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ON A GENERALIZED JORDAN–VON NEUMANN TYPE CONSTANT AND NORMAL STRUCTURE

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Abstract. In this paper, we introduce a new geometric constant $C_{-\infty}^{(p)}(a,X)$, which is closely related to the generalized Jordan-von Neumann type constant. We show that 2 and $\frac{(a+2)^p}{2^{p-2}(2^p+a^p)}$ are the upper and lower bound for $C_{-\infty}^{(p)}(a,X)$, respectively. Moreover, we obtain that $C_{-\infty}^{(p)}(a,X) = C_{-\infty}^{(p)}\left(a,\widetilde{X}\right)$, where $\widetilde{X}$ is the ultrapower space of X. Subsequently, we give some sufficient conditions for normal structure of a Banach space with different constants, such as the generalized James constant, Domínguez-Benavides coefficient and the coefficient of weak orthogonality.

1. Introduction

The concept of normal structure was firstly proposed by Brodskii and Milman in [1]. Subsequently, Kirk proved that the singlevalued nonexpansive mapping of Banach space with normal structure has fixed point property in [10]. Since then, the study of normal structure has been an important means to study the fixed point theory (see [20, 25]).

Whether a Banach space has normal structure, it depended on the geometry of its unit ball and its unit sphere. However, it is very difficult to give the characterization of geometry structure of Banach space. In recent years, various geometrical constants have been introduced by many scholars, which can describe the geometry structure of Banach space.

Among them the James and Jordan-von Neumann constant are two widely studied constants. Throughout the paper, X is a Banach space, and X^* denotes the dual space of X. The closed unit ball and unit sphere of X are denoted by B_X and S_X, respectively. The following two constants,

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x,y\in X, \|x\|+\|y\|>0\right\},$$

$$J(X) = \sup\{\min\{\|x+y\|,\|x-y\|\} : x,y\in S_X\}.$$

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are called the Jordan-von Neumann constant and James constant in [8, 14], respectively. They have the following properties:

(1) For any Banach space X, $\sqrt{2} \leqslant J(X) \leqslant 2$ and $1 \leqslant C_{NJ}(X) \leqslant 2$.

(2) If X is an inner product space, then $J(X) = \sqrt{2}$. $C_{NJ}(X) = 1$ if and only if X is an inner product space.

In order to promote the results of $C_{NJ}(X)$ and $J(X)$, Dhompongsa introduced the constants $J(a,X)$ and $C_{NJ}(a,X)$ (see [3, 4]), where $a \geqslant 0$.

$$J(a,X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x,y,z \in B_X, \|y-z\| \leqslant a\|x\|\}.$$

$$C_{NJ}(a,X) = \sup\left\{\frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x,y,z \in X\right\}$$

where x,y,z are not all zero and $\|y-z\| \leqslant a\|x\|$. They have the following properties :

(1) $J(0,X) = J(X)$, $C_{NJ}(0,X) = C_{NJ}(X)$;

(2) $C_{NJ}(a,X)$ and $J(a,X)$ are nondecreasing continuous function with respect to a;

(3) For any Banach space X, $1 + \frac{4a}{4+a^2} \leqslant C_{NJ}(a,X) \leqslant 2$ for all $a \geqslant 0$. If H is a Hilbert space, Then $C_{NJ}(a,X) = 1 + \frac{4a}{4+a^2}$ for all $a \in [0,2]$.

Cui [2] and Dinarvand [5] introduced the constants $C_{NJ}^{(p)}(X)$ and $C_{NJ}^{(p)}(a,X)$, respectively, and gave some sufficient conditions for the normal structure, where $a \geqslant 0, 1 \leqslant p < \infty$.

$$C_{NJ}^{(p)}(X) = \sup\left\{\frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}(\|x\|^p + \|y\|^p)} : x,y \in X, (x,y) \neq (0,0)\right\}.$$

$$C_{NJ}^{(p)}(a,X) = \sup\left\{\frac{\|x+y\|^p + \|x-z\|^p}{2^{p-1}\|x\|^p + 2^{p-2}(\|y\|^p + \|z\|^p)} : x,y,z \in X\right\}$$

where x,y,z are not all zero and $\|y-z\| \leqslant a\|x\|$.

To further describe the geometric properties of Banach space, such as uniform non-square and normal structure, some scholars introduced the constants (see [19, 22, 23]) $C_{-\infty}(X)$, $C_{-\infty}(a,X)$ and $C_{-\infty}^{(p)}(X)$, where $a \geqslant 0$ and $1 \leqslant p < \infty$. For more details on these geometric constants,

$$C_{-\infty}(X) = \sup\left\{\frac{\min\left\{\|x+y\|^2, \|x-y\|^2\right\}}{\|x\|^2 + \|y\|^2} : x,y \in X, \|x\| + \|y\| > 0\right\}.$$

$$C_{-\infty}^{(p)}(X) = \sup\left\{\frac{\min\{\|x+y\|^p, \|x-y\|^p\}}{2^{p-2}(\|x\|^p + \|y\|^p)} : x,y \in X, (x,y) \neq (0,0)\right\}.$$

$$C_{-\infty}(a,X) = \sup\left\{\frac{2\min\left\{\|x+y\|^2, \|x-z\|^2\right\}}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x,y,z \in X\right\}$$

where x,y,z are not all zero and $\|y-z\| \leqslant a\|x\|$.

