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SOME GEOMETRIC CONSTANTS RELATED TO ρ–ORTHOGONALITY AND ