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SHARP WEIGHTED HÖLDER MEAN BOUNDS FOR SEIFFERT'S MEANS

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Abstract. Let $P(a,b)$ and $T(a,b)$ be the first and second Seiffert's means for two positive numbers a and b, in this paper, for any fixed $p \in \mathbb{R}$, we present the optimal parameters $\alpha_p, \beta_p, \lambda_p, \mu_p \in [0,1]$ such that the inequalities

$$H_p(a,b;\alpha_p) \leqslant P(a,b) \leqslant H_p(a,b;\beta_p), \qquad H_p(a,b;\lambda_p) \leqslant T(a,b) \leqslant H_p(a,b;\mu_p)$$

hold true for all $a,b>0$, where $H_p(a,b;\omega)$ is the weighted p-order Hölder (power) mean with the weight $\omega \in [0,1]$. As applications, various sharp inequalities for $P(a,b)$ and $T(a,b)$, including the sharp power mean bounds, will be established.

1. Introduction

Throughout this paper, we denote by $\mathbb{R}^*$ the set of all non-zero real numbers. For $a,b>0$, the first and second Seiffert's means of a and b are respectively defined by

$$P := P(a,b) = \begin{cases} \dfrac{a-b}{2\arcsin\left(\dfrac{a-b}{a+b}\right)}, & a \neq b, \\ a, & a = b \end{cases}$$

and

$$T := T(a,b) = \begin{cases} \dfrac{a-b}{2\arctan\left(\dfrac{a-b}{a+b}\right)}, & a \neq b, \\ a, & a = b \end{cases}$$

(cf. [2, 10, 11, 15, 16]).

In the past twenty years, $P(a,b)$ and $T(a,b)$ have been the hot research topic in the theory of means. By comparing the Seiffert's means with the other classical means, or bounding some combinations of the Seiffert's means and other means, many remarkable inequalities for $P(a,b)$ and $T(a,b)$ have been established in [4, 5, 6, 7, 8, 9, 18, 19, 21, 25]. For instance, A, G, H, Q and H_p stand for the arithmetic, geometric,

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harmonic, root-square and Hölder means of two positive numbers a and b respectively, which are given by

$$A(a,b)=\frac{a+b}{2},\quad G(a,b)=\sqrt{ab},\quad H(a,b)=\frac{2ab}{a+b},\quad Q(a,b)=\sqrt{\frac{a^2+b^2}{2}}$$

and

$$H_p(a,b)=\begin{cases}\left(\frac{a^p+b^p}{2}\right)^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0.\end{cases}$$

As we know, the following chain of inequalities chain involving the above means

$$\begin{aligned}\min\{a,b\}&<H(a,b)=H_{-1}(a,b)<G(a,b)=H_0(a,b)<P(a,b)\\ &<A(a,b)=H_1(a,b)<T(a,b)<Q(a,b)=H_2(a,b)<\max\{a,b\}\end{aligned}$$

holds true for all $a,b>0$ with $a\neq b$.

In 2019, Chen and Sándor [4] solved the conjectures for $P(a,b)$ and $T(a,b)$ raised by Vukšić in [18, Conjecture 3.7], and obtained the inequalities

$$\begin{aligned}\frac{3G(a,b)+2T(a,b)}{5}&<P(a,b)<\frac{G(a,b)+T(a,b)}{2},\\ \frac{P(a,b)+3Q(a,b)}{4}&<T(a,b)<\frac{\pi-2\sqrt{2}}{\pi-\sqrt{2}}P(a,b)+\frac{\sqrt{2}}{\pi-\sqrt{2}}Q(a,b)\end{aligned}$$

for all $a,b>0$ with $a\neq b$. Moreover, the optimal estimates for $P(a,b)$ and $T(a,b)$ by the Hölder means $H_p(a,b)$ were obtained in [6, 7, 9]: for $a,b>0$ with $a\neq b$,

$$H_{\frac{\log 2}{\log\pi}}(a,b)<P(a,b)<H_{\frac{2}{3}}(a,b),\tag{1.1}$$

$$H_{\frac{\log 2}{\log(\pi/2)}}(a,b)<T(a,b)<H_{\frac{5}{3}}(a,b).\tag{1.2}$$

In 2015, Witkowski [22] extended the Seiffert's means to the Seiffert-like means $M_f(a,b)$, which are the means of the form

$$M_f(a,b)=\begin{cases}\dfrac{|a-b|}{2f\left(\frac{|a-b|}{a+b}\right)}, & a\neq b,\\ a, & a=b,\end{cases}\tag{1.3}$$

where $f:(0,1)\to\mathbb{R}$ is the so-called Seiffert function of $M_f(a,b)$ satisfying

$$\frac{z}{1+z}\leqslant f(z)\leqslant\frac{z}{1-z}.$$

In particular, $P(a,b)=M_{\arcsin}(a,b)$, $T(a,b)=M_{\arctan}(a,b)$. And the author also showed that every symmetric homogenous mean can be represented in the form (1.3), and thus a one-to-one correspondence between means and Seiffert functions was derived as follows

$$f(z)=\frac{z}{M(1-z,1+z)},\quad z=\frac{|a-b|}{a+b}.$$

Besides, four new Seiffert-like means $M_{\sin}(a,b)$, $M_{\sinh}(a,b)$, $M_{\tan}(a,b)$, $M_{\tanh}(a,b)$ were introduced and studied in [22]. Since then, a lot of various optimal bounds for these Seiffert-like means in terms of $A(a,b)$, $G(a,b)$, $H(a,b)$ have been established by Nowicka and Witkowski in [12, 13, 14], and Zhu and Malešević in [29, 30, 31].

The initial thread for this investigation begins with the following linear bounds in exponential type for two Seiffert-like means $M_{\tan}(a,b)$ and $M_{\sinh}(a,b)$. More precisely, for some fixed $p\in\mathbb{R}^*$, Zhu[29] determined the best constants such that

$$\alpha_pA^p(a,b)+(1-\alpha_p)H^p(a,b)<M_{\tan}^p(a,b)<\beta_pA^p(a,b)+(1-\beta_p)H^p(a,b),\quad (1.4)$$
$$\lambda_pA^p(a,b)+(1-\lambda_p)H^p(a,b)<M_{\sinh}^p(a,b)<\mu_pA^p(a,b)+(1-\mu_p)H^p(a,b)\quad (1.5)$$

are valid for all $a,b>0$ with $a\neq b$. The analogous bounds for $M_{\sin}(a,b)$ and $M_{\tanh}$ by arithmetic and centroidal means were also established in [30, 31]. In the light of (1.4) and (1.5), it is a matter of fact that for fixed $p\in\mathbb{R}^*$, the author find the optimal weights of $H_p(A,H;\omega)$ approximating to $M_{\tan}$ and $M_{\sinh}$, where

$$H_p(a,b;\omega)=\begin{cases}\left[\omega a^p+(1-\omega)b^p\right]^{1/p}, & p\neq 0,\\ a^{\omega}b^{1-\omega}, & p=0\end{cases}\quad (1.6)$$

is the weighted p-order Hölder mean of two positive numbers a and b with the weight $\omega\in[0,1]$.