In general, it's difficult to know that whether Banach space has normal structure. In recent years, some scholars have found that the relationship between geometric constants can give sufficient conditions for normal structure. Since then, many scholars have focused on getting sufficient conditions for the normal structure. We can see some results in [24].

In this paper, we introduce a new constant $C_{-\infty}^{(p)}(a,X)$, which is closely related to the generalized Jordan-von Neumann type constant and analyze some properties of this new constant. In next section, the study is focussed on the relation between normal structure and the constant $C_{-\infty}^{(p)}(a,X)$. Using ultrapower techniques, we establish the relationship between generalized James constant (Domínguez-Benavides coefficient, the coefficient of weak orthogonality) and $C_{-\infty}^{(p)}(a,X)$ to get a sufficient condition for a normal structure. Accordingly, we can improve the results which have been obtained in [22, 24].

2. Preliminaries

In following part, we give some necessary notations and definitions.

DEFINITION 1. ([11]) A Banach space X is said to be uniformly non-square if there exists $\delta > 0$ such that for any $x,y \in S_X$, we have $\min\{\|x+y\|, \|x-y\|\} < 2(1-\delta)$.

DEFINITION 2. ([1]) A Banach space X is said to have (weak) normal structure if for every (weakly compact) closed bounded convex subset K in X that contains more than one point, there exists a point $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < \sup\{\|x-y\| : x,y \in K\}.$$

If there exists $c \in (0,1)$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c\sup\{\|x-y\| : x,y \in K\}.$$

then X is said to have uniform normal structure.

In the sequel, we need to some basic facts about ultrapowers of Banach space in [9, 18, 21].

Let $\mathscr{F}$ be a filter on $\mathbb{N}$ and let X be a Banach space. A sequence $\{x_n\}$ in X converges to x with respect to $\mathscr{F}$, denoted by $\lim_{\mathscr{F}} x_n = x$, if for each neighborhood U of x, $\{i \in : x_i \in U\} \in \mathscr{F}$. A filter $\mathscr{U}$ on $\mathbb{N}$ is called an ultrafilter if it is maximal with respect to set inclusion. An ultrafilter is called trivial if it is of the form $\{A \subset : i_0 \in A\}$ for some fixed $i_0 \in \mathbb{N}$, otherwise, it is called nontrivial. Let $l_\infty(X)$ denote the subspace of the product space $\prod_{n\in\mathbb{N}} X$ equipped with the norm

$$\|(x_n)\|_\infty = \sup_{n\in\mathbb{N}} \|x_n\| < \infty.$$

Let $\mathscr{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathscr{U}}(X)=\left\{x=(x_n)\in l_\infty(X):\lim_{\mathscr{U}}\|x_n\|=0\right\}.$$

The ultrapower of X, denoted by $\widetilde{X}$ is the quotient space $\frac{l_\infty(X)}{N_{\mathscr{U}}(X)}$ equipped with the quotient norm. Write $(x_n)_{\mathscr{U}}$ to denote the elements of the ultrapower. It follows from the definition of the quotient norm that

$$\|(x_n)_{\mathscr{U}}\|=\lim_{\mathscr{U}}\|x_n\|.$$

Note that if $\mathscr{U}$ is nontrivial, then x can be embedded into $\widetilde{X}$ isometrically. An important result about ultrapower space is that if X is super-reflexive, i.e., $\widetilde{X^*}=\left(\widetilde{X}\right)^*$, then X has uniform normal structure if and only if $\widetilde{X}$ has normal structure (see [9]).

3. The constant $C_{-\infty}^{(p)}(a,X)$

We define the generalized constant $C_{-\infty}^{(p)}(a,X)$ as

$$C_{-\infty}^{(p)}(a,X)=\sup\left\{\frac{\min\{\|x+y\|^p,\|x-z\|^p\}}{2^{p-2}\|x\|^p+2^{p-3}(\|y\|^p+\|z\|^p)}:x,y,z\in X\right\}$$

for all $1\leqslant p<\infty$ and $a\geqslant 0$, where x,y,z are not all zero and $\|y-z\|\leqslant a\|x\|$. By analogy with the definition of the constant $C_{NJ}^{(p)}(a,X)$, the following is the equivalent definition of the generalized constant $C_{-\infty}^{(p)}(a,X)$:

$$C_{-\infty}^{(p)}(a,X)=\sup\left\{\frac{\min\{\|x+y\|^p,\|x-z\|^p\}}{2^{p-2}\|x\|^p+2^{p-3}(\|y\|^p+\|z\|^p)}:x,y,z\in B_X\right\}$$

where x,y,z are not all zero and $\|y-z\|\leqslant a\|x\|$. Now let us collect some useful properties of this new constant.

