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CONE PROPERTY AND MEASURE DENSITY CONDITION

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Abstract. We prove the existence of open set Ω in the Euclidean space, satisfying the measure density condition, such that the boundary $\partial\Omega$ is a graph and Ω does not satisfy the cone condition. In this way we give an answer to the conjecture formulated by V. Burenkov. Some of the results are formulated in the setting of metric and metric-measure spaces. In particular, for Ω, which is a subset of a metric space, we study the relationships between the measure density condition of Ω and the growth of the measure $\mu(\Omega\cap B(x,r))$, where x is taken from the boundary $\partial\Omega$. Moreover, similar issue is studied for cone condition.

1. Introduction

Let Ω be a domain in $\mathbb{R}^n$. It is well known (see e.g., [1]) that if Ω satisfies the cone condition,[1] then for $1 < p < n$ the Sobolev embedding

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

holds with $p^* = np/(n-p)$. On the other hand, (see e.g., [6]) if Ω is such that the above embedding holds, then Ω satisfies the measure density condition. It is easy to see that every domain satisfying the cone condition, satisfies also the measure density condition. Moreover, the class of domains satisfying the measure density condition is strictly bigger than the class of domains satisfying the cone condition [4]. Recently, Burenkov [3, 5] conjectured that every open set $\Omega \subset \mathbb{R}^n$, satisfying the n-measure density condition, such that the boundary $\partial\Omega$ is a graph has to satisfy the cone condition. The main aim of this paper is to give an answer to Burenkov's conjecture.

The remainder of the paper is structured as follows. In Section 2 we formulate and prove two topological results which will play a crucial role in the subsequent sections. In Section 3 we recall the notion of the measure density condition and the cone condition. Furthermore, for $\Omega \subset X$, where X is a metric space, we shall study the relationships between the measure density condition of Ω and the growth of the measure $\mu(\Omega\cap B(x,r))$, where x is taken from the boundary $\partial\Omega$ (see Theorem 3.1 and Theorem 3.2). Moreover, similar issue is studied for cone condition (see Theorem 3.3). In the last section we prove Theorem 4.1 which will give an answer to Burenkov's conjecture.

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[1] We refer to Section 3 for appropriate definitions.

2. Preliminaries

In this work a metric-measure space (X,d,μ) is a metric space (X,d) equipped with a Borel measure μ. We assume throughout the paper that the measure of every open non empty set is positive and the measure of every bounded set is finite. Let us note to construct the example we only need to consider the Euclidean setting. Nevertheless, the metric setting will be needed in Section 3. Moreover, we hope that the general results presented in the present section will be of interest to readers interested in the analysis and geometry of metric spaces.

LEMMA 2.1. *Let (X,d) be a metric space such that open balls are path-connected. If $\Omega \subset X$ is non-empty, open and $\Omega \neq X$, then for every $x \in \Omega$*

$$\mathrm{dist}\,(x,\partial\Omega) \leqslant \mathrm{dist}\left(x, X\setminus\Omega\right).$$

Proof. Let us note that since every ball is path-connected, we have in particular that X is connected. Therefore, every non-empty subset D of X, such that $D \neq X$ has non-empty boundary. Let us take $y \notin \Omega$ and $\varepsilon > 0$. We denote by r the distance between x and y. Since $y \in B(x, r+\varepsilon)$ and balls are path connected, there exists a continuous function

$$\gamma_\varepsilon \colon\ [0,1] \to B(x,r+\varepsilon)$$

such that $\gamma_\varepsilon(0) = x$ and $\gamma_\varepsilon(1) = y$. Openness of Ω and continuity of γ_ε imply that the quantity

$$\tau_\varepsilon \colon = \inf\left\{p\in[0,1]\colon\ \gamma_\varepsilon(p)\notin\Omega\right\}$$

is positive and $\gamma_\varepsilon(\tau_\varepsilon) \in \overline{\Omega}$. We shall prove that $\gamma_\varepsilon(\tau_\varepsilon) \in \partial\Omega$. Suppose that $\gamma_\varepsilon(\tau_\varepsilon) \in \Omega$. Thus $\tau_\varepsilon < 1$ and since Ω is open and γ_ε continuous, there exist $\eta > 0$ and $\xi \in (0, 1-\tau_\varepsilon)$ such that

$$\gamma_\varepsilon\left(\left(\tau_\varepsilon-\xi,\tau_\varepsilon+\xi\right)\right) \subset B\left(\gamma_\varepsilon(\tau_\varepsilon),\eta\right) \subset \Omega.$$

Combining those facts

$$\tau_\varepsilon = \inf\left\{p\in[0,1]\colon\ \gamma_\varepsilon(p)\notin\Omega\right\} \geqslant \tau_\varepsilon + \xi/2.$$

This contradiction shows that

$$\gamma_\varepsilon(\tau_\varepsilon) \in \overline{\Omega}\setminus\Omega = \partial\Omega.$$

Therefore, we have proved that for every $\varepsilon > 0$ there exists $z_\varepsilon \in \partial\Omega \cap B(x, r+\varepsilon)$. Hence,

$$\mathrm{d}(x,y)+\varepsilon \geqslant \mathrm{d}(x,z_\varepsilon) \geqslant \mathrm{dist}\,(x,\partial\Omega).$$

Finally, since $y \notin \Omega$ and $\varepsilon > 0$ are arbitrary, we easily get

$$\mathrm{dist}\left(x,X\setminus\Omega\right) \geqslant \mathrm{dist}\,(x,\partial\Omega). \quad \square$$

For the convenience of the reader, we recall the notion of the Hausdorff distance. Let (X,d) be a metric space, by $\mathfrak{C}(X)$ we denote the family of all compact and non-empty subsets of X. For $A,B \in \mathfrak{C}(X)$ we define the Hausdorff distance between A and B as follows

$$d_H(A,B) = \inf\{\varepsilon > 0 : A \subset (B)_\varepsilon, B \subset (A)_\varepsilon\},$$

where

$$(A)_\varepsilon = \bigcup_{a\in A} \bar{B}(a,\varepsilon).$$

It is well known that $(\mathfrak{C}(X),d_H)$ is a metric space. Moreover, $(\mathfrak{C}(X),d_H)$ inherits some properties from (X,d), e.g., compactness and completness (see e.g., [2]). Now, we are in position to formulate and prove the result which will play a crucial role in the next section.