Inspired by Zhu's work, it is natural to ask that, for any fixed $p\in\mathbb{R}$, what are the best parameters $\alpha=\alpha_p,\beta=\beta_p,\lambda=\lambda_p,\mu=\mu_p\in[0,1]$ depending on p such that the inequalities

$$H_p(a,b;\alpha)\leqslant P(a,b)\leqslant H_p(a,b;\beta)\quad\text{and}\quad H_p(a,b;\lambda)\leqslant T(a,b)\leqslant H_p(a,b;\mu)\quad (1.7)$$

hold for all $a,b>0$ with $a\neq b$. It is worth mentioning that $H_p(a,b;\omega)$ is non-symmetric mean of a and b unless $\omega=1/2$ and $\omega\mapsto H_p(a,b;\omega)$ is strictly increasing (decreasing) on $[0,1]$ for arbitrarily given $p\in\mathbb{R}$ and $a>b>0$ ($b>a>0$). In particular, $H_p(a,b;1/2)=H_p(a,b)$ is the Hölder mean and $H_p(a,b;\omega)=H_p(b,a;1-\omega)$. For some basic properties for the (weighted) Hölder mean, and recent research results see [2, 3, 20, 26, 27, 28]. According to this, once we find the best parameters $\alpha_p,\beta_p,\lambda_p$ and μ_p such that the inequalities (1.7) hold for $a>b>0$, then the best parameters making (1.7) hold for $b>a>0$ should be $1-\alpha_p$, $1-\beta_p$, $1-\lambda_p$ and $1-\mu_p$. In other words, it suffices to focus on the case of $a>b>0$.

The aim of this paper is to solve the above question in the case of $a>b>0$, and our main theorems are stated in Section 3. As applications, several sharp inequalities for the Seiffert's means $P(a,b)$ and $T(a,b)$, including (1.1) and (1.2), will be given.

2. Preliminaries

LEMMA 2.1. ([1, Theorem 1.25]) *Let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b) for $-\infty<a<b<\infty$. If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strict monotone, then the monotonicity in the conclusion is also strict.

However, f'/g' is not always monotone in the whole interval but piecewise monotone. Now we introduce a useful auxiliary function $H_{f,g}$, which is called H-function (cf. [17, 24]) and makes a bridge between the derivatives of the ratios f/g and f'/g'. For $-\infty\leqslant a<b\leqslant\infty$, let f and g be differentiable on (a,b) and $g'\neq 0$ on (a,b). Then the function $H_{f,g}$ is defined by

$$H_{f,g}:=\frac{f'}{g'}g-f. \tag{2.1}$$

For some basic properties of $H_{f,g}$, see [23, Properties 1,2]. In particular, if f and g are twice differentiable on (a,b), then we have

$$\left(\frac{f}{g}\right)'=\frac{g'}{g^2}\left(\frac{f'}{g'}g-f\right)=\frac{g'}{g^2}H_{f,g}, \tag{2.2}$$

$$H'_{f,g}=\left(\frac{f'}{g'}\right)'g. \tag{2.3}$$

LEMMA 2.2. *Let $x\in(0,1)$. Then the function*

$$h(x)=\frac{x\left[(2-x^2)\arcsin x-2x\sqrt{1-x^2}\right]}{2x\arcsin x-(x^2+\arcsin^2 x)\sqrt{1-x^2}}$$

is strictly increasing from $(0,1)$ onto $(1/3,1/2)$.

Proof. Let

$$h_1(x)=\frac{x\left[(2-x^2)\arcsin x-2x\sqrt{1-x^2}\right]}{\sqrt{1-x^2}},\quad h_2(x)=\frac{2x\arcsin x}{\sqrt{1-x^2}}-(x^2+\arcsin^2 x),$$

$$h_3(x)=\frac{(2x^4-3x^2+2)\arcsin x+(3x^2-2)x\sqrt{1-x^2}}{2x^2},\quad h_4(x)=\arcsin x+x\sqrt{1-x^2}.$$

Then it can be verified that

$$h(x)=\frac{h_1(x)}{h_2(x)},\quad h_1(0)=h_2(0)=0 \tag{2.4}$$

and

$$\frac{h_1'(x)}{h_2'(x)} = \frac{(2x^4-3x^2+2)\arcsin x + (3x^2-2)x\sqrt{1-x^2}}{2x^2\left(\arcsin x + x\sqrt{1-x^2}\right)} = \frac{h_3(x)}{h_4(x)},$$
$$h_3(0^+) = h_4(0) = 0. \tag{2.5}$$

Moreover, we can also obtain by differentiation that

$$\left[\frac{h_3'(x)}{h_4'(x)}\right]' = \left[\frac{(1+x^2)(x-\sqrt{1-x^2}\arcsin x)}{x^3}\right]' = \frac{(3-x^2)}{x^4\sqrt{1-x^2}}h_5(x), \tag{2.6}$$

where

$$h_5(x) = \arcsin x - \frac{x(x^2+3)\sqrt{1-x^2}}{3-x^2}.$$

It can be easily seen that $h_5(x) > 0$ for all $x \in (0,1)$, due to

$$h_5(0) = 0, \quad h_5'(x) = \frac{2x^4(7-x^2)}{\sqrt{1-x^2}(3-x^2)^2} > 0.$$

This together with (2.6) implies that $h_3'(x)/h_4'(x)$ is strictly increasing on $(0,1)$ and so is $h_1'(x)/h_2'(x)$ by (2.5) and Lemma 2.1. According to this with (2.4), it follows from Lemma 2.1 that $h(x)$ is strictly increasing on $(0,1)$. For the limiting values, it is easy to see that $h(1^-) = 1/2$ and by l'Hôpital rule,

$$h(0^+) = \lim_{x\to 0^+}\frac{h_3'(x)}{h_4'(x)} = \lim_{x\to 0^+}\frac{x-\sqrt{1-x^2}\arcsin x}{x^3} = \lim_{x\to 0^+}\frac{x\arcsin x}{3x^2\sqrt{1-x^2}} = \frac{1}{3}.$$

This completes the proof. □

LEMMA 2.3. *Let $x \in (0,1)$. Then the function*

$$g(x) = \frac{x^5-4x^3+3x+(x^6+7x^4+3x^2-3)\arctan x}{x^2(1+x^2)[x-(1-x^2)\arctan x]}$$

is strictly decreasing from $(0,1)$ onto $(\pi, 24/5)$.