REMARK 1. X is a Banach space and $1\leqslant p<\infty$, the following statements hold.

(1) $C_{-\infty}^{(p)}(0,X)=C_{-\infty}^{(p)}(X)$, $C_{-\infty}^{(2)}(a,X)=C_{-\infty}(a,X)$, $C_{-\infty}^{(2)}(0,X)=C_{-\infty}(X)$;

(2) $C_{-\infty}^{(p)}(a,X)$ is a non-decreasing and continuous function for $a\geqslant 0$;

(3) If $C_{-\infty}^{(p)}(a,X)<2$, then $C_{-\infty}^{(p)}(X)<2$, and consequently, X is uniformly non-square in [23] ;

(4) For all $a\geqslant 0$, we have $C_{-\infty}^{(p)}(a,X)\leqslant C_{NJ}^{(p)}(a,X)\leqslant 2$.

THEOREM 1. *X is a Banach space and $1\leqslant p<\infty$, we have*

(1) $\frac{(a+2)^p}{2^{p-2}(2^p+a^p)}\leqslant C_{-\infty}^{(p)}(a,X)\leqslant 2$ *for all* $a\in[0,2]$;

(2) $C_{-\infty}^{(p)}(a,X)=2$ *for all* $a\geqslant 2$.

Proof. (1) Suppose that $a \in [0,2]$ and from Remark 1, we have $C_{-\infty}^{(p)}(a,X) \leqslant 2$. Let $x \in S_X$, $y = -z = \frac{a}{2}x$ such that $y - z = ax$. In addition,

$$\|x+y\| = \left\|x + \frac{a}{2}x\right\| = \frac{a+2}{2},$$

$$\|x-z\| = \left\|x + \frac{a}{2}x\right\| = \frac{a+2}{2}.$$

Then

$$C_{-\infty}^{(p)}(a,X) \geqslant \frac{\min\{\|x+y\|^p, \|x-z\|^p\}}{2^{p-2}\|x\|^p + 2^{p-3}(\|y\|^p + \|z\|^p)} = \frac{\left(\frac{a+2}{2}\right)^p}{2^{p-2} + 2^{p-2}\left(\frac{a}{2}\right)^p} = \frac{(a+2)^p}{2^{p-2}(2^p + a^p)}.$$

(2) We observe that the function $a \to \frac{(a+2)^p}{2^{p-2}(2^p+a^p)}$ is strictly increasing on $[0,2]$ and attains its maximum at $a = 2$, which means $C_{-\infty}^{(p)}(a,X) \geqslant 2$. From Remark 1, we have $C_{-\infty}^{(p)}(a,X) = 2$. □

EXAMPLE 1. Let X be $\mathbb{R}^2$ endowed with the $l_1 - l_\infty$ norm

$$\|(x_1,x_2)\| = \begin{cases} \|(x_1,x_2)\|_1, & x_1x_2 \geqslant 0, \\ \|(x_1,x_2)\|_\infty, & x_1x_2 \leqslant 0. \end{cases}$$

Then $C_{-\infty}^{(p)}(1,X) = 2$.

Proof. From Theorem 1, we have $C_{-\infty}^{(p)}(1,X) \leqslant 2$. Let $x = (1,-1)$, $y = (1,0)$, $z = (0,1)$. We have $y - z = (1,-1) = x$, $\|x+y\| = \|(2,-1)\|_\infty = 2$, $\|x-z\| = \|(1,-2)\|_\infty = 2$. From the defition of the $C_{-\infty}^{(p)}(a,X)$, we have

$$C_{-\infty}^{(p)}(1,X) \geqslant \frac{\min\{\|x+y\|^p, \|x-z\|^p\}}{2^{p-2}\|x\|^p + 2^{p-3}(\|y\|^p + \|z\|^p)} = \frac{2^p}{2^{p-2} + 2^{p-2}} = 2$$

Hence, $C_{-\infty}^{(p)}(1,X) = 2$. □

EXAMPLE 2. Let X be $\mathbb{R}^2$ endowed with the $l_1 - l_2$ norm

$$\|(x_1,x_2)\| = \begin{cases} \|(x_1,x_2)\|_1, & x_1x_2 \geqslant 0, \\ \|(x_1,x_2)\|_2, & x_1x_2 \leqslant 0. \end{cases}$$

Then $C_{-\infty}^{(p)}(2,X) = 2$.