THEOREM 2.2. *Let (X,d) be a metric space, $C \in \mathfrak{C}(X)$ and $\varphi_k : C \to \varphi_k(C) \subset X$ be a sequence of isometries such that $\varphi_k(C) \to D$ in $(\mathfrak{C}(X),d_H)$. Then:*

a) D is isometric to C;

b) If $\varphi_k(x) \overset{k\to\infty}{\to} y$ for some $x \in C$, $y \in D$, then there exists an isometry $f : C \to D$ such that $f(x) = y$;

c) If (X,d) is such that every open ball is path-connected and every isometry defined on subsets of X is the restriction of an isometry defined on the whole space X, $C \neq X$, then for every set $\Omega \subset X$ such that $\bigcup_{k=1}^{\infty} \mathrm{Int}\big(\varphi_k(C)\big) \subset \Omega$ we have $\mathrm{Int}(D) \subset \Omega$.

Proof. We shall construct an isometry from C onto D. For this purpose we fix $l \in \mathbb{N}$, then from the very definition of the Hausdorff distance there exists $k_l \in \mathbb{N}$ such that

$$\varphi_{k_l}(C) \subset (D)_{\frac{1}{l}}, \tag{2.1}$$

$$D \subset \big(\varphi_{k_l}(C)\big)_{\frac{1}{l}}. \tag{2.2}$$

Therefore, from (2.1) for every $x \in C$ there exists $f_l(x) \in D$ such that

$$\mathrm{d}(\varphi_{k_l}(x), f_l(x)) \leqslant 1/l. \tag{2.3}$$

Hence, taking into account the above inequality, we have for every $x_1,x_2 \in C$

$$\begin{aligned}\mathrm{d}(f_l(x_1), f_l(x_2)) &\leqslant \mathrm{d}(f_l(x_1), \varphi_{k_l}(x_1)) + \mathrm{d}(\varphi_{k_l}(x_1), \varphi_{k_l}(x_2)) + \mathrm{d}(\varphi_{k_l}(x_2), f_l(x_2))\\ &\leqslant 1/l + \mathrm{d}(\varphi_{k_l}(x_1), \varphi_{k_l}(x_2)) + 1/l\\ &= 2/l + \mathrm{d}(x,y),\end{aligned}$$

and in the same fashion we get

$$\begin{aligned}\mathrm{d}(x_1,x_2) &= \mathrm{d}(\varphi_{k_l}(x_1), \varphi_{k_l}(x_2))\\ &\leqslant \mathrm{d}(\varphi_{k_l}(x_1), f_l(x_1)) + \mathrm{d}(f_l(x_1), f_l(x_2)) + \mathrm{d}(f_l(x_2), \varphi_{k_l}(x_2))\\ &\leqslant \mathrm{d}(f_l(x_1), f_l(x_2)) + 2/l.\end{aligned}$$

So, finally we have

$$\mathrm{d}(x,y) - 2/l \leqslant \mathrm{d}(f_l(x), f_l(y)) \leqslant \mathrm{d}(x,y) + 2/l. \tag{2.4}$$

Next, we shall pass to the limit in the above inequalities. For this purpose we fix $\{x_k\}_{k=1}^{\infty}$ a countable and dense subset[2] of C. Hence, by compactness of D and the Tychonoff Theorem, there exists a subsequence $\{f_{l_m}\}_{m=1}^{\infty}$ such that for every $k \in \mathbb{N}$

$$\{f_{l_m}(x_k)\}_{m=1}^{\infty} \text{ converges.} \tag{2.5}$$

Now, we shall show that for $x \in C$

$$\{f_{l_m}(x)\}_{m=1}^{\infty} \text{ converges.} \tag{2.6}$$

Indeed, let $x \in C$ and $\varepsilon > 0$, then from the density of $\{x_k\}_{k=1}^{\infty}$ there exists $k \in \mathbb{N}$ such that

$$\mathrm{d}(x, x_k) \leqslant \varepsilon/3. \tag{2.7}$$

Therefore,

$$\begin{aligned} \mathrm{d}(f_{l_m}(x), f_{l_n}(x)) &\leqslant \mathrm{d}(f_{l_m}(x), f_{l_m}(x_k)) + \mathrm{d}(f_{l_m}(x_k), f_{l_n}(x_k)) + \mathrm{d}(f_{l_n}(x_k), f_{l_n}(x)) \\ &\overset{(2.4)}{\leqslant} \mathrm{d}(x, x_k) + 2/l_m + \mathrm{d}(f_{l_m}(x_k), f_{l_n}(x_k)) + \mathrm{d}(x, x_k) + 2/l_n \\ &\overset{(2.7)}{\leqslant} 2\varepsilon/3 + 2/l_n + 2/l_m + \mathrm{d}(f_{l_m}(x_k), f_{l_n}(x_k)). \end{aligned}$$

