{13}$$

here $\phi_R(x) = \phi\left(\frac{|x|}{R}\right)$. Notice that, by [13, Lemma 2.2],

$$(-\Delta)^s(\rho_{N+2s}\phi_R)(x) \longrightarrow (-\Delta)^s(\rho_{N+2s})(x) \text{ as } R \longrightarrow +\infty.$$

From this and the monotone convergence theorem, we obtain from (13) that

$$(q+1)\int_{\mathbb{R}^N} u^q(x)v^{p+1}(x)\rho_{N+2s}(x)dx = \int_{\mathbb{R}^N} u(x)(-\Delta)^s\rho_{N+2s}(x)dx.$$

Combining this and the fact that, see [13, Lemma 2.1],

$$|(-\Delta)^s\rho_{N+2s}(x)| \leqslant C\rho_{N+2s}(x),$$

we deduce (12). □

We next prove some nonlinear integral estimates.

Step 2. For $1 \leqslant \alpha < q+\sqrt{q^2-q}$, it holds

$$\int_{\mathbb{R}^N} u^{2\alpha+q-1}v^{p+1}\rho_{N+2s}\phi_R^2 dx \leqslant C\int_{\mathbb{R}_+^{N+1}} U^{2\alpha}|\nabla(\zeta\Phi_R)|^2 t^{1-2s}dxdt, \tag{14}$$

where

$$\Phi_R(x,t) = \phi\left(\frac{|(x,t)|}{R}\right)$$

and

$$\zeta(x,t) = (1+|x|^2+t^2)^{-\frac{N+2s}{4}} \text{ for } x \in \mathbb{R}^N \text{ and } t \geqslant 0.$$

Proof. Recall that U is the extension of u in the sense of Theorem A. It follows from (6) with test function $U^{2\alpha-1}\zeta^2\Phi_R^2$ that

$$\begin{aligned}
\kappa_s(q+1)\int_{\mathbb{R}^N} u^{2\alpha+q-1}v^{p+1}\rho_{N+2s}\phi_R^2 dx &= \int_{\mathbb{R}_+^{N+1}} \nabla U \cdot \nabla(U^{2\alpha-1}\zeta^2\Phi_R^2)t^{1-2s}dxdt \\
&= (2\alpha-1)\int_{\mathbb{R}_+^{N+1}} |\nabla U|^2 U^{2\alpha-2}\zeta^2\Phi_R^2 t^{1-2s}dxdt \\
&\quad + 2\int_{\mathbb{R}_+^{N+1}} \nabla U\cdot\nabla(\zeta\Phi_R)U^{2\alpha-1}\zeta\Phi_R t^{1-2s}dxdt \\
&= \frac{2\alpha-1}{\alpha^2}\int_{\mathbb{R}_+^{N+1}} |\nabla(U^\alpha)|^2(\zeta\Phi_R)^2 t^{1-2s}dxdt \\
&\quad + \frac{2}{\alpha}\int_{\mathbb{R}_+^{N+1}} \nabla(U^\alpha)\cdot\nabla(\zeta\Phi_R)\zeta\Phi_R U^\alpha t^{1-2s}dxdt.
\end{aligned} \tag{15}$$

The first integral in the right hand side of (15) is computed as

$$\begin{aligned}
\int_{\mathbb{R}_+^{N+1}} |\nabla(U^\alpha)|^2(\zeta\Phi_R)^2 t^{1-2s}dxdt &= \int_{\mathbb{R}_+^{N+1}} |\nabla(U^\alpha\zeta\Phi_R)|^2 t^{1-2s}dxdt \\
&\quad -2\int_{\mathbb{R}_+^{N+1}} \nabla(U^\alpha)\cdot\nabla(\zeta\Phi_R)\zeta\Phi_R U^\alpha t^{1-2s}dxdt \\
&\quad -\int_{\mathbb{R}_+^{N+1}} U^{2\alpha}|\nabla(\zeta\Phi_R)|^2 t^{1-2s}dxdt.
\end{aligned} \tag{16}$$