Proof. From Theorem 1, we have $C_{-\infty}^{(p)}(2,X) \leqslant 2$. Let $x = \left(\frac{1}{2}, \frac{1}{2}\right)$, $y = (0,1)$, $z = (-1,0)$. We have $y - z = (1,1) = 2\left(\frac{1}{2}, \frac{1}{2}\right) = 2x$, $\|x+y\| = \left\|\left(\frac{1}{2}, \frac{3}{2}\right)\right\|_1 = 2$, $\|x-z\| = \left\|\left(\frac{3}{2}, \frac{1}{2}\right)\right\|_1 = 2$. From the defition of the $C_{-\infty}^{(p)}(a,X)$, we have

$$C_{-\infty}^{(p)}(2,X) \geqslant \frac{\min\{\|x+y\|^p, \|x-z\|^p\}}{2^{p-2}\|x\|^p + 2^{p-3}(\|y\|^p + \|z\|^p)} = \frac{2^p}{2^{p-2} + 2^{p-2}} = 2.$$

Hence, $C_{-\infty}^{(p)}(2,X)=2$. $\square$

From the definition of the constant $J(a,X)$ and $C_{-\infty}^{(p)}(a,X)$, we can easily obtained the following lemma.

LEMMA 1. *Let X be a Banach space and $1<p<\infty$. For all $a\geqslant 0$, we have*

$$J(a,X)\leqslant 2^{\frac{p-1}{p}}\sqrt[p]{C_{-\infty}^{(p)}(a,X)}.$$

LEMMA 2. *Let X be a Banach space and $1\leqslant p<\infty$. Then*

$$C_{-\infty}^{(p)}(a,X)=C_{-\infty}^{(p)}\left(a,\widetilde{X}\right).$$

Proof. For $1\leqslant p<\infty$, we have

$$C_{-\infty}^{(p)}(a,X)\leqslant C_{-\infty}^{(p)}\left(a,\widetilde{X}\right).$$

Next, we prove that $C_{-\infty}^{(p)}(a,X)\geqslant C_{-\infty}^{(p)}\left(a,\widetilde{X}\right)$. Let $\delta>0$, $\alpha\in[0,a]$. Assume that $\widetilde{x},\widetilde{y},\widetilde{z}\in\widetilde{X}$ satisfy $\|\widetilde{y}-\widetilde{z}\|=\alpha\|\widetilde{x}\|$, not all of them are zero. If $\widetilde{x}=0$, then

$$\frac{\min\{\|\widetilde{x}+\widetilde{y}\|^p,\|\widetilde{x}-\widetilde{z}\|^p\}}{2^{p-2}\|\widetilde{x}\|^p+2^{p-3}(\|\widetilde{y}\|^p+\|\widetilde{z}\|^p)}=\frac{\min\{\|\widetilde{y}\|^p,\|\widetilde{z}\|^p\}}{2^{p-3}(\|\widetilde{y}\|^p+\|\widetilde{z}\|^p)}=2^{2-p}.$$

thus $2^{2-p}\leqslant C_{-\infty}^{(p)}(a,X)$.

If $\widetilde{x}\neq 0$, then choose $\varepsilon>0$ such that $\varepsilon<\delta\|\widetilde{x}\|$. Since $\|\widetilde{x}\|=\lim_{\mathscr{U}}\|x_n\|$ and

$$c=\frac{\min\{\|\widetilde{x}+\widetilde{y}\|^p,\|\widetilde{x}-\widetilde{z}\|^p\}}{2^{p-2}\|\widetilde{x}\|^p+2^{p-3}(\|\widetilde{y}\|^p+\|\widetilde{z}\|^p)}=\lim_{\mathscr{U}}\frac{\min\{\|x_n+y_n\|^p,\|x_n-z_n\|^p\}}{2^{p-2}\|x_n\|^p+2^{p-3}(\|y_n\|^p+\|z_n\|^p)}=\lim_{\mathscr{U}}c_n.$$

Then it follows that the set

$$E=\{n\in\mathbb{N}:|c_n-c|<\delta,\|y_n-z_n\|\leqslant\alpha\|x_n\|+\varepsilon<(\alpha+\delta)\|x_n\|\}$$

belongs to $\mathscr{U}$. In particular, notice that $x_n\neq 0$ for all $n\in E$, there exists n such that

$$c<\frac{\min\{\|x_n+y_n\|^p,\|x_n-z_n\|^p\}}{2^{p-2}\|x_n\|^p+2^{p-3}(\|y_n\|^p+\|z_n\|^p)}+\delta\leqslant C_{-\infty}^{(p)}(a+\delta,X)+\delta.$$

Hence, the inequality $C_{-\infty}^{(p)}\left(a,\widetilde{X}\right)\leqslant C_{-\infty}^{(p)}(a,X)$ follows from the arbitrariness of δ and the continuity of $C_{-\infty}^{(p)}(\cdot,X)$. $\square$