Hence, gathering the above inequality with (2.5), we get for m and n big enough

$$\mathrm{d}(f_{l_m}(x), f_{l_n}(x)) \leqslant \varepsilon,$$

and this shows that $\{f_{l_m}(x)\}_{m=1}^{\infty}$ is a Cauchy sequence in D. Thus, compactness of D yields that $\{f_{l_m}(x)\}_{m=1}^{\infty}$ converges and (2.6) follows. Let us denote by $f(x)$ the quantity $\lim_{m\to\infty} f_{l_m}(x)$. Hence, we pass to the limit in (2.4) and we obtain

$$\mathrm{d}(f(x), f(y)) = \mathrm{d}(x,y).$$

In other words, we proved the existence of a distance preserving transformation $f : C \to D$. We shall need to show that f is surjective. By (2.2), for every $x \in D$ there exists $h_l(x) \in \varphi_{k_l}(C)$ such that

$$\mathrm{d}(x, h_l(x)) < 1/l.$$

This leads us to the following inequalities

$$\mathrm{d}(x,y) - 2/l \leqslant \mathrm{d}(h_l(x), h_l(y)) \leqslant \mathrm{d}(x,y) + 2/l.$$

[2] Separability of C follows from compactness of C.

Therefore, if we define $g_l\colon D\to C$ as $g_l(x)\colon=\left(\varphi_{k_l}^{-1}\circ h_l\right)(x)$, then using the fact that $\varphi_{k_l}^{-1}$ is an isometry we can rewrite the above inequality in the following manner

$$\mathrm{d}(x,y)-2/l\leqslant \mathrm{d}(g_l(x),g_l(y))\leqslant \mathrm{d}(x,y)+2/l. \tag{2.8}$$

As before, we can prove that from (2.8) follows the existence of a subsequence of g_l, which converges to $g\colon D\to C$. Moreover, using inequality (2.8) we obtain that g preserves the distance. Thus, the map $f\circ g\colon D\to D$ preserves the distance. Finally, since D is compact, from the Freudenthal-Hurewicz Theorem (see [7, Theorem 3.3.3]) we get that $f\circ g$ is an isometry. Therefore, f is surjective and this shows that f is isometry.

Now, we shall prove the second part of the theorem. Let f be defined as a point-wise limit of f_{l_m}, where f_l was defined in the first part of the proof. We have proved that f is an isometry between C and D. By (2.3) we have

$$\begin{aligned}\mathrm{d}(f_{l_m}(x),y)&\leqslant \mathrm{d}(f_{l_m}(x),\varphi_{k_{l_m}}(x))+\mathrm{d}(\varphi_{k_{l_m}}(x),y)\\&\leqslant 1/l_m+\mathrm{d}(\varphi_{k_{l_m}}(x),y).\end{aligned}$$

Therefore, since $\varphi_{k_{l_m}}(x)\to y$ as $m\to\infty$, we can pass to the limit in the above inequality and we get $f(x)=y$.

Finally, we will prove the last part of the theorem. Since the case $\mathrm{Int}(D)=\emptyset$ is trivial, we assume $\mathrm{Int}(D)\neq\emptyset$ and let $z\in\mathrm{Int}(D)$. Combining connectedness with compactness one can show that $\partial\mathrm{Int}(D)\neq\emptyset$. We define the isometry $\alpha_l : X\to X$ by the formula $\alpha_l=\hat{\varphi}_{k_l}\circ\hat{f}^{-1}$, where $\hat{\varphi}_{k_l},\hat{f} : X\to X$ are isometries such that $\hat{\varphi}_{k_l}|_C=\varphi_{k_l},\hat{f}|_C=f$. Subsequently, we put $z_l=\alpha_l(z)$. Henc, by (2.3), we have

$$\mathrm{d}(z_l,z)\leqslant 1/l+\mathrm{d}(f_l\circ f^{-1}(z),z)\xrightarrow{l\to\infty}\mathrm{d}(z,z)=0.$$

Let $\varepsilon=\mathrm{dist}\left(z,\partial\mathrm{Int}(D)\right)/2>0$, then there exists k such that $z\in B(z_k,\varepsilon)$. Moreover, since α_k is a homeomorphism, we have

$$\mathrm{dist}\left(z,\partial\mathrm{Int}(D)\right)=\mathrm{dist}\left(z_k,\partial\mathrm{Int}\left(\alpha_k(D)\right)\right).$$

Therefore, we get

$$z\in B(z_k,\varepsilon)\subset\mathrm{Int}\left(\alpha_k(D)\right)\subset\Omega.$$

Indeed, if we suppose that some $w\in B(z_k,\varepsilon)$ does not belong to $\mathrm{Int}\left(\alpha_k(D)\right)$, then by Lemma 2.1 we get

$$\begin{aligned}\mathrm{dist}\left(z_k,X\setminus\mathrm{Int}\left(\alpha_k(D)\right)\right)\leqslant \mathrm{d}(z_k,w)<\varepsilon&=\mathrm{dist}\left(z_k,\partial\mathrm{Int}\left(\alpha_k(D)\right)\right)/2\\&\leqslant\mathrm{dist}\left(z_k,X\setminus\mathrm{Int}\left(\alpha_k(D)\right)\right)/2,\end{aligned}$$

which is a contradiction. □

3. Measure density condition and cone property

3.1. Measure density condition

Let (X,d,μ) be a metric-measure space and $s>0$. We shall say that the measure μ is *lower Ahlfors s-regular* if there is a constant $D>0$ such that

$$\mu(B(z,r)) \geqslant Dr^s \text{ for } r\in(0,1],\ z\in X.$$

Let (X,d,μ) be a metric-measure space and $\alpha>0$, we say that a measurable set $\Omega\subset X$ is *α-regular* if there is a constant C such that

$$\mu(B(x,r)\cap\Omega) \geqslant Cr^\alpha \text{ for } r\in(0,1],\ x\in\Omega.$$

In the case of $(\mathbb{R}^n,\|.\|,\lambda_n)$, we shall say that a λ_n-measurable subset Ω of $\mathbb{R}^n$ satisfies the *α-measure density condition* if Ω is α-regular.