Plugging (16) into (15), we obtain

$$\begin{aligned}
\kappa_s(q+1)\int_{\mathbb{R}^N} u^{2\alpha+q-1}v^{p+1}\rho_{N+2s}\phi_R^2 dx &= \frac{2\alpha-1}{\alpha^2}\int_{\mathbb{R}_+^{N+1}} |\nabla(U^\alpha\zeta\Phi_R)|^2 t^{1-2s}dxdt \\
&\quad -\frac{2\alpha-2}{\alpha^2}\int_{\mathbb{R}_+^{N+1}} \nabla(U^\alpha)\cdot\nabla(\zeta\Phi_R)\zeta\Phi_R U^\alpha t^{1-2s}dxdt \\
&\quad -\frac{2\alpha-1}{\alpha^2}\int_{\mathbb{R}_+^{N+1}} U^{2\alpha}|\nabla(\zeta\Phi_R)|^2 t^{1-2s}dxdt \\
&= \frac{2\alpha-1}{\alpha^2}\int_{\mathbb{R}_+^{N+1}} |\nabla(U^\alpha\zeta\Phi_R)|^2 t^{1-2s}dxdt \\
&\quad -\frac{2\alpha-2}{\alpha^2}\int_{\mathbb{R}_+^{N+1}} \nabla(U^\alpha\zeta\Phi_R)\cdot\nabla(\zeta\Phi_R)U^\alpha t^{1-2s}dxdt \\
&\quad -\frac{1}{\alpha^2}\int_{\mathbb{R}_+^{N+1}} U^{2\alpha}|\nabla(\zeta\Phi_R)|^2 t^{1-2s}dxdt.
\end{aligned} \tag{17}$$

The Young inequality implies for $\varepsilon > 0$ that

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} \nabla(U^\alpha \zeta \Phi_R)\cdot\nabla(\zeta\Phi_R) U^\alpha t^{1-2s} dxdt \\
&\leqslant \varepsilon \int_{\mathbb{R}^{N+1}_+} |\nabla(U^\alpha \zeta\Phi_R)|^2 t^{1-2s} dxdt \\
&\quad + \frac{1}{4\varepsilon}\int_{\mathbb{R}^{N+1}_+} U^{2\alpha}|\nabla(\zeta\Phi_R)|^2 t^{1-2s} dxdt.
\end{aligned} \tag{18}$$

As a consequence of (18) and (17), it holds

$$\begin{aligned}
&\kappa_s(q+1)\int_{\mathbb{R}^N} u^{2\alpha+q-1} v^{p+1}\rho_{N+2s}\phi_R^2 dx \\
&\geqslant \left(\frac{2\alpha-1}{\alpha^2} - \varepsilon\frac{2\alpha-2}{\alpha^2}\right)\int_{\mathbb{R}^{N+1}_+} |\nabla(U^\alpha\zeta\Phi_R)|^2 t^{1-2s} dxdt \\
&\quad - \left(\frac{1-\alpha}{2\alpha^2\varepsilon} + \frac{1}{\alpha^2}\right)\int_{\mathbb{R}^{N+1}_+} U^{2\alpha}|\nabla(\zeta\Phi_R)|^2 t^{1-2s} dxdt.
\end{aligned} \tag{19}$$

Next, it results from the stability inequality (5) with a test function $u^\alpha \rho_{N+2s}^{\frac{1}{2}}\phi_R$ that

$$\kappa_s q(q+1)\int_{\mathbb{R}^N} u^{2\alpha+q-1} v^{p+1}\rho_{N+2s}\phi_R^2 dx \leqslant \kappa_s \|u^\alpha \rho_{N+2s}^{\frac{1}{2}}\phi_R\|_{\dot{H}^s(\mathbb{R}^N)}. \tag{20}$$