LEMMA 3. ([6]) *If X is a super-reflexive Banach space and fails to have normal structure, then for $r\in(0,1]$ there are $\widetilde{x_1},\widetilde{x_2},\widetilde{x_3}\in S_{\widetilde{X}}$, $\widetilde{f_1},\widetilde{f_2},\widetilde{f_3}\in S_{(\widetilde{X})^*}$ such that*

(1) $\left\|\widetilde{x_i}-\widetilde{x_j}\right\|=1$ and $\widetilde{f_i}(\widetilde{x_j})=0$ for all $i\neq j$, $(i,j=(1,2,3))$;
(2) $\widetilde{f_i}(\widetilde{x_i})=1$, $(i=1,2,3)$;
(3) $\|\widetilde{x_3}-(\widetilde{x_2}+r\widetilde{x_1})\|\geqslant\|\widetilde{x_2}+r\widetilde{x_1}\|$.

THEOREM 2. *X is a Banach space, if there exists $a \in [0,1]$ such that*

$$C_{-\infty}^{(p)}(a,X) < \frac{1}{2^{p-1}}\left(1+a+\frac{1-a^2}{J(a,X)+2a}\right)^p.$$

where $1 < p < \infty$, then X has normal structure.

Proof. From and $1 < p < \infty$, we know that $(J(a,X))^p \leqslant 2^{p-1}C_{-\infty}^{(p)}(a,X)$. So X is uniform non-square according to the inequality, consequently, X is also super-reflexive (see [11]). Assume that X fails to have normal structure and the inequality holds for some $a \in [0,1]$. Then from there are $\widetilde{x_1},\widetilde{x_2},\widetilde{x_3} \in S_{\widetilde{X}}, \widetilde{f_1},\widetilde{f_2},\widetilde{f_3} \in S_{(\widetilde{X})^*}$ satisfying all the conditions in Lemma 3. We put

$$\begin{cases} \widetilde{x} = \widetilde{x_3} - \widetilde{x_1} \\ \widetilde{y} = a\widetilde{x_3} + (1-a)\frac{\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|} \\ \widetilde{z} = a\widetilde{x_1} + (1-a)\frac{\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|} \end{cases}$$

Then $\|\widetilde{x}\| = 1$, $\|\widetilde{y}\| \leqslant 1$, $\|\widetilde{z}\| \leqslant 1$, $\widetilde{y}-\widetilde{z} = a\widetilde{x}$. Hence, we have

$$\begin{aligned} \|\widetilde{x}+\widetilde{y}\| &= \left\|\widetilde{x_3} - \widetilde{x_1} + a\widetilde{x_3} + (1-a)\frac{\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|}\right\| \\ &\geqslant \widetilde{f_3}\left(\widetilde{x_3} - \widetilde{x_1} + a\widetilde{x_3} + (1-a)\frac{\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|}\right) \\ &= 1+a+\frac{1-a}{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|}. \end{aligned}$$

$$\begin{aligned} \|\widetilde{x}-\widetilde{z}\| &= \left\|\widetilde{x_3} - \widetilde{x_1} + a\widetilde{x_1} + (1-a)\frac{\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|}\right\| \\ &\geqslant \left(-\widetilde{f_1}\right)\left(\widetilde{x_3} - \widetilde{x_1} + a\widetilde{x_1} + (1-a)\frac{\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|}\right) \\ &= 1+a+\frac{1-a}{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|} \end{aligned}$$

Therefore,

$$\begin{aligned} C_{-\infty}^{(p)}(a,X) &= C_{-\infty}^{(p)}\left(a,\widetilde{X}\right) \\ &\geqslant \frac{\min\{\|\widetilde{x}+\widetilde{y}\|^p, \|\widetilde{x}-\widetilde{z}\|^p\}}{2^{p-2}\|\widetilde{x}\|^p + 2^{p-3}(\|\widetilde{y}\|^p + \|\widetilde{z}\|^p)} \\ &\geqslant \frac{1}{2^{p-1}}\left(1+a+\frac{1-a}{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|}\right)^p. \end{aligned}$$

We put

$$\begin{cases} \widetilde{x} = \widetilde{x_2} - \widetilde{x_1} \\ \widetilde{y} = a\widetilde{x_2} + (1-a)\frac{-\widetilde{x_3}+\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_3}-\widetilde{x_2}-\widetilde{x_1}\|} \\ \widetilde{z} = a\widetilde{x_1} + (1-a)\frac{-\widetilde{x_3}+\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_3}-\widetilde{x_2}-\widetilde{x_1}\|} \end{cases}$$