THEOREM 3.1. *Let (X,d,μ) be a metric-measure space and $\Omega\subset X$. Then:*

(i) *If Ω is α-regular, where $\alpha>0$, then there exists C such that*

$$\mu(B(x,r)\cap\Omega) \geqslant Cr^\alpha \text{ for } r\in(0,1],\ x\in\partial\Omega. \tag{3.1}$$

(ii) *If (X,d,μ) is a metric-measure space such that open balls are path-connected and the measure μ is lower Ahlfors s-regular, $s\leqslant\alpha$ and $\Omega\subset X$ is open, non-empty and there exists C such that (3.1) holds, then Ω is α-regular.*

Proof. (i) Let $r\in(0,1]$, $y\in\partial\Omega$ and $\{y_n\}_{n=1}^\infty\subset\Omega$ be a sequence converging to y. For sufficiently large n we have that $\mathrm{d}(y_n,y)<r$. Therefore, for such n we have

$$B\left(y_n, r-\mathrm{d}(y_n,y)\right)\subset B(y,r).$$

Hence,

$$\begin{aligned}\mu\left(B(y,r)\cap\Omega\right) &\geqslant \mu\left(B\left(y_n,r-\mathrm{d}(y_n,y)\right)\cap\Omega\right)\\ &\geqslant C(r-\mathrm{d}(y_n,y))^\alpha,\end{aligned}$$

where the last inequality follows from the assumption on Ω. Finally, since n is arbitrary, we can pass to the limit and we get

$$\mu\left(B(y,r)\cap\Omega\right)\geqslant \lim_{n\to\infty} C(r-\mathrm{d}(y_n,y))^\alpha = Cr^\alpha.$$

(ii) Since μ is lower Ahlfors s-regular, there exists D such that

$$\mu(B(z,r)) \geqslant Dr^s \text{ for } r\in(0,1],\ z\in X.$$

Let $x\in\Omega$ and $r\in(0,1]$. We shall consider two cases.

1. $\operatorname{dist}(x,\partial\Omega) > r/2$. In this case, by Lemma 2.1, we obtain that $B\left(x,r/2\right) \subset \Omega$. Therefore, since the measure μ is lower Ahlfors s-regular, we have

$$\begin{aligned}\mu\left(B(x,r)\cap\Omega\right) &\geqslant \mu\left(B\left(x,r/2\right)\cap\Omega\right)\\ &= \mu\left(B\left(x,r/2\right)\right) \geqslant \frac{D}{2^{\alpha}}r^{s}.\end{aligned}$$

2. $\operatorname{dist}(x,\partial\Omega) \leqslant r/2$. Let us take $y\in\partial\Omega$ such that $\mathrm{d}(x,y) < 2r/3$. Then, gathering the inclusion

$$B\left(y,r-\mathrm{d}(x,y)\right) \subset B(x,r),$$

with (3.1), we obtain

$$\begin{aligned}\mu\left(B(x,r)\cap\Omega\right) &\geqslant \mu\left(B\left(y,r-\mathrm{d}(x,y)\right)\cap\Omega\right)\\ &\geqslant C\left(r-\mathrm{d}(x,y)\right)^{\alpha}\\ &\geqslant \frac{C}{3^{\alpha}}r^{\alpha}.\end{aligned}$$

Therefore, we proved that Ω is α-regular with constant $\tilde{C} = \min\left\{C/3^{\alpha}, D/2^{\alpha}\right\}$. □

As a corollary we get.

THEOREM 3.2. *Let $\Omega\subset\mathbb{R}^n$ be an open set and $\alpha\geqslant n$. Then Ω satisfies the α-measure density condition if and only if there exists C such that*

$$\lambda_n(B(x,r)\cap\Omega) \geqslant Cr^{\alpha} \text{ for } r\in(0,1],\ x\in\partial\Omega.$$

In particular, if Ω satisfies the n-measure density condition, then $\bar{\Omega}$ satisfies the n-measure density condition as well.

EXAMPLE 1. Let $\{\alpha_i\}_{i=1}^{\infty} = \mathbb{Q}^n$ and $\Omega = \bigcup_{i=1}^{\infty} B\left(\alpha_i, 1/2^i\right)$. Then, $\overline{\Omega}$ satisfies n–measure density condition, but Ω does not.

3.2. Cone property

Now we recall the definition of cone condition [1]. Let v be a vector in $\mathbb{R}^n$ with $\|v\| = 1$, $y\in\mathbb{R}^n$, $\rho > 0$ and κ satisfying $0 < \kappa < \pi$, the set

$$C_y(v,\kappa,\rho) = y + \left\{x\in\mathbb{R}^n:\ x = 0 \text{ or } 0 < \|x\| \leqslant \rho,\ \angle(x,v) \leqslant \kappa/2\right\}$$

is called a *finite cone* of height ρ, axis direction v and aperture angle κ with vertex at y.[3]

Furthermore, we shall say that cones $C_{x_1} := C_{x_1}(v_1,\kappa_1,\rho_1)$ and $C_{x_2} := C_{x_2}(v_2,\kappa_2,\rho_2)$ are *congruent* if there exists an isometry $\psi\colon C_{x_1}\to C_{x_2}$, such that $\psi(x_1) = x_2$. One can

[3]For $x\neq 0$ and $z\neq 0$ we denote by $\angle(x,z)$ the angle between vectors x and z.

easily convince oneself that cones $C_{x_1}(v_1,\kappa_1,\rho_1)$, $C_{x_2}(v_2,\kappa_2,\rho_2)$ are congruent if and only if $\kappa_1=\kappa_2$ and $\rho_1=\rho_2$.