Proof. Let

$$g_1(x) = \frac{x^5-4x^3+3x+(x^6+7x^4+3x^2-3)\arctan x}{x^2(1+x^2)(1-x^2)}, \quad g_2(x) = \frac{x}{1-x^2} - \arctan x,$$

$$g_3(x) = \arctan x - \frac{x(9x^6-11x^4-x^2+3)}{(7x^4-6x^2+3)(1+x^2)^2}, \quad g_4(x) = \frac{2x^5}{(7x^4-6x^2+3)(1+x^2)}.$$

Then it can be easily verified that

$$g(x) = \frac{g_1(x)}{g_2(x)}, \quad g_1(0) = g_2(0) = 0 \tag{2.7}$$

and

$$\frac{g_1'(x)}{g_2'(x)}=\frac{(7x^4-6x^2+3)(1+x^2)^2\arctan x-x(9x^6-11x^4-x^2+3)}{2x^5(1+x^2)}=\frac{g_3(x)}{g_4(x)}, \quad (2.8)$$
$$g_3(0)=g_4(0)=0.$$

By differentiation, we obtain

$$g_3'(x)=\frac{16x^4(1-x^2)(9-x^2+15x^4-7x^6)}{(1+x^2)^3(7x^4-6x^2+3)^2},\ g_4'(x)=\frac{2x^4(1-x^2)(7x^4+6x^2+15)}{(1+x^2)^2(7x^4-6x^2+3)^2}$$

and thereby

$$\left[\frac{g_3'(x)}{g_4'(x)}\right]'=\left[\frac{8(9-x^2+15x^4-7x^6)}{(1+x^2)(7x^4+6x^2+15)}\right]'=-\frac{64x(7x^4-6x^2+3)(7x^4+16x^2+17)}{(1+x^2)^2(7x^4+6x^2+15)^2}<0$$

for all $x\in(0,1)$. This gives $g_3'(x)/g_4'(x)$ is strictly decreasing on $(0,1)$. Combining this with Lemma 2.1, it follows from (2.7) and (2.8) that $g(x)$ is strictly decreasing on $(0,1)$. Moreover, $g(1^-)=\pi$, and by l'Hôpital's rule,

$$g(0^+)=\lim_{x\to0^+}\frac{g_3'(x)}{g_4'(x)}=\lim_{x\to0^+}\frac{8(9-x^2+15x^4-7x^6)}{(1+x^2)(7x^4+6x^2+15)}=\frac{24}{5}.$$

This completes the proof. □

LEMMA 2.4. *Let $x\in(0,1)$. Then the function*

$$f(x)=\frac{2x(1-x^2)(x-\arctan x)}{[x(1+x)-(1+x^2)\arctan x]\,[(1+x^2)\arctan x-x(1-x)]}$$

is strictly decreasing from $(0,1)$ onto $(0,2/3)$.

Proof. Let

$$f_1(x)=\frac{2x(1-x^2)(x-\arctan x)}{(1+x^2)^2},\quad f_2(x)=-\arctan^2 x+\frac{2x\arctan x}{1+x^2}-\frac{x^2(1-x^2)}{(1+x^2)^2},$$
$$f_3(x)=\frac{(x^4-6x^2+1)\arctan x+5x^3-x}{2x^3},\quad f_4(x)=\frac{(1+x^2)\arctan x}{x}-2.$$

Then we clearly see that

$$f(x)=\frac{f_1(x)}{f_2(x)},\quad f_1(0)=f_2(0)=0, \quad (2.9)$$

and

$$\frac{f_1'(x)}{f_2'(x)}=\frac{(x^4-6x^2+1)\arctan x+5x^3-x}{2x^2[(1+x^2)\arctan x-2x]}=\frac{f_3(x)}{f_4(x)}. \quad (2.10)$$

Further, by differentiation,

$$f_3'(x) = \frac{x^5 - 4x^3 + 3x + (x^6 + 7x^4 + 3x^2 - 3)\arctan x}{2x^4(1+x^2)}, \quad f_4'(x) = \frac{x - (1-x^2)\arctan x}{x^2}$$

and thereby

$$\frac{f_3'(x)}{f_4'(x)} = \frac{x^5 - 4x^3 + 3x + (x^6 + 7x^4 + 3x^2 - 3)\arctan x}{2x^2(1+x^2)[x - (1-x^2)\arctan x]} = \frac{g(x)}{2},$$

where $g(x)$ is defined in Lemma 2.3. It is easy to see that $f_4'(x) > 0$ for all $x \in (0,1)$ due to

$$\frac{d}{dx}\left[\frac{x}{1-x^2} - \arctan x\right] = \frac{4x^2}{(1-x^2)^2(1+x^2)} > 0$$

and so $f_4(x)$ is strictly increasing for $x \in (0,1)$ and $f_4(x) < f_4(1) = \pi/2 - 2 < 0$. This together with Lemma 2.3, (2.1) and (2.3) implies that

$$H'_{f_3,f_4}(x) = \left[\frac{f_3'(x)}{f_4'(x)}\right]' f_4(x) = \frac{g'(x)}{2} f_4(x) > 0$$

for all $x \in (0,1)$. That is to say, $H_{f_3,f_4}(x)$ is strictly increasing on $(0,1)$ and thereby

$$\begin{aligned} H_{f_3,f_4}(x) < H_{f_3,f_4}(1^-) &= \lim_{x\to 1^-}\left[\frac{f_3'(x)}{f_4'(x)} f_4(x) - f_3(x)\right] \\ &= \lim_{x\to 1^-}\left[\frac{g(x) f_4(x)}{2} - f_3(x)\right] \\ &= -\frac{(\pi+2)(4-\pi)}{4} < 0. \end{aligned}$$

By (2.2), for all $x \in (0,1)$,

$$\left[\frac{f_3(x)}{f_4(x)}\right]' = \frac{f_4'(x)}{f_4^2(x)} H_{f_3,f_4}(x) < 0.$$

Combining this with (2.9), (2.10) and Lemma 2.1, we deduce the monotonicity of $f(x)$ on $(0,1)$. Moreover, clearly $f(1) = 0$, and by l'Hôpital's rule,

$$f(0^+) = \lim_{x\to 0^+} \frac{f_1'(x)}{f_2'(x)} = \lim_{x\to 0^+} \frac{(x^2-6)\dfrac{\arctan x}{x} + 5 - \dfrac{x - \arctan x}{x^3}}{2\left[(1+x^2)\dfrac{\arctan x}{x} - 2\right]} = \frac{2}{3}.$$

This completes the proof. □

3. Main results

Before proving the main results, we first show the following monotonicity of theorems.