Thanks to the fact that $U^\alpha\zeta\Phi_R$ has trace $u^\alpha\rho_{N+2s}^{\frac{1}{2}}\phi_R$ on $\partial\mathbb{R}^{N+1}_+$, it holds

$$\|u^\alpha\rho_{N+2s}^{\frac{1}{2}}\phi_R\|_{H^s(\mathbb{R}^N)} \leqslant \int_{\mathbb{R}^{N+1}_+} |\nabla(U^\alpha\zeta\Phi_R)|^2 t^{1-2s} dxdt. \tag{21}$$

We then deduce from (19) , (20) and (21) that

$$\begin{aligned}
&\left(\frac{2\alpha-1}{\alpha^2} + \varepsilon\frac{2\alpha-2}{\alpha^2} - \frac{1}{q}\right)\int_{\mathbb{R}^{N+1}_+} |\nabla(U^\alpha\zeta\Phi_R)|^2 t^{1-2s} dxdt \\
&\leqslant \left(\frac{1-\alpha}{2\alpha^2\varepsilon} + \frac{1}{\alpha^2}\right)\int_{\mathbb{R}^{N+1}_+} U^{2\alpha}|\nabla(\zeta\Phi_R)|^2 t^{1-2s} dxdt.
\end{aligned}$$

Recall that $1 \leqslant \alpha < q + \sqrt{q^2-q}$, then $\frac{2\alpha-1}{\alpha^2} - \frac{1}{q} > 0$. Therefore, there is ε small enough such that

$$\frac{2\alpha-1}{\alpha^2} + \varepsilon\frac{2\alpha-2}{\alpha^2} - \frac{1}{q} > 0.$$

This follows that

$$\int_{\mathbb{R}^{N+1}_+} |\nabla(U^\alpha\zeta\Phi_R)|^2 t^{1-2s} dxdt \leqslant C\int_{\mathbb{R}^{N+1}_+} U^{2\alpha}|\nabla(\zeta\Phi_R)|^2 t^{1-2s} dxdt. \tag{22}$$

Combining (20), (21) and (22), we obtain (14). $\square$

Step 3. For $1 \leqslant \alpha < q + \sqrt{q^2 - q}$, it holds

$$\int_{\mathbb{R}^N} u^{2\alpha+p+q}(x)\rho_{N+2s}(x)dx \leqslant C. \tag{23}$$

Proof. By using the same argument as in [10, Formula (2.14)], we also obtain the estimate

$$\int_{\mathbb{R}^{N+1}_+} U^{2\alpha}|\nabla(\zeta\Phi_R)|^2 t^{1-2s}dxdt \leqslant C\int_{\mathbb{R}^N} u^{2\alpha}(x)\rho_{N+2s}(x)dx. \tag{24}$$

Combining (14) and (24) we get

$$\int_{\mathbb{R}^N} u^{2\alpha+q-1}v^{p+1}\rho_{N+2s}\phi_R^2 dx \leqslant C\int_{\mathbb{R}^N} u^{2\alpha}\rho_{N+2s}dx.$$

Letting $R \to +\infty$ in this inequality, we arrive at

$$\int_{\mathbb{R}^N} u^{2\alpha+q-1}v^{p+1}\rho_{N+2s}dx \leqslant C\int_{\mathbb{R}^N} u^{2\alpha}\rho_{N+2s}dx.$$

This and the comparison between u and v (11) imply that

$$\int_{\mathbb{R}^N} u^{2\alpha+q+p}\rho_{N+2s}dx \leqslant C\int_{\mathbb{R}^N} u^{2\alpha}\rho_{N+2s}dx. \tag{25}$$

In particular, (12) implies that the right hand side of (25) is finite with $\alpha = \frac{p+q+1}{2}$. Then the left hand side of (25) is also finite. Repeating this argument and using the Hölder inequality, we obtain that the right hand side of (25) is finite for all $1 \leqslant \alpha < q + \sqrt{q^2 - q}$.