Let us observe that if $C\subset\mathbb{R}^n$ is a finite cone with axis direction v and vertex at y, and $\psi\colon C\to\psi(C)\subset\mathbb{R}^n$ is some isometry, then $\psi(C)$ is a finite cone with vertex at $\psi(y)$, axis direction $\psi(v+y)-\psi(y)$ and the same height and aperture angle as C.[4] In other words

$$\psi(C)=\psi(y)+\left\{x\in\mathbb{R}^n\colon\ x=0 \text{ or } 0<\|x\|\leqslant\rho,\ \angle(x,\psi(v+y)-\psi(y))\leqslant\kappa/2\right\}.$$

Let us remark that since we deal with the Euclidean space, every isometry defined on subsets of $\mathbb{R}^n$ is the restriction of an isometry defined on the whole $\mathbb{R}^n$ (see [8, Theorem 11.4]).

Finally, we shall say that $\Omega\subset\mathbb{R}^n$ satisfies the *cone condition* if there exists a finite cone C such that each $x\in\Omega$ is the vertex of a finite cone $C_x\subset\Omega$ congruent to C.

THEOREM 3.3. *Let Ω be an open subset of $\mathbb{R}^n$. If Ω satisfies the cone condition, then $\overline{\Omega}$ also satisfies the cone condition. Moreover, there exists a finite cone C such that for every $x\in\overline{\Omega}$ there exists a cone C_x with vertex at x congruent to C and* $\operatorname{Int}(C_x)\subset\Omega$.

Proof. By assumption there exists a finite cone $C=C_y(v,\kappa,\rho)$ such that for every $x\in\Omega$ there exists a finite cone $C_x\subset\Omega$ with vertex at x and C_x is congruent to C. It is sufficient to show that the cone condition is satisfied for $x\in\partial\Omega$. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of elements from Ω, converging to x. Therefore, there exists a sequence of finite cones $C_{x_k}\subset\Omega$ with vertex at x_k, which are congruent to C. In other words, for every $k\in\mathbb{N}$ there exists an isometry $\psi_k\colon C\to C_{x_k}$ such that $\psi_k(y)=x_k$. Without loss of generality we can assume that $C\subset\Omega$. Let us take $R>0$ such that cones C,C_{x_k} are containd in the ball $B(x,R)$. Then,

$$C_{x_k}\subset\Omega\cap B(x,R)\subset\bar{\Omega}\cap\bar{B}(x,R):=X.$$

Since X is compact and $C_{x_k}\in\mathfrak{C}(X)$, by the Blaschke compactness theorem (see [2, Theorem 7.3.8]) there exists a subsequence $C_{x_{k_l}}$ and $D\in\mathfrak{C}(X)$ such that $C_{x_{k_l}}\to D$ in $(\mathfrak{C}(X),d_H)$. Finally, since $\psi_{k_l}(y)\to x$, Theorem 2.2 yields the existence of isometry $\psi:C\to D$ such that $\psi(y)=x$. Therefore, we get $D\subset\overline{\Omega}$ is a finite cone with vertex at x which is congruent to C. Moreover, since $\operatorname{Int}\left(C_{x_k}\right)\subset\Omega\cap B(x,R)$, by Theorem 2.2, we get that $\operatorname{Int}(D)\subset\Omega\cap B(x,R)\subset\Omega$. $\square$

EXAMPLE 2. Let $\Omega=\bigcup_{i=1}^{\infty}B\left(\alpha_i,1/2^i\right)$, where $\{\alpha_i\}_{i=1}^{\infty}=\mathbb{Q}^n$. Then $\overline{\Omega}$ satisfies the cone condition, but Ω does not.

EXAMPLE 3. Let $\Omega=\mathbb{R}^2\setminus\Gamma$, where

$$\Gamma:=\left\{(x_1,x_2)\in[0,\infty)\times\mathbb{R}:|x_2|=x_1^2\right\}.$$

[4] It easily follows from the fact that the map $x\mapsto\psi(x+y)-\psi(y)$ preserves distances and angles.

Then, there exists a finite cone C such that for every $x \in \partial\Omega$ there exists a cone C_x with vertex at x congruent to C and $\mathrm{Int}(C_x) \subset \Omega$, but Ω does not satisfy the cone condition. Therefore, we can not expect that for the cone property the analog of Theorem 3.2 holds.

4. Burenkov's question

It is easy to see that if λ_n-measurable set $\Omega \subset \mathbb{R}^n$ satisfies the cone condition, then Ω satisfies the n-measure density condition. On the other hand Burenkov conjectured [3] that every open set $\Omega \subset \mathbb{R}^n$, satisfying the n-measure density condition, such that the boundary $\partial\Omega$ is a graph has to satisfy the cone condition. In the next theorem we give an answer to Burenkov's conjecture.

THEOREM 4.1. *Let $n \geqslant 2$, then there exists an open set $\Omega \subset \mathbb{R}^n$, satisfying the n-measure density condition, such that $\partial\Omega$ is a graph and Ω does not satisfy the cone condition.*

Proof. For $k \geqslant 1$, let $d_k = \mathbf{c_d}(n)/5^k$, $r_k = \mathbf{c_r}(n)/3^k$, $h_k = \mathbf{c_h}(n)/2^k$, where $\mathbf{c_d}(n) = \alpha(n)$, $\mathbf{c_r}(n) = \mathbf{c_h}(n) = 2^n$ and $\alpha(n)$ denotes the n-dimensional Lebesgue measure of the unit ball. For $k \geqslant 2$ let us define the following sequences:

$$B_k^1 = \sum_{i=1}^{k-1} 2 \cdot d_i + r_i, \qquad B_k^2 = B_k^1 + d_k, \qquad B_k^3 = B_k^2 + d_k.$$