THEOREM 3.1. *Let* $p \in \mathbb{R}^*$ *and define the function* $x \mapsto F_p$ *on* $(0,1)$ *by*

$$F_p(x) = \frac{(1+x)^p - (x/\arcsin x)^p}{(1+x)^p - (1-x)^p}. \tag{3.1}$$

Then the following statements hold true:

(1) If $p \geqslant 2/3$*, then* F_p *is strictly increasing from* $(0,1)$ *onto* $(1/2, 1-(1/\pi)^p)$*;*

(2) If $p \leqslant 1/2$*, then* F_p *is strictly decreasing on* $(0,1)$*. Furthermore, in this case,* $F_p(0^+) = 1/2$ *and* $F_p(1^-) = 1-(1/\pi)^p$ *if* $0 < p \leqslant 1/2$ *and* $F_p(1^-) = 0$ *if* $p < 0$*;*

(3) If $1/2 < p < 2/3$*, then there exists a unique point* $x_0^* = x_0^*(p) \in (0,1)$ *such that* F_p *is strictly decreasing on* $(0, x_0^*)$*, and strictly increasing on* $(x_0^*, 1)$*. Consequently, inequalities*

$$\sigma_0^* \leqslant F_p(x) < 1/2, \qquad \text{if } p \in (1/2, p_0], \tag{3.2}$$
$$\sigma_0^* \leqslant F_p(x) < 1-(1/\pi)^p, \quad \text{if } p \in (p_0, 2/3) \tag{3.3}$$

hold for all $x \in (0,1)$*, where* $p_0 = \log 2/\log \pi$ *and* $\sigma_0^* = F_p(x_0^*) < \min\{1/2, 1-(1/\pi)^p\}$*. The right-hand side of* (3.2) *(res.* (3.3)*) can be arrived at as* $r \to 0^+$ *(*$r \to 1^-$*, res.).*

Proof. Let

$$\phi_1(x) = 1 - \left[\frac{x}{(1+x)\arcsin x}\right]^p, \quad \phi_2(x) = 1 - \left(\frac{1-x}{1+x}\right)^p.$$

Then we clearly see that

$$F_p(x) = \frac{\phi_1(x)}{\phi_2(x)}, \quad \phi_1(0^+) = \phi_2(0) = 0, \tag{3.4}$$
$$\phi_1'(x) = p\left[\frac{x}{(1+x)\arcsin x}\right]^{p-1} \frac{x(1+x) - \sqrt{1-x^2}\arcsin x}{\sqrt{1-x^2}(1+x)^2 \arcsin^2 x},$$
$$\phi_2'(x) = 2p\left(\frac{1-x}{1+x}\right)^{p-1} \frac{1}{(1+x)^2}$$

and thereby

$$\frac{\phi_1'(x)}{\phi_2'(x)} = \left[\frac{x}{(1-x)\arcsin x}\right]^{p-1} \frac{x(1+x) - \sqrt{1-x^2}\arcsin x}{2\sqrt{1-x^2}\arcsin^2 x} := \phi_3(x). \tag{3.5}$$

By logarithmic differentiation, we obtain

$$\begin{aligned}\frac{\phi_3'(x)}{\phi_3(x)} &= (p-1)\frac{\sqrt{1-x^2}\arcsin x - x(1-x)}{x(1-x)\sqrt{1-x^2}\arcsin x} \\ &\quad + \frac{(2-x^2)\arcsin x - 2x\sqrt{1-x^2}}{(1-x)\arcsin x[x(1+x)-\sqrt{1-x^2}\arcsin x]} \\ &= \frac{\sqrt{1-x^2}\arcsin x - x(1-x)}{x(1-x)\sqrt{1-x^2}\arcsin x}\big[p-1+h(x)\big], \end{aligned} \tag{3.6}$$

where $h(x)$ is defined in Lemma 2.2.

We now divide into three cases to complete the proof.

Case 1.1 $p \geqslant 2/3$. It is easy to verify that the function $x \mapsto \sqrt{1-x^2}\arcsin x - x(1-x)$ is positive on $(0,1)$. This together with (3.6) and Lemma 2.2 leads to the conclusion that $\phi_3'(x) > 0$ for all $x \in (0,1)$ and so $\phi_3(x)$, as well as $\phi_1'(x)/\phi_2'(x)$, is strictly increasing on $(0,1)$ due to (3.5). Hence the monotonicity of F_p follows from (3.4) and Lemma 2.1. Moreover, $F_p(1^-) = 1-(1/\pi)^p$, and

$$\begin{aligned} F_p(0^+) &= \lim_{x\to 0^+}\frac{\phi_1'(x)}{\phi_2'(x)} = \lim_{x\to 0^+}\frac{x(1+x)-\sqrt{1-x^2}\arcsin x}{2x^2} \\ &= \lim_{x\to 0^+}\left[\frac{1}{2}+\frac{\arcsin x}{4\sqrt{1-x^2}}\right] = \frac{1}{2}. \end{aligned}$$

Case 1.2 $p \leqslant 1/2$. In this case, it follows from (3.6) and Lemma 2.2 that $\phi_3'(x) < 0$ for all $x \in (0,1)$, and therefore $\phi_3(x)$ is strictly decreasing on $(0,1)$ and so is $F_p(x)$ by (3.4), (3.5) and application of Lemma 2.1. Similarly, $F_p(0^+) = 1/2$, while $F_p(1^-) = 1-(1/\pi)^p$ if $0 < p \leqslant 1/2$ and $F_p(1^-) = 0$ if $p < 0$.

Case 1.3 $1/2 < p < 2/3$. Simple calculations with (2.2) and (2.3) yield

$$F_p'(x) = \left[\frac{\phi_1(x)}{\phi_2(x)}\right]' = \frac{\phi_2'(x)}{\phi_2^2(x)} H_{\phi_1,\phi_2}(x), \tag{3.7}$$

$$H_{\phi_1,\phi_2}'(x) = \left[\frac{\phi_1'(x)}{\phi_2'(x)}\right]' \phi_2(x) = \phi_3'(x)\phi_2(x). \tag{3.8}$$

Moreover, in this case, by (3.5),

$$\lim_{x\to 0^+}\frac{\phi_1'(x)}{\phi_2'(x)} = \frac{1}{2} \quad \text{and} \quad \lim_{x\to 1^-}\frac{\phi_1'(x)}{\phi_2'(x)} = \lim_{x\to 1^-}\frac{(2/\pi)^{p+1}}{\sqrt{2}(1-x)^{p-1/2}} = +\infty,$$

which together with (2.1) and (3.4) gives

$$H_{\phi_1,\phi_2}(0^+) = 0, \quad H_{\phi_1,\phi_2}(1^-) = +\infty. \tag{3.9}$$

