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CORRIGENDUM TO "A HARMONIC MEAN INEQUALITY FOR THE POLYGAMMA FUNCTION"

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Abstract. We point out that the proof of Lemma 5 in the article "A harmonic mean inequality for the polygamma function" by S. Das and A. Swaminathan contains a small mistake. Giving a new and proper proof of Lemma 5 is the main aim of this paper.

1. Introduction

Recently, S. Das and A. Swaminathan published a proof of the following Lemma in [2]:

LEMMA 5. [2] *For $x>0$, $x\neq 1$, and $m\in\mathbb{N}$, we have*

$$\psi^{(m)}(x)\psi^{(m)}\left(\frac{1}{x}\right)>\left[\psi^{(m)}(1)\right]^2.$$

The proof of Lemma 5 contains a small mistake which can be explained as follows: In the proof of Lemma 5, Das and Swaminathan applied that the function

$$|\psi^{(m)}(x)|=(-1)^{m+1}\psi^{(m)}(x)$$

is decreasing by Theorem 1 in reference [3]. This means

$$(-1)^{m+1}\psi^{(m)}(x)>(-1)^{m+1}\psi^{(m)}(1),\ for\ x<1. \tag{1}$$

On the other hand,

$$(-1)^{m+1}\psi^{(m)}(y)>(-1)^{m+1}\psi^{(m)}(1),\ for\ y>1. \tag{2}$$

Combing these two inequalities, the authors get the required result.

A careful observation shows the inequality (2) should be

$$(-1)^{m+1}\psi^{(m)}(y)<(-1)^{m+1}\psi^{(m)}(1),\ for\ y>1. \tag{3}$$

Therefore, the available evidence does not lead to this conclusion that the function $\psi^{(m)}(x)\psi^{(m)}(\frac{1}{x})$ gets the minimum value at 1. Here, we will give a new proof of Lemma 5. The following Lemma is useful.

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LEMMA 1. [1] *For positive real numbers x, and $m \geqslant 2$, $m \in \mathbb{N}$, the following holds*

$$\frac{m}{m+1} > \frac{\left[\psi^{(m)}(x)\right]^2}{\psi^{(m+1)}(x)\psi^{(m-1)}(x)} > \frac{m-1}{m}. \tag{4}$$

The lower and upper bounds are best possible.

Proof of Lemma 5. Let

$$N(x) = \psi^{(m)}(x)\psi^{(m)}\left(\frac{1}{x}\right),$$

then

$$N'(x) = \psi^{(m+1)}(x)\psi^{(m)}\left(\frac{1}{x}\right) - \frac{1}{x^2}\psi^{(m)}(x)\psi^{(m+1)}\left(\frac{1}{x}\right), \tag{5}$$

$$N''(x) = \psi^{(m+2)}(x)\psi^{(m)}\left(\frac{1}{x}\right) - \frac{2}{x^2}\psi^{(m+1)}(x)\psi^{(m+1)}\left(\frac{1}{x}\right) + \frac{1}{x^4}\psi^{(m)}(x)\psi^{(m+2)}\left(\frac{1}{x}\right). \tag{6}$$

So, $N'(1) = 0$ and

$$N''(1) = 2\psi^{(m+2)}(1)\psi^{(m)}(1) - 2\psi^{(m+1)}(1)\psi^{(m+1)}(1).$$

In view of the Lemma 1, we can get $N''(1) > 0$, which indicates that $N(1)$ is the minimum of $N(x)$. □

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