Note that $B_k^j \overset{k\to\infty}{\to} \frac{\mathbf{c_d}(n)+\mathbf{c_r}(n)}{2} =: \mathbf{C}(n)$, for $j = 1,2,3$. Next, for $k \geqslant 1$, we define maps $f_k : \mathbb{R} \to \mathbb{R}$ as follows

$$f_k(x) = L_k(x) \cdot \chi_{[B_k^1, B_k^2]}(x) + R_k(x) \cdot \chi_{(B_k^2, B_k^3]}(x),$$

where $L_k, R_k : \mathbb{R} \to \mathbb{R}$ are linear functions such that $L_k(B_k^1) = 0$, $L_k(B_k^2) = h_k$ and $R_k(B_k^2) = h_k$, $R_k(B_k^3) = 0$. Finally, we define $g\colon \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R}^{n-1} \to \mathbb{R}$ in the following manner

$$g(x) = \sum_{k=1}^{\infty} f_k(x)$$

and

$$h(x_1, x_2, \ldots, x_{n-1}) = g(x_1).$$

It is obvious that g and h are continuous maps. Let

$$\Omega = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > h(x_1, \ldots, x_{n-1})\right\}.$$

We shall show that Ω satisfies the n-measure density condition, and Ω does not satisfy the cone condition. We divide the proof into two steps.

Step 1: Ω satisfies the n-measure density condition.

We shall prove that there exists $C>0$ such that

$$\lambda_n(B(\mathbf{x},r)\cap\Omega)\geqslant Cr^n$$

for $r\in(0,1]$ and $\mathbf{x}=(x_1,x_2,\dots,x_n)\in\Omega$. Having in mind Theorem 3.2, we need only to consider $\mathbf{x}\in\partial\Omega$.

Let

$$G=\left\{(x_1,x_2,\dots,x_{n-1},g(x_1)):(x_1,\dots,x_{n-1})\in\left[B_2^2,\mathbf{C}(n)\right)\times\mathbb{R}^{n-2}\right\},\ \text{for } n>2$$

and

$$G=\left\{(x_1,g(x_1)):x_1\in\left[B_2^2,\mathbf{C}(2)\right)\right\}\ \text{for } n=2.$$

Case 1. If $\mathbf{x}\in\partial\Omega\setminus G$ and $r\in(0,1]$, then at least quarter of $B(\mathbf{x},r)$ is contained in Ω. Therefore, for $\mathbf{x}\in\partial\Omega\setminus G$ and $r\in(0,1]$ we have

$$\lambda_n(B(\mathbf{x},r)\cap\Omega)\geqslant\frac{\alpha(n)}{4}r^n.$$

Case 2. If $\mathbf{x}\in G$ and $r\in(0,1]$, then there exists m such that $x_1\in\left[B_m^2,B_{m+1}^2\right]$. We shall consider two subcases.

Case 2*a*. If $r\leqslant r_m/2$, then at least quarter of $B(\mathbf{x},r)$ is contained in Ω. Indeed, if $x_1\in[B_m^2,B_m^3+r_m/2]$, then $B(\mathbf{x},r)\cap\{\mathbf{y}\in\mathbb{R}^n: y_1>x_1\wedge y_n>x_n\}\subset\Omega$, and if $x_1\in(B_m^3+r_m/2,B_{m+1}^2]$, then $B(\mathbf{x},r)\cap\{\mathbf{y}\in\mathbb{R}^n: y_1<x_1\wedge y_n>x_n\}\subset\Omega$. Therefore,

$$\lambda_n(B(\mathbf{x},r)\cap\Omega)\geqslant\frac{\alpha(n)}{4}r^n.$$

Case 2*b*. $r>r_m/2$. Let us denote

$$T_k=\left\{(y_1,\dots,y_n)\in\mathbb{R}^n:\ y_1\in\left[B_k^1,B_k^3\right]\wedge g(y_1)\geqslant y_n\geqslant 0\right\}.$$

Then, by the Fubini Theorem we have

$$\begin{aligned}&\lambda_n\left(T_k\cap B(\mathbf{x},r)\right)\\&\leqslant\lambda_n\left(\left\{\mathbf{y}\in\mathbb{R}^n:\ y_1\in\left[B_k^1,B_k^3\right]\wedge g(y_1)\geqslant y_n\geqslant 0\wedge\forall_{2\leqslant l\leqslant n-1}\,y_l\in[x_l-r,x_l+r]\right\}\right)\\&=(2r)^{n-2}\int_{B_k^1}^{B_k^3}\int_0^{g(x_1)}\mathrm{d}x_n\mathrm{d}x_1=\frac{2^{n-2}\mathbf{c_d}(n)\mathbf{c_h}(n)}{10^k}r^{n-2}.\end{aligned}\tag{4.1}$$

Since

$$\Omega = \{\mathbf{y} \in \mathbb{R}^n : y_n > 0\} \setminus \bigcup_{k=2}^{\infty} T_k$$

and

$$\bigcup_{k=2}^{m} T_k \cap \left\{\mathbf{y} \in B(\mathbf{x}, r) : y_1 > x_1 \wedge y_n > x_n\right\} = \emptyset,$$

we have

$$\begin{aligned} &\left\{\mathbf{y} \in B(\mathbf{x}, r) : y_1 > x_1 \wedge y_n > x_n\right\} \cap \Omega \\ &= \left\{\mathbf{y} \in B(\mathbf{x}, r) : y_1 > x_1 \wedge y_n > x_n\right\} \setminus \bigcup_{k=m+1}^{\infty} T_k \cap B(\mathbf{x}, r). \end{aligned} \tag{4.2}$$