By (3.6) and Lemma 2.2, we conclude that there exists $x_0 \in (0,1)$ such that $\phi_3'(x) < 0$ for $x \in (0,x_0)$ and $\phi_3'(x) > 0$ for $x \in (x_0,1)$. Since $\phi_2(x)$ is strictly increasing and positive on $(0,1)$ for $1/2 < p < 2/3$, it follows from (3.8) that $H_{\phi_1,\phi_2}(x)$ is strictly decreasing on $(0,x_0)$ and strictly increasing on $(x_0,1)$, and so there exists $x_0^* \in (x_0,1)$ such that $H_{\phi_1,\phi_2}(x) < 0$ for $x \in (0,x_0^*)$ and $H_{\phi_1,\phi_2}(x) > 0$ for $x \in (x_0^*,1)$ by (3.9). Hence F_p is also first decreasing then increasing due to (3.7). Consequently, the inequality

$$\sigma_0^* := F_p(x_0^*) \leqslant F_p(x) < \max\{F_p(0^+), F_p(1^-)\} = \max\{1/2, 1-(1/\pi)^p\}$$

takes place for all $x \in (0,1)$. It was observed that $p \mapsto 1-(1/\pi)^p$ is strictly increasing for $p \in \mathbb{R}$ and so the equation $1-(1/\pi)^p = 1/2$ has a unique root $p_0 = \log 2/\log \pi = 0.6055\cdots$ on $(1/2,2/3)$. In other words, $F_{p_0}(0^+) = F_{p_0}(1^-) = 1/2$ and $F_p(0^+) < F_p(1^-)$ if $p \in (1/2,p_0)$ and $F_p(0^+) < F_p(1^-)$ if $p \in (p_0,2/3)$, and therefore inequalities (3.2) and (3.3) follows. The remaining assertions in Theorem 3.1 are clear. □

THEOREM 3.2. *Let $p \in \mathbb{R}^*$ and define the function $x \mapsto G_p$ on $(0,1)$ by*

$$G_p(x) = \frac{(1+x)^p - (x/\arctan x)^p}{(1+x)^p - (1-x)^p}. \tag{3.10}$$

Then the following statements hold

(1) If $p \geqslant 5/3$, then G_p is strictly increasing from $(0,1)$ onto $(1/2, 1-(2/\pi)^p)$;

(2) If $p \leqslant 1$, then G_p is strictly decreasing on $(0,1)$. Furthermore, in this case $G_p(0^+) = 1/2$, and $G_p(1^-) = 1-(2/\pi)^p$ if $0 < p \leqslant 1$ and $G_p(1^-) = 0$ if $p < 0$;

(3) If $1 < p < 5/3$, then there exists a unique $x_1^* = x_1^*(p) \in (0,1)$ such that G_p is strictly decreasing on $(0,x_1^*)$, and strictly increasing on $(x_1^*,1)$. Consequently, inequalities

$$\sigma_1^* \leqslant G_p(x) < 1/2, \qquad \text{if } p \in (1,p_1], \tag{3.11}$$
$$\sigma_1^* \leqslant G_p(x) < 1-(2/\pi)^p, \quad \text{if } p \in (p_1,5/3) \tag{3.12}$$

hold for all $x \in (0,1)$, where $p_1 = \log 2/[\log(\pi/2)]$ and $\sigma_1^* = G_p(x_1^*) < \min\{1/2, 1-(2/\pi)^p\}$. The right-hand side of (3.11) (res. (3.12)) can be arrived at as $r \to 0^+$ ($r \to 1^-$, respectively).

Proof. Let

$$\varphi_1(x) = 1 - \left[\frac{x}{(1+x)\arctan x}\right]^p, \quad \varphi_2(x) = 1 - \left(\frac{1-x}{1+x}\right)^p.$$

Then it is easy to see that

$$G_p(x) = \frac{\varphi_1(x)}{\varphi_2(x)}, \quad \varphi_1(0^+) = \varphi_2(0) = 0, \tag{3.13}$$

$$\varphi_1'(x) = p\left[\frac{x}{(1+x)\arctan x}\right]^{p-1} \frac{x(1+x)-(1+x^2)\arctan x}{(1+x^2)(1+x)^2\arctan^2 x},$$

$$\varphi_2'(x) = 2p\left(\frac{1-x}{1+x}\right)^{p-1} \frac{1}{(1+x)^2}$$

and thereby

$$\frac{\varphi_1'(x)}{\varphi_2'(x)} = \left[\frac{x}{(1-x)\arctan x}\right]^{p-1} \frac{x(1+x)-(1+x^2)\arctan x}{2(1+x^2)\arctan^2 x} := \varphi_3(x). \tag{3.14}$$

Logarithmic differentiation of φ_3 leads to

$$\begin{aligned}\frac{\varphi_3'(x)}{\varphi_3(x)} =&(p-1)\left[\frac{1}{x}+\frac{1}{1-x}-\frac{1}{(1+x^2)\arctan x}\right]+\frac{2x-2x\arctan x}{x(1+x)-(1+x^2)\arctan x}\\ &-\frac{2x}{1+x^2}-\frac{2}{(1+x^2)\arctan x}\\ =&(p-1)\frac{(1+x^2)\arctan x - x(1-x)}{x(1-x)(1+x^2)\arctan x}\\ &+\frac{2(1+x)(\arctan x - x)}{(1+x^2)\arctan x[x(1+x)-(1+x^2)\arctan x]}\\ =&\frac{(1+x^2)\arctan x - x(1-x)}{x(1-x)(1+x^2)\arctan x}\left[p-1-f(x)\right],\end{aligned} \tag{3.15}$$

where $f(x)$ is defined as in Lemma 2.4.

Following we also divide the proof into three cases.

Case 2.1 $p \geqslant 5/3$. Since it can be easily known that the function $x \mapsto (1+x^2)\arctan x - x(1-x)$ is strictly increasing and positive on $(0,1)$, we conclude from (3.15) and Lemma 2.4 that $\varphi_3'(x) > 0$ for all $x \in (0,1)$. Hence $\varphi_3(x)$, as well as $\varphi_1'(x)/\varphi_2'(x)$, is strictly increasing on $(0,1)$ due to (3.14). Therefore, the monotonicity of G_p directly follows from (3.13) and Lemma 2.1. Moreover, $G_p(1^-) = 1-(2/\pi)^p$, and

$$\begin{aligned}G_p(0^+) &= \lim_{x\to 0^+}\frac{\varphi_1'(x)}{\varphi_2'(x)} = \lim_{x\to 0^+}\frac{x(1+x)-(1+x^2)\arctan x}{2x^2}\\ &= \lim_{x\to 0^+}\frac{1-\arctan x}{2} = \frac{1}{2}.\end{aligned}$$

Case 2.2 $p \leqslant 1$. In this case, it follows from (3.15) and Lemma 2.4 that $\varphi_3'(x) < 0$ for all $x \in (0,1)$, and therefore $\varphi_3(x)$ is strictly decreasing on $(0,1)$. So is $G_p(x)$ by applying Lemma 2.1 together with (3.13) and (3.14). Further, in this case we have $G_p(0^+) = 1/2$, and while $G_p(1^-) = 1-(2/\pi)^p$ if $0 < p \leqslant 1$ and $G_p(1^-) = 0$ if $p < 0$.