Then $\|\widetilde{x}\| = 1$, $\|\widetilde{y}\| \leqslant 1$, $\|\widetilde{z}\| \leqslant 1$, $\widetilde{y} - \widetilde{z} = a\widetilde{x}$. Similarly, we get

$$\begin{aligned} C_{-\infty}^{(p)}(a,X) &= C_{-\infty}^{(p)}\left(a,\widetilde{X}\right) \\ &\geqslant \frac{\min\{\|\widetilde{x}+\widetilde{y}\|^p, \|\widetilde{x}-\widetilde{z}\|^p\}}{2^{p-2}\|\widetilde{x}\|^p + 2^{p-3}(\|\widetilde{y}\|^p + \|\widetilde{z}\|^p)} \\ &\geqslant \frac{1}{2^{p-1}}\left(1+a+\frac{1-a}{\|\widetilde{x_3}-\widetilde{x_2}-\widetilde{x_1}\|}\right)^p. \end{aligned}$$

We put

$$\begin{cases} \widetilde{x} = \widetilde{x_3} - \widetilde{x_2} \\ \widetilde{y} = a\widetilde{x_3} + (1-a)\widetilde{x_1} \\ \widetilde{z} = a\widetilde{x_2} + (1-a)\widetilde{x_1} \end{cases}$$

We have $\|\widetilde{x}\| = 1$, $\|\widetilde{y}\| \leqslant 1$, $\|\widetilde{z}\| \leqslant 1$, $\widetilde{y} - \widetilde{z} = a\widetilde{x}$. Hence,

$$\|\widetilde{x}+\widetilde{y}\| = \|\widetilde{x_3} - \widetilde{x_2} + a\widetilde{x_3} + (1-a)\widetilde{x_1}\| \geqslant (1+a)\|\widetilde{x_3} - \widetilde{x_2} + \widetilde{x_1}\| - a\|\widetilde{x_2} - 2\widetilde{x_1}\|.$$

$$\|\widetilde{x}-\widetilde{z}\| = \|\widetilde{x_3} - (1+a)\widetilde{x_2} - (1-a)\widetilde{x_1}\| \geqslant (1+a)\|\widetilde{x_3} - \widetilde{x_2} - \widetilde{x_1}\| - a\|\widetilde{x_3} - 2\widetilde{x_1}\|.$$

$$\|\widetilde{x_2} - 2\widetilde{x_1}\| \leqslant \|\widetilde{x_2} - \widetilde{x_1}\| + \|\widetilde{x_1}\| = 2, \|\widetilde{x_3} - 2\widetilde{x_1}\| \leqslant 2.$$

From the defition of the $J(a,X)$, we have

$$\begin{aligned} J(a,X) &= J\left(a,\widetilde{X}\right) \\ &\geqslant \min\{\|\widetilde{x}+\widetilde{y}\|, \|\widetilde{x}-\widetilde{z}\|\} \\ &\geqslant (1+a)\min\{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|, \|\widetilde{x_3}-\widetilde{x_2}-\widetilde{x_1}\|\} - 2a. \end{aligned}$$

It follows that

$$\frac{1}{\min\{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|, \|\widetilde{x_3}-\widetilde{x_2}-\widetilde{x_1}\|\}} \geqslant \frac{1+a}{J(a,X)+2a}.$$

Consequently, we obtain

$$\begin{aligned} C_{-\infty}^{(p)}(a,X) &\geqslant \frac{1}{2^{p-1}}\left(1+a+\frac{1-a}{\min\{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|, \|\widetilde{x_3}-\widetilde{x_2}-\widetilde{x_1}\|\}}\right)^p \\ &\geqslant \frac{1}{2^{p-1}}\left(1+a+\frac{1-a}{J(a,X)+2a}\right)^p. \end{aligned}$$

which contradicts with the hypothesis. So Banach space X has normal structure. □

Next, we get the following corollary.

COROLLARY 1. *X is a Banach space. If*

$$C_{-\infty}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{J(X)}\right)^p$$

or

$$C_{-\infty}^{(p)}(1,X) < 2.$$

where $1 < p < \infty$, then X has normal structure.

We know that if $C_{-\infty}(X) < 1+\frac{1}{J(X)^2}$, then X has normal structure (see [24]). From Corollary 1, we have $C_{-\infty}(X) < \frac{1}{2}\left(1+\frac{1}{J(X)}\right)^2$ when $p=2$, since $\sqrt{2} \leqslant J(X) \leqslant 2$ and $\frac{1}{2}\left(1+\frac{1}{J(X)}\right)^2 < 1+\frac{1}{J(X)^2}$, so we have improved the results.

4. Domínguez-Benavides coefficient

The constant $R(a,X)$ was introduced in 1996 by Domínguez-Benavides [7], for $a \geqslant 0$,

$$R(a,X) = \sup\left\{\liminf_{n\to\infty}\{\|x_n+x\|\}\right\}.$$

where the supremum is taken over all $x \in X$ with $\|x\| \leqslant a$. All weakly null sequences $\{x_n\}$ in B_X satisfy

$$D[(x_n)] = \limsup_{n\to\infty}\limsup_{m\to\infty}\|x_n - x_m\| \leqslant 1.$$

In particular, if $a=1$, we have $1 \leqslant R(1,X) \leqslant 2$.