Therefore, using (4.1) with (4.2), we obtain the following string of inequalities

$$\begin{aligned} \lambda_n(B(\mathbf{x}, r) \cap \Omega) &\geqslant \lambda_n\left(\left\{\mathbf{y} \in B(\mathbf{x}, r) : y_1 > x_1 \wedge y_n > x_n\right\} \setminus \bigcup_{k=m+1}^{\infty} T_k \cap B(\mathbf{x}, r)\right) \\ &\geqslant \lambda_n\left(\left\{\mathbf{y} \in B(\mathbf{x}, r) : y_1 > x_1 \wedge y_n > x_n\right\}\right) - \sum_{k=m+1}^{\infty} \lambda_n\left(T_k \cap B(\mathbf{x}, r)\right) \\ &\geqslant \frac{\alpha(n) r^n}{4} - 2^{n-2} \mathbf{c_d}(n) \mathbf{c_h}(n) r^{n-2} \sum_{k=m+1}^{\infty} \frac{1}{10^k} \\ &\geqslant \alpha(n) r^n \left(\frac{1}{4} - \frac{4 \cdot 2^{n-2} \mathbf{c_d}(n) \mathbf{c_h}(n)}{\alpha(n) r_m^2} \cdot \frac{1}{9 \cdot 10^m}\right) \\ &= \alpha(n) r^n \left(\frac{1}{4} - \frac{4 \cdot 2^{n-2} \mathbf{c_d}(n) \mathbf{c_h}(n)}{9 \mathbf{c_r}(n)^2 \alpha(n)} \cdot \left(\frac{9}{10}\right)^m\right) \\ &\geqslant \frac{\alpha(n) r^n}{10}. \end{aligned}$$

Step 2: Ω does not satisfy the cone condition.

Let $e_1, \dots, e_n$ be a canonical basis of $\mathbb{R}^n$. Denote by $\tau_k = \frac{B_k^3 + B_{k+1}^1}{2}$ and $x_k = \tau_k e_1 \in \mathbb{R}^n$. By the definition $x_k \in \partial\Omega$.

Let us suppose that Ω satisfies the cone condition. Then, by Proposition 3.3 there exists a finite cone C of height ρ and aperture angle κ such that for every $x \in \partial\Omega$, there exists a cone $C_x \subset \overline{\Omega}$ with vertex at x congruent with C. In particular for every $k \in \mathbb{N}$, there exists a cone $C_k \subset \overline{\Omega}$ with vertex at x_k congruent to C. For $k \in \mathbb{N}$ we denote by $v^k = \sum_{s=1}^n v_s^k e_s \in \mathbb{R}^n$ a direction of cone C_k, such that $\left\|v^k\right\| = 1$. Therefore, we can write

$$C_k = C_{x_k}\left(v^k, \kappa, \rho\right).$$

Since $C_k \subset \overline{\Omega} \subset \mathbb{R}^n_+$, for every $z \in C_k$ we have $z_n = \langle z, e_n \rangle \geqslant 0$. Let us observe

$$v_n^k \geqslant \sin\left(\kappa/2\right). \tag{4.3}$$

Indeed, let us suppose that $v_n^k < \sin\left(\kappa/2\right)$, and let $\omega = x_k + \frac{\rho}{2\|u_k\|}u_k$, where $u_k = v^k - \sin\left(\kappa/2\right)e_n$. Then, using the fact that for every $A \in (0,1)$ the map

$$[0,A] \ni x \mapsto \frac{1-Ax}{\sqrt{1+A^2-2Ax}}$$

is non-increasing we get

$$\begin{aligned}\angle\left(\omega - x_k, v^k\right) = \angle\left(u_k, v^k\right) &= \arccos\left(\frac{\left\langle u_k, v^k\right\rangle}{\|u_k\|\|v^k\|}\right)\\ &= \arccos\left(\frac{1-\sin\left(\kappa/2\right)v_n^k}{\sqrt{1+\sin^2\left(\kappa/2\right)-2v_n^k\sin\left(\kappa/2\right)}}\right)\\ &\leqslant \arccos\left(\frac{1-\sin^2\left(\kappa/2\right)}{\sqrt{1+\sin^2\left(\kappa/2\right)-2\sin^2\left(\kappa/2\right)}}\right)\\ &= \arccos\Big(\cos\left(\kappa/2\right)\Big) = \kappa/2,\end{aligned}$$

and

$$\|\omega - x_k\| = \left\|\frac{\rho}{2\|u_k\|}u_k\right\| = \rho/2 \in (0,\rho].$$

This implies $\omega \in C_k$. Therefore,

$$0 \leqslant \langle \omega, e_n\rangle = \frac{\rho}{2\|u_k\|}\langle u_k, e_n\rangle = \frac{\rho}{2\|u_k\|}\left(v_n^k - \sin\left(\kappa/2\right)\right) < 0,$$

and this contradiction finishes the proof of (4.3).

Let $A_k \colon \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation[5], such that

$$\begin{aligned}&A_k(e_1) = e_1\\ &A_k\left(\left[0, v_2^k, \ldots, v_n^k\right]^{\mathrm{T}}\right) = \left(\sqrt{\sum_{i=2}^n \left(v_i^k\right)^2}\right)e_n = \left(\sqrt{1-\left(v_1^k\right)^2}\right)e_n\end{aligned}$$

and let $\Pi \colon \mathbb{R}^n \to \mathbb{R}^2$ be a projection given by the formula $\Pi(x_1, \ldots, x_n) = (x_1, x_n)$.

[5]The existence of such map follows from the basic linear algebra.