Case 2.3 $1 < p < 5/3$. In this case, by making use of (2.1), (2.2) and (2.3), it is a matter of simple computations that

$$G_p'(x) = \left[\frac{\varphi_1(x)}{\varphi_2(x)}\right]' = \frac{\varphi_2'(x)}{\varphi_2^2(x)} H_{\varphi_1,\varphi_2}(x), \tag{3.16}$$

$$H_{\varphi_1,\varphi_2}'(x) = \left[\frac{\varphi_1'(x)}{\varphi_2'(x)}\right]' \varphi_2(x) = \varphi_3'(x)\varphi_2(x), \tag{3.17}$$

$$H_{\varphi_1,\varphi_2}(0^+) = 0, \quad H_{\varphi_1,\varphi_2}(1^-) = +\infty. \tag{3.18}$$

It is apparent from Lemma 2.4 and (3.15) that there exists $x_1 \in (0,1)$ such that $\varphi_3'(x) < 0$ for $x \in (0,x_1)$ and $\varphi_3'(x) > 0$ for $x \in (x_1,1)$. Note that, for $1 < p < 5/3$, $\varphi_2(x)$ is strictly increasing and positive on $(0,1)$. Then from (3.17) and (3.18) it follows that there exists $x_1^* \in (x_1,1)$ such that $H_{\varphi_1,\varphi_2}(x) < 0$ for $x \in (0,x_1^*)$ and $H_{\varphi_1,\varphi_2}(x) > 0$ for $x \in (x_1^*,1)$, so that G_p is piecewise monotone on $(0,1)$, first decreasing then increasing due to (3.16). Consequently, the inequality

$$\sigma_1^* := G_p(x_1^*) \leqslant G_p(x) < \max\{G_p(0^+), G_p(1^-)\} = \max\{1/2, 1-(2/\pi)^p\}$$

is valid for all $x \in (0,1)$. Observed that $p \mapsto 1-(2/\pi)^p$ is strictly increasing on $(-\infty,+\infty)$ and the equation $1-(2/\pi)^p = 1/2$ has a unique root $p_1 = \log 2/\log(\pi/2) = 1.534\cdots$ on $(1,5/3)$. This implies that $G_{p_1}(0^+) = G_{p_1}(1^-) = 1/2$, $G_p(0^+) \geqslant G_p(1^-)$ if $p \in (1,p_1]$ and $G_p(0^+) < G_p(1^-)$ if $p \in (p_1,5/3)$, and therefore inequalities (3.11) and (3.12) follows. The remaining asserted results in Theorem 3.2 can be derived easily. □

We are now in a position to give the proofs of our main results.

THEOREM 3.3. *Let $\alpha,\beta \in (0,1)$, $p_0 = \log 2/\log\pi = 0.6055\cdots$, $F_p(x)$ and σ_0^* be defined in Theorem 3.1. Then for each fixed $p \in \mathbb{R}$, the inequality*

$$H_p(a,b;\alpha) < P(a,b) \leqslant H_p(a,b;\beta) \tag{3.19}$$

holds for all $a > b > 0$ if and only if $\alpha \leqslant \alpha^*(p)$ and $\beta \geqslant \beta^*(p)$, where

$$\alpha^*(p) = \begin{cases} 1/2, & p \in (-\infty, p_0], \\ (1/\pi)^p, & p \in (p_0,\infty) \end{cases} \quad \text{and} \quad \beta^*(p) = \begin{cases} 1, & p \in (-\infty,0], \\ (1/\pi)^p, & p \in (0,1/2], \\ 1-\sigma_0^*, & p \in (1/2,2/3), \\ 1/2, & p \in [2/3,\infty). \end{cases} \tag{3.20}$$

In particular, the equality of (3.19) holds only for $p \in (1/2,2/3)$, $\beta = \beta^*(p)$ and some (a,b) satisfying $F_p'\left(\frac{a-b}{a+b}\right) = 0$.

Proof. Since both $P(a,b)$ and $H_p(a,b;\omega)$ are homogenous of degree one means of a and b, without loss of generality, we may assume that $a = 1+x > b = 1-x$ for $x \in (0,1)$. In the following we divide into two cases $p = 0$ and $p \neq 0$ to complete the proof.

Case 3.1 $p = 0$. In this case, the inequality (3.19) can be written as

$$\alpha < 1 - \frac{\log\left[\dfrac{x}{(1+x)\arcsin x}\right]}{\log\left(\dfrac{1-x}{1+x}\right)} \leqslant \beta, \qquad x \in (0,1). \tag{3.21}$$

We will show that the function $x \mapsto \log\left[\dfrac{x}{(1+x)\arcsin x}\right] \Big/ \log\left(\dfrac{1-x}{1+x}\right)$ is strictly decreasing from $(0,1)$ onto $(0,1/2)$, and thereby it can be taken the best constants as $\alpha^*(0) = 1/2$ and $\beta^*(0) = 1$.

In fact, we denote

$$\eta(x) = \frac{\log\left[\dfrac{x}{(1+x)\arcsin x}\right]}{\log\left(\dfrac{1-x}{1+x}\right)} \triangleq \frac{\eta_1(x)}{\eta_2(x)}, \quad \text{for} \quad x \in (0,1).$$

Then we clearly see that $\eta_1(0^+) = \eta_2(0^+) = 0$ and

$$\frac{\eta_1'(x)}{\eta_2'(x)} = \left[\frac{x}{(1-x)\arcsin x}\right]^{-1} \frac{x(1+x) - \sqrt{1-x^2}\arcsin x}{2\sqrt{1-x^2}\arcsin^2 x},$$

which is (3.5) in the case of $p = 0$. It follows from the *Case 1.2* of Theorem 3.1 that $\eta_1'(x)/\eta_2'(x)$ is strictly decreasing on $(0,1)$ and so is $\eta(x)$ by Lemma 2.1. Moreover, $\eta(1^-) = 0$, and by l'Hôpital's rule,

$$\eta(0^+) = \lim_{x\to 0^+} \frac{\eta_1'(x)}{\eta_2'(x)} = \lim_{x\to 0^+} \frac{x(1+x) - \sqrt{1-x^2}\arcsin x}{2x^2} = \frac{1}{2}.$$

Case 3.2 $p \neq 0$. In this case, it is a matter of simple transformation that the inequality (3.19) can be rewritten as

$$\alpha < 1 - F_p(x) \leqslant \beta \tag{3.22}$$

for $x \in (0,1)$, where $F_p(x)$ is defined in (3.1). Hence, the best constants $\alpha^*(p)$ and $\beta^*(p)$ in (3.20), and their sharpness can be derived immediately by Theorem 3.1 and (3.22). In particular, when $p \in (1/2, 2/3)$, $F_p(x)$ attains a unique minimal value on $(0,1)$. In other words, the right-side equality of (3.22) holds when $p \in (1/2, 2/3)$ and $F_p'(x) = 0$. □

THEOREM 3.4. *Let $\lambda, \mu \in (0,1)$, $p_1 = \log 2/\log(\pi/2) = 1.534\cdots$, $G_p(x)$ and σ_1^* be defined in Theorem 3.2. Then for each fixed $p \in \mathbb{R}$, the inequality*

$$H_p(a,b;\lambda) < T(a,b) \leqslant H_p(a,b;\mu) \tag{3.23}$$

holds for all $a > b > 0$ if and only if $\lambda \leqslant \lambda^*(p)$ and $\mu \geqslant \mu^*(p)$, where

$$\lambda^*(p) = \begin{cases} 1/2, & p \in (-\infty, p_1], \\ (2/\pi)^p, & p \in (p_1, \infty), \end{cases} \quad \mu^*(p) = \begin{cases} 1, & p \in (-\infty, 0], \\ (2/\pi)^p, & p \in (0,1], \\ 1 - \sigma_1^*, & p \in (1, 5/3), \\ 1/2, & p \in [5/3, \infty). \end{cases} \tag{3.24}$$

In particular, the equality of (3.23) holds only for $p \in (1, 5/3)$, $\mu = \mu^*(p)$ and some (a,b) satisfying $G_p'\left(\frac{a-b}{a+b}\right) = 0$.