LEMMA 4. ([6]) *If a super-reflexive Banach space X fails to have normal structure, then there are $\widetilde{x_1},\widetilde{x_2} \in S_{\widetilde{X}}, \widetilde{f_1},\widetilde{f_2} \in S_{(\widetilde{X})^*}$ such that*

(1) $\|\widetilde{x_1}-\widetilde{x_2}\| = 1$ *and* $\widetilde{f_i}(\widetilde{x_j}) = 0$ *for* $i \neq j$, $(i,j=(1,2))$;
(2) $\widetilde{f_i}(\widetilde{x_i}) = 1$, $(i=1,2)$;
(3) $\|\widetilde{x_2}+\widetilde{x_1}\| \leqslant R(1,X)$.

THEOREM 3. *X is a Banach space, if there exists $a \in [0,1]$ such that*

$$C_{-\infty}^{(p)}(a,X) < \frac{1}{2^{p-1}}\left(1+a+\frac{1-a}{R(1,X)}\right)^p.$$

where $1 < p < \infty$, then X has normal structure.

Proof. Suppose that $1 < p < \infty$, if $a=1$, from Corollary 1, we know that Banach space X has normal structure.

If $0 \leqslant a < 1$, since $R(1,X) \geqslant 1$, we have $C_{-\infty}^{(p)}(a,X) < 2$, then X is uniform non-square, consequently, X is super-reflexive.

Assume that X fails to have normal structure. From Lemma 4, there are $\widetilde{x_1},\widetilde{x_2} \in S_{\widetilde{X}}$, $\widetilde{f_1},\widetilde{f_2} \in S_{(\widetilde{X})^*}$ satisfying all the conditions in Lemma 4. We put

$$\begin{cases} \widetilde{x} = \widetilde{x_2} - \widetilde{x_1} \\ \widetilde{y} = a\widetilde{x_2} + (1-a)\frac{\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_2}+\widetilde{x_1}\|} \\ \widetilde{z} = a\widetilde{x_1} + (1-a)\frac{\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_2}+\widetilde{x_1}\|} \end{cases}$$

Then $\|\widetilde{x}\|=1$, $\|\widetilde{y}\|\leqslant 1$, $\|\widetilde{z}\|\leqslant 1$, and $\|\widetilde{x_2}+\widetilde{x_1}\|\geqslant \widetilde{f_1}(\widetilde{x_2}+\widetilde{x_1})=1$. We can easily obtain that

$$\begin{aligned}\|\widetilde{x}+\widetilde{y}\| &= \left\|\widetilde{x_2}-\widetilde{x_1}+a\widetilde{x_2}+(1-a)\frac{\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_2}+\widetilde{x_1}\|}\right\| \\ &\geqslant \widetilde{f_2}\left(\widetilde{x_2}-\widetilde{x_1}+a\widetilde{x_2}+(1-a)\frac{\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_2}+\widetilde{x_1}\|}\right) \\ &= 1+a+\frac{1-a}{\|\widetilde{x_2}+\widetilde{x_1}\|}.\end{aligned}$$

$$\begin{aligned}\|\widetilde{x}-\widetilde{z}\| &= \left\|\widetilde{x_2}-\widetilde{x_1}-a\widetilde{x_1}-(1-a)\frac{\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_2}+\widetilde{x_1}\|}\right\| \\ &\geqslant \left(-\widetilde{f_1}\right)\left(\widetilde{x_2}-\widetilde{x_1}-a\widetilde{x_1}-(1-a)\frac{\widetilde{x_2}+\widetilde{x_1}}{\|\widetilde{x_2}+\widetilde{x_1}\|}\right) \\ &= 1+a+\frac{1-a}{\|\widetilde{x_2}+\widetilde{x_1}\|}.\end{aligned}$$

From the definition of the $C_{-\infty}^{(p)}(a,X)$, it follows

$$\begin{aligned}C_{-\infty}^{(p)}(a,X) &= C_{-\infty}^{(p)}\left(a,\widetilde{X}\right) \\ &\geqslant \frac{\min\{\|\widetilde{x}+\widetilde{y}\|^p,\|\widetilde{x}-\widetilde{z}\|^p\}}{2^{p-2}\|\widetilde{x}\|^p+2^{p-3}(\|\widetilde{y}\|^p+\|\widetilde{z}\|^p)} \\ &\geqslant \frac{1}{2^{p-1}}\left(1+a+\frac{1-a}{\|\widetilde{x_2}+\widetilde{x_1}\|}\right)^p \\ &\geqslant \frac{1}{2^{p-1}}\left(1+a+\frac{1-a}{R(1,X)}\right)^p,\end{aligned}$$

which contradicts with the hypothesis. This completes the proof. □

If $a=0$, we get the following corollary.