Applying the Hölder inequality to the right hand side of (25), one has

$$\int_{\mathbb{R}^N} u^{2\alpha+q+p}\rho_{N+2s}dx \leqslant C\left(\int_{\mathbb{R}^N} u^{2\alpha+q+p}\rho_{N+2s}dx\right)^{\frac{2\alpha}{2\alpha+q+p}} < \infty.$$

A simplification of this estimate follows (23). □

End of Proof of Theorem 2. For $R > 0$ sufficiently large, put

$$u_R(x) = R^{\frac{2s}{p+q}}u(Rx) > 0 \text{ and } v_R(x) = R^{\frac{2s}{p+q}}v(Rx) > 0.$$

By some straightforward computation, (u_R, v_R) is also u-stable solution of the system (1). From (23), it holds

$$\int_{\mathbb{R}^N} u_R^{2\alpha+q+p}(x)\rho_{N+2s}(x)dx < C$$

or equivalently

$$R^{\frac{2s(2\alpha+p+q)}{p+q}}\int_{\mathbb{R}^N} u^{2\alpha+q+p}(Rx)\rho_{N+2s}(x)dx < C.$$

Using a change of variable, we arrive at

$$R^{-N+\frac{2s(2\alpha+p+q)}{p+q}} \int_{\mathbb{R}^N} u^{2\alpha+q+p}(x)\rho_{N+2s}(x/R)dx < C$$

or equivalently

$$\int_{\mathbb{R}^N} u^{2\alpha+q+p}(x)\rho_{N+2s}(x/R)dx < CR^{N-\frac{2s(2\alpha+p+q)}{p+q}}. \tag{26}$$

By the assumption

$$N < 2s + \frac{4s}{p+q}\left(q+\sqrt{q^2-q}\right),$$

we can choose α close to $q+\sqrt{q^2-q}$ such that the exponent in the right hand side of (26) is negative. By taking $R \to +\infty$ we obtain $u \equiv 0$ which implies a contradiction since $u > 0$. The proof is finished. $\square$

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REFERENCES

[1] J. BERTOIN, *Lévy processes*, vol. 121 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1996.

[2] A. BISWAS, *Liouville type results for systems of equations involving fractional Laplacian in exterior domains*, Nonlinearity **32**, 6 (2019), 2246–2268.

[3] J.-P. BOUCHAUD AND A. GEORGES, *Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications*, Phys. Rep. **195**, 4–5 (1990), 127–293.

[4] L. CAFFARELLI AND L. SILVESTRE, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32**, 7–9 (2007), 1245–1260.

[5] L. A. CAFFARELLI AND A. VASSEUR, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math. (2) **171**, 3 (2010), 1903–1930.

[6] P. CONSTANTIN, *Euler equations, Navier-Stokes equations and turbulence*, In Mathematical foundation of turbulent viscous flows, vol. 1871 of Lecture Notes in Math., Springer, Berlin, 2006, pp. 1–43.

[7] C. COWAN, *Liouville theorems for stable Lane-Emden systems with biharmonic problems*, Nonlinearity **26**, 8 (2013), 2357–2371.

[8] J. DÁVILA, L. DUPAIGNE AND J. WEI, *On the fractional Lane-Emden equation*, Trans. Amer. Math. Soc. **369**, 9 (2017), 6087–6104.

[9] Y. DU AND Z. GUO, *Positive solutions of an elliptic equation with negative exponent: stability and critical power*, J. Differential Equations **246**, 6 (2009), 2387–2414.

[10] A. T. DUONG, V. T. LUONG AND T. Q. NGUYEN, *Classification of stable solutions to a fractional singular elliptic equation with weight*, Acta Appl. Math. **170** (2020), 579–591.

[11] A. T. DUONG, T. Q. NGUYEN AND T. H. A. VU, *A note on positive supersolutions of the fractional Lane-Emden system*, J. Pseudo-Differ. Oper. Appl. **11**, 4 (2020), 1719–1730.

[12] A. T. DUONG, AND V. H. NGUYEN, *Liouville type theorems for some fractional elliptic problems*, Nonlinear Anal. **210** (2021), Paper No. 112383, 20.

[13] A. T. DUONG, AND V. H. NGUYEN, *A Liouville-type theorem for fractional elliptic equation with exponential nonlinearity*, Mediterr. J. Math. **19**, 2 (2022), Paper No. 91, 16.