Proof. Without loss of generality, we assume that $a=1+x>b=1-x$ for $x\in(0,1)$. The proof will be split into two cases.

Case 4.1 $p=0$. In this case, the inequality (3.23) is equivalent to

$$\lambda < 1-\zeta(x) \leqslant \mu \tag{3.25}$$

for $x\in(0,1)$, where

$$\zeta(x)=\frac{\log\left[\dfrac{x}{(1+x)\arctan x}\right]}{\log\left(\dfrac{1-x}{1+x}\right)}.$$

Let

$$\zeta_1(x)=\log\left[\frac{x}{(1+x)\arctan x}\right] \quad \text{and} \quad \zeta_2(x)=\log\left(\frac{1-x}{1+x}\right).$$

Then we clearly see that

$$\zeta(x)=\frac{\zeta_1(x)}{\zeta_2(x)}, \quad \zeta_1(0^+)=\zeta_2(0^+)=0, \tag{3.26}$$

and

$$\frac{\zeta_1'(x)}{\zeta_2'(x)}=\left[\frac{x}{(1-x)\arctan x}\right]^{-1}\frac{x(1+x)-(1+x^2)\arctan x}{2(1+x^2)\arctan^2 x},$$

which is the same as (3.14) with $p=0$. This together with (3.14), (3.15) and Lemma 2.4 shows that $\zeta_1'(x)/\zeta_2'(x)$ is strictly decreasing on $(0,1)$, so is $\zeta(x)$ by (3.26) and Lemma 2.1. For the limiting values, it is clear that $\zeta(1^-)=0$, and by l'Hôpital's rule,

$$\zeta(0^+)=\lim_{x\to0^+}\frac{\zeta_1'(x)}{\zeta_2'(x)}=\lim_{x\to0^+}\frac{x(1+x)-(1+x^2)\arctan x}{2x^2}=\lim_{x\to0^+}\frac{1-\arctan x}{2}=\frac{1}{2}.$$

Case 4.2 $p\neq0$. In this case, we rewrite the inequality (3.23) as

$$\lambda < 1-G_p(x) \leqslant \mu, \quad x\in(0,1), \tag{3.27}$$

where $G_p(x)$ is defined in (3.10). Therefore, the best constants $\lambda^*(p)$ and $\mu^*(p)$ in (3.24) can be obtained immediately by Theorem 3.2 and (3.27). Moreover, as in the proof of Theorem 3.3, the right-side equality of (3.27) holds only when $p\in(1,5/3)$ and $G_p'(x)=0$. $\square$

The following corollary can be derived from Theorem 3.3, which is an alternative proof of (1.1).

COROLLARY 3.1. *Let $q_1,q_2\in\mathbb{R}$. Then the double inequality*

$$H_{q_1}(a,b)=H_{q_1}(a,b;1/2)<P(a,b)<H_{q_2}(a,b;1/2)=H_{q_2}(a,b) \tag{3.28}$$

holds for all $a,b>0$ with $a\neq b$ with the best constants $q_1=\log 2/\log\pi=0.6055\cdots$ and $q_2=2/3$.

Proof. Since both $P(a,b)$ and $H_p(a,b)$ are the symmetric means of their variables a and b, without loss of generality, we may assume that $a > b > 0$. First, it is easy to see from (3.20) that $\alpha^*(p_0) = \beta^*(2/3) = 1/2$. Taking $(\alpha^*(p_0), p_0)$ and $(\beta^*(2/3), 2/3)$ into (3.19) deduces, by applying Theorem 3.3, that the inequality (3.28) holds for all $a > b > 0$ with $q_1 = p_0$ and $q_2 = 2/3$.

In the following we show that $H_{p_0}(a,b)$ and $H_{2/3}(a,b)$ as the Hölder mean bounds of $P(a,b)$ are sharp. Indeed, for any given $p \in (p_0, 2/3)$, it follows from Theorem 3.3 that the double inequality

$$H_p\left(a,b;(1/\pi)^p\right) < P(a,b) \leqslant H_p(a,b;1-\sigma_0^*) \tag{3.29}$$

takes place for all $a > b > 0$ with the best weight $(1/\pi)^p$ and $1-\sigma_0^*$. From Theorem 3.1(3), it is easy to see that $1-(1/\pi)^p > 1/2 > \sigma_0^*$, that is, $(1/\pi)^p < 1/2 < 1-\sigma_0^*$ for $p \in (p_0, 2/3)$ and thereby

$$H_p\left(a,b;(1/\pi)^p\right) < H_p\left(a,b;1/2\right) < H_p(a,b;1-\sigma_0^*)$$

for all $a > b > 0$. This together with (3.29), implies that there exist (a_1, b_1) and (a_2, b_2) with $a_1 > b_1 > 0$, $a_2 > b_2 > 0$ such that

$$H_p(a_1,b_1;1/2) > P(a_1,b_1), \quad P(a_2,b_2) > H_p(a_2,b_2;1/2).$$

Hence the proof of Corollary 3.1 is completed. □

With the similar argument of Corollary 3.1, the following corollary, namely (1.2), can also be derived by Theorem 3.4, and its proof will be omitted.

COROLLARY 3.2. *Let $q_3, q_4 \in \mathbb{R}$. Then the double inequality*

$$H_{q_3}(a,b) = H_{q_3}(a,b;1/2) < T(a,b) < H_{q_4}(a,b;1/2) = H_{q_4}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ with the best possible constants $q_3 = \log 2/\log(\pi/2) = 1.534\cdots$ and $q_4 = 5/3$.

LEMMA 3.1. *For any fixed $a > b > 0$ and $\tau \in (0,1)$, the function $p \mapsto H_p(a,b;\tau^p)$ is strictly decreasing on $(0,\infty)$.*

Proof. Without loss of generality, we may assume that $a = 1$ and $b = x \in (0,1)$. It suffices to prove that, for any given $\tau \in (0,1)$ and $x \in (0,1)$, $p \mapsto H_p(1,x;\tau^p)$ is strictly decreasing on $(0,\infty)$.