COROLLARY 2. *X is a Banach space, if*

$$C_{-\infty}^{(p)}(X)<\frac{1}{2^{p-1}}\left(1+\frac{1}{R(1,X)}\right)^p.$$

where $1<p<\infty$, then X has normal structure.

We know that if $C_{-\infty}(a,X)<1+\left(\frac{(1+a)}{\min\{R(1,X)+a,2\}}\right)^2$, then X has normal structure [22]. From Theorem 3, it has $C_{-\infty}(a,X)<\frac{1}{2}\left(1+a+\frac{1-a}{R(1,X)}\right)^2$ when $p=2$. Hence, for $a\in[0,1]$, $1\leqslant R(1,X)\leqslant 2$, we have

$$\frac{1}{2}\left(1+a+\frac{1-a}{R(1,X)}\right)^2\in\left[\frac{9}{8},2\right],\quad 1+\left(\frac{(1+a)}{\min\{R(1,X)+a,2\}}\right)^2\in\left[\frac{5}{4},2\right].$$

In particular, when $p=2$, according to the Corollary 2, we know

$$C_{-\infty}(X) < \frac{1}{2}\left(1+\frac{1}{R(1,X)}\right)^2.$$

Since $\frac{1}{2}\left(1+\frac{1}{R(1,X)}\right)^2 < 1+\frac{1}{R(1,X)^2}$ holds, therefore we have improved the results.

5. The coefficient of weak orthogonality

The weak orthogonality property, which is denoted as the WORTH-property, was introduced by Sims [17]. Recall that a Banach space X has the WORTH-property if

$$\lim_{n\to\infty} |\|x_n+x\|-\|x_n-x\|| = 0.$$

for all $x\in X$ and all weakly null sequences $\{x_n\}$. In order to characterize the WORTH-property, Jimenez-Melado et al introduced the coefficient of weak orthogonality[12], defined by

$$\mu(X) = \inf\left\{\lambda : \limsup_{n\to\infty}\|x_n+x\| \leqslant \lambda \limsup_{n\to\infty}\|x_n-x\|\right\}.$$

where the infimum is taken over all $x\in X$ and all weakly null sequences $\{x_n\}$ in X. It is well known from that [12], [13]

(1) $1\leqslant \mu(X) \leqslant 3$.
(2) X has normal structure if and only if $\mu(X)=1$.
(3) If X is a reflexive Banach space, then $\mu(X)=\mu(X^*)$.

LEMMA 5. ([15]) *If a super-reflexive Banach space X fails to have normal structure, then there are $\widetilde{x_1},\widetilde{x_2}\in S_{\widetilde{X}}, \widetilde{f_1},\widetilde{f_2}\in S_{(\widetilde{X})^*}$ such that*

(1) $\|\widetilde{x_1}-\widetilde{x_2}\| = 1,\ \widetilde{f_i}(\widetilde{x_j}) = 0$ *for all* $i\neq j,\ (i,j=(1,2))$;
(2) $\widetilde{f_i}(\widetilde{x_i}) = 1,\ (i=1,2)$;
(3) $\|\widetilde{x_2}+\widetilde{x_1}\| \leqslant \mu(X)$.

THEOREM 4. *X is a Banach space, if there exists $a\in[0,1]$ such that*

$$C_{-\infty}^{(p)}(a,X) < \frac{1}{2^{p-1}}\left(1+a+\frac{1-a}{\mu(X)}\right)^p.$$

where $1<p<\infty$, then X has normal structure.

Proof. It can be obtained by using similar arguments as those be given in the proof of Theorem 3 and Lemma 5. So we omit it. □

If $a=0$, we get the following corollary.

COROLLARY 3. *X is a Banach space, if*

$$C_{-\infty}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{\mu(X)}\right)^{p}.$$

where $1<p<\infty$, then X has normal structure.

We know that if $C_{-\infty}(a,X) < 1+\left(\frac{(1+a)}{\min\{\mu(X)+a,2\}}\right)^2$, then X has normal structure [22]. From Theorem 4, it has $C_{-\infty}(a,X) < \frac{1}{2}\left(1+a+\frac{1-a}{\mu(X)}\right)^2$ when $p=2$. Hence, $a\in[0,1]$, $1\leqslant\mu(X)\leqslant 3$, we have

$$\frac{1}{2}\left(1+a+\frac{1-a}{\mu(X)}\right)^2 \in \left[\frac{8}{9},2\right],$$

$$1+\left(\frac{(1+a)}{\min\{\mu(X)+a,2\}}\right)^2 \in \left[\frac{10}{9},2\right].$$

In particular, when $p=2$, according to the Corollary 3, we know $C_{-\infty}(X) < \frac{1}{2}\left(1+\frac{1}{\mu(X)}\right)^2$. Since $\frac{1}{2}\left(1+\frac{1}{\mu(X)}\right)^2 \leqslant 1+\frac{1}{\mu(X)^2}$ holds, so we have improved the results.

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