[14] L. DUPAIGNE AND A. FARINA, *Stable solutions of* $-\Delta u = f(u)$ *in* $\mathbb{R}^N$, J. Eur. Math. Soc. (JEMS) **12**, 4 (2010), 855–882.

[15] A. FARINA, *On the classification of solutions of the Lane-Emden equation on unbounded domains of* $\mathbb{R}^N$, J. Math. Pures Appl. (9) **87**, 5 (2007), 537–561.
[16] A. FARINA, *Stable solutions of* $-\Delta u = e^u$ *on* $\mathbb{R}^N$, C. R. Math. Acad. Sci. Paris **345**, 2 (2007), 63–66.
[17] M. FAZLY, Y. HU AND W. YANG, *On stable and finite Morse index solutions of the nonlocal Hénon-Gelfand-Liouville equation*, Calc. Var. Partial Differential Equations **60**, 1 (2021), Paper No. 11, 26.
[18] M. FAZLY AND J. WEI, *On stable solutions of the fractional Hénon-Lane-Emden equation*, Commun. Contemp. Math. **18**, 5 (2016), 1650005, 24.
[19] M. FAZLY AND J. WEI, *On finite Morse index solutions of higher order fractional Lane-Emden equations*, Amer. J. Math. **139**, 2 (2017), 433–460.
[20] H. HAJLAOUI, A. HARRABI AND F. MTIRI, *Liouville theorems for stable solutions of the weighted Lane-Emden system*, Discrete Contin. Dyn. Syst. **37**, 1 (2017), 265–279.
[21] L.-G. HU AND J. ZENG, *Liouville type theorems for stable solutions of the weighted elliptic system*, J. Math. Anal. Appl. **437**, 2 (2016), 882–901.
[22] H. A. LEVINE, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form* $Pu_t = -Au + F(u)$, Arch. Rational Mech. Anal. **51** (1973), 371–386.
[23] Y. LI AND Y. LEI, *On existence and nonexistence of positive solutions of an elliptic system with coupled terms*, Commun. Pure Appl. Anal. **17**, 5 (2018), 1749–1764.
[24] P. QUITTNER AND P. SOUPLET, *Symmetry of components for semilinear elliptic systems*, SIAM J. Math. Anal. **44**, 4 (2012), 2545–2559.
[25] B. RAHAL AND C. ZAIDI, *On the classification of stable solutions of the fractional equation*, Potential Anal. **50**, 4 (2019), 565–579.
[26] V. E. TARASOV AND G. M. ZASLAVSKY, *Fractional dynamics of systems with long-range interaction*, Commun. Nonlinear Sci. Numer. Simul. **11**, 8 (2006), 885–898.
[27] S. TERRACINI, G. VERZINI AND A. ZILIO, *Uniform Hölder regularity with small exponent in competition-fractional diffusion systems*, Discrete Contin. Dyn. Syst. **34**, 6 (2014), 2669–2691.
[28] S. TERRACINI, G. VERZINI AND A. ZILIO, *Uniform Hölder bounds for strongly competing systems involving the square root of the laplacian*, J. Eur. Math. Soc. (JEMS) **18**, 12 (2016), 2865–2924.
[29] G. VERZINI AND A. ZILIO, *Strong competition versus fractional diffusion: the case of Lotka-Volterra interaction*, Comm. Partial Differential Equations **39**, 12 (2014), 2284–2313.
[30] H. YANG AND W. ZOU, *Symmetry of components and Liouville-type theorems for semilinear elliptic systems involving the fractional Laplacian*, Nonlinear Anal. **180** (2019), 208–224.
[31] C. ZAIDI, *Liouville results for fractional elliptic equations with Hardy potential*, J. Dynam. Differential Equations **32**, 4 (2020), 1983–1995.
[32] R. ZHUO AND Y. LI, *Nonexistence and symmetry of solutions for Schrödinger systems involving fractional Laplacian*, Discrete Contin. Dyn. Syst. **39**, 3 (2019), 1595–1611.

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