Let us denote

$$Y(p,x) = H_p(1,x;\tau^p) = \left[\tau^p + (1-\tau^p)x^p\right]^{1/p}$$

for $(p,x) \in (0,\infty) \times (0,1)$. Then logarithmic differentiation of $Y(p,x)$ with respect to p yields

$$\frac{1}{Y(p,x)} \frac{\partial Y(p,x)}{\partial p} = -\frac{\hat{Y}(p,x^p)}{p^2\left[\tau^p + (1-\tau^p)x^p\right]}, \tag{3.30}$$

where

$$\hat{Y}(p,x) = [\tau^p + (1-\tau^p)x]\log[\tau^p + (1-\tau^p)x] - (1-x)\tau^p \log\tau^p - (1-\tau^p)x\log x,$$

$$\hat{Y}(p,0^+) = \hat{Y}(p,1^-) = 0. \tag{3.31}$$

Differentiating $\hat{Y}(p,x)$ with respect to x leads to

$$\frac{\partial \hat{Y}(p,x)}{\partial x} = (1-\tau^p)\left[\log\left(\tau^p + (1-\tau^p)x\right) - \log x\right] + \tau^p \log\tau^p := \hat{Y}_0(p,x)$$

and

$$\hat{Y}_0(p,0^+) = +\infty, \quad \hat{Y}_0(p,1^-) = \tau^p\log\tau^p < 0,$$
$$\frac{\partial \hat{Y}_0(p,x)}{\partial x} = -\frac{\tau^p(1-\tau^p)}{x[\tau^p + (1-\tau^p)x]} < 0$$

for $x \in (0,1)$. This demonstrates that $\hat{Y}(p,x)$ is piecewise monotone for $x \in (0,1)$, first increasing then decreasing. According to this with (3.31), it follows that $\hat{Y}(p,x) > 0$ for $x \in (0,1)$ and so the monotonicity of $p \mapsto Y(p,x)$ follows from (3.30).

Taking $\alpha^*(p) = \beta^*(p) = (1/\pi)^p$, Theorem 3.3 and Lemma 3.1 deduce the sharp bounds for $P(a,b)$ in the form of $H_p(a,b;(1/\pi)^{1/p})$. The proof is very similar to that in Corollary 3.1, which is left to the reader for details.

COROLLARY 3.3. *Let $q_1^*, q_2^* \in (0,\infty)$. Then the double inequality*

$$H_{q_1^*}(a,b;(1/\pi)^{q_1^*}) < P(a,b) < H_{q_2^*}(a,b;(1/\pi)^{q_2^*})$$

holds for all $a > b > 0$ if and only if $q_1^* \geqslant p_0 = \log 2/\log\pi = 0.6055\cdots$ and $q_2^* \leqslant 1/2$.

Similarly, we can obtain the following corollary from Theorem 3.4.

COROLLARY 3.4. *Let $q_3^*, q_4^* \in (0,\infty)$. Then the double inequality*

$$H_{q_3^*}(a,b;(2/\pi)^{q_3^*}) < T(a,b) < H_{q_4^*}(a,b;(2/\pi)^{q_4^*})$$

holds for all $a > b > 0$ if and only if $q_3^* \geqslant p_1 = \log 2/\log(\pi/2) = 1.534\cdots$ and $q_4^* \leqslant 1$.

REMARK 1. We remark that it has been shown that the best constants $\beta^*(p) = 1-\sigma_0^*$ for $p \in (1/2,2/3)$ in (3.19) and $\mu^*(p) = 1-\sigma_1^*$ for $p \in (1,5/3)$ in (3.23) but the values of σ_0^* and σ_1^* are not simple computable. However, with the help of mathematical software MAPLE 13, the approximate values of $\beta^*(p)$ and $\mu^*(p)$ on a series of discrete points are listed in Table 1 and the numerical simulation graphs of $p \mapsto \beta^*(p)$ on $(1/2,1/3)$ and $p \mapsto \mu^*(p)$ on $(1,5/3)$ are illustrated in Figure 1.

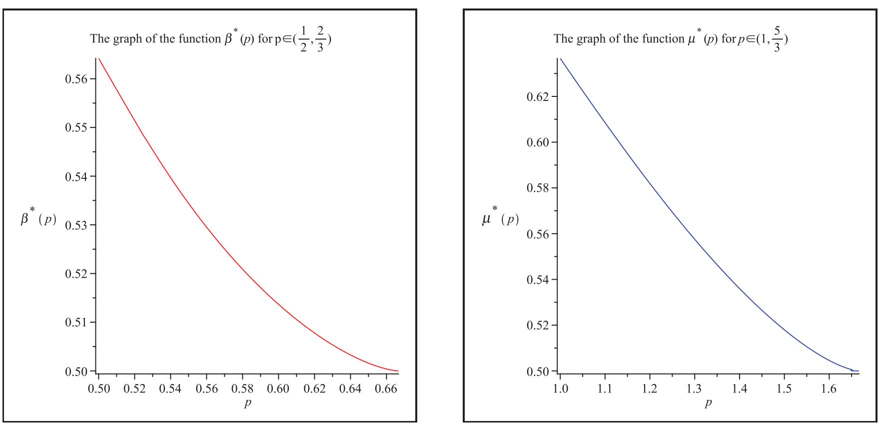
Approximations for $\beta^*(p)$

p	$\beta^*(p)$	p	$\beta^*(p)$
0.51	$0.5577\cdots$	0.59	$0.5171\cdots$
0.52	$0.5514\cdots$	0.60	$0.5136\cdots$
0.53	$0.5453\cdots$	0.61	$0.5105\cdots$
0.54	$0.5396\cdots$	0.62	$0.5077\cdots$
0.55	$0.5344\cdots$	0.63	$0.5053\cdots$
0.56	$0.5295\cdots$	0.64	$0.5032\cdots$
0.57	$0.5250\cdots$	0.65	$0.5016\cdots$
0.58	$0.5208\cdots$	0.66	$0.5004\cdots$

Approximations for $\mu^*(p)$

p	$\mu^*(p)$	p	$\mu^*(p)$
1.01	$0.6337\cdots$	1.35	$0.5463\cdots$
1.05	$0.6224\cdots$	1.40	$0.5360\cdots$
1.10	$0.6085\cdots$	1.45	$0.5264\cdots$
1.15	$0.5949\cdots$	1.50	$0.5179\cdots$
1.20	$0.5819\cdots$	1.55	$0.5105\cdots$
1.25	$0.5694\cdots$	1.60	$0.5045\cdots$
1.30	$0.5575\cdots$	1.65	$0.5005\cdots$

Table 1: *Approximate values of* $\beta^*(p)$ *and* $\mu^*(p)$ *on a series of discrete points*



(a) $\beta^*(p)$ (b) $\mu^*(p)$

Figure 1: *The visualized graph of* $\beta^*(p)$ *and* $\mu^*(p)$

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