Denote by $\Phi_k = \operatorname{span}\left(e_1, v^k\right)$, then we see that

$$\overline{\Omega} \cap \Phi_k = \left\{ \left[t, hv_2^k, \ldots, hv_n^k \right]^{\mathrm{T}} \in \mathbb{R}^n \colon t \in \mathbb{R} \wedge h \geqslant \frac{g(t)}{v_n^k} \right\},$$

and

$$\begin{aligned}&\left(C_k \setminus \{x_k\}\right) \cap \Phi_k \\ &= \left\{ y \in \mathbb{R}^n : \exists_{h,t \in \mathbb{R}}\ y = \left[t, hv_2^k, \ldots, hv_n^k \right]^{\mathrm{T}} \wedge 0 < \|y - x_k\| \leqslant \rho \wedge \angle\left(y - x_k, v^k\right) \leqslant \kappa/2 \right\}.\end{aligned}$$

Moreover, let $\widetilde{\Omega}_k \colon = A_k\left(\overline{\Omega} \cap \Phi_k\right)$ and $\widetilde{\Gamma}_k \colon = A_k\left(C_k \cap \Phi_k\right)$. Then,

$$\widetilde{\Omega}_k = \left\{ te_1 + he_n \in \mathbb{R}^n \colon t \in \mathbb{R} \wedge h \geqslant \frac{g(t)}{v_n^k} \sqrt{1 - \left(v_1^k\right)^2} \right\},$$

and

$$\begin{aligned}&\widetilde{\Gamma}_k = \{\tau_k e_1\} \\ &\cup \left\{ y \in \mathbb{R}^n \colon \exists_{t,h \in \mathbb{R}}\ y = te_1 + he_n \wedge 0 < \left\| \left[t - \tau_k, h \right]^{\mathrm{T}} \right\| \leqslant \rho \wedge \angle\left(\left[t - \tau_k, h \right]^{\mathrm{T}}, \overline{v^k} \right) \leqslant \kappa/2 \right\},\end{aligned}$$

where $\overline{v^k} = \left[v_1^k, \sqrt{1 - (v_1^k)^2} \right]^{\mathrm{T}}$. Let

$$\Omega_k \colon = \Pi\left(\widetilde{\Omega}_k\right) = \left\{ (t,h) \in \mathbb{R}^2 \colon t \in \mathbb{R} \wedge h \geqslant \frac{g(t)}{v_n^k} \sqrt{1 - \left(v_1^k\right)^2} \right\},$$

and

$$\Gamma_k \colon = \Pi\left(\widetilde{\Gamma}_k\right) = C_{(\tau_k, 0)}\left(\overline{v^k}, \kappa, \rho\right),$$

where the second equality means, that Γ_k is a 2-D cone of height ρ, axis direction $\overline{v^k}$ and aperture angle κ with vertex at $(\tau_k, 0)$. Since $C_k \subset \overline{\Omega}$, we have $\Gamma_k \subset \Omega_k$.

Let $M_k := \sqrt{\left(r_k/2 + d_k\right)^2 + \left(1 - \left(v_1^k\right)^2\right) \frac{h_k^2}{\left(v_n^k\right)^2}}$. Since $r_k, h_k, d_k \overset{k \to \infty}{\to} 0$ and $\frac{\sqrt{1 - \left(v_1^k\right)^2}}{v_n^k} \leqslant \frac{1}{v_n^k} \leqslant \frac{1}{\sin \kappa/2}$, for $k \in \mathbb{N}$ large enough, we have $M_k < \rho$. For such $k \in \mathbb{N}$, by an elementary geometry (see Figure 1) we see that

$$\beta_k + \gamma_k + \kappa \leqslant \pi, \tag{4.4}$$

where

$$\beta_k = \arctan\left(\frac{\sqrt{1-\left(v_1^k\right)^2}}{v_n^k}\cdot\frac{h_k}{r_k/2+d_k}\right)$$

and

$$\gamma_k = \arctan\left(\frac{\sqrt{1-\left(v_1^k\right)^2}}{v_n^k}\cdot\frac{h_{k+1}}{r_k/2+d_{k+1}}\right).$$

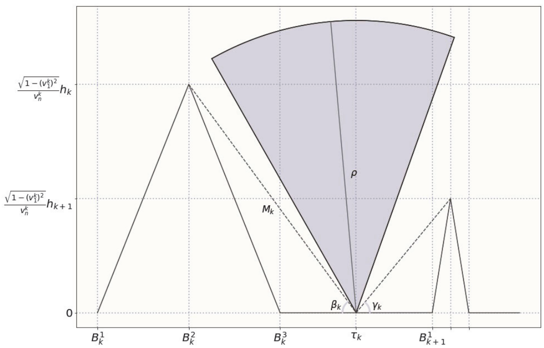

Figure 1: *Possible position of the cone* Γ_k.

Using the following inequalities

$$\frac{\sqrt{1-\left(v_1^k\right)^2}}{v_n^k} \leqslant \frac{1}{v_n^k} \leqslant \frac{1}{\sin\kappa/2},$$

$$\frac{\sqrt{1-\left(v_1^k\right)^2}}{v_n^k} \geqslant \frac{\sqrt{(v_n^k)^2}}{v_n^k} = 1,$$

we get

$$\arctan\left(\frac{h_k}{r_k/2+d_k}\right) \leqslant \beta_k \leqslant \arctan\left(\frac{1}{\sin\kappa/2}\cdot\frac{h_k}{r_k/2+d_k}\right),$$

$$\arctan\left(\frac{h_{k+1}}{r_k/2+d_{k+1}}\right) \leqslant \gamma_k \leqslant \arctan\left(\frac{1}{\sin\kappa/2}\cdot\frac{h_{k+1}}{r_k/2+d_{k+1}}\right).$$

Therefore, since

$$\frac{h_{k+1}}{r_k/2+d_{k+1}} \xrightarrow{k\to\infty} \infty \qquad \text{and} \qquad \frac{h_k}{r_k/2+d_k} \xrightarrow{k\to\infty} \infty,$$

we have $\beta_k \to \pi/2$ and $\gamma_k \to \pi/2$. Finally, exploring the above converges in (4.4) we get $\kappa \leqslant 0$. This is an obvious contradiction with assumption $\kappa > 0$. $\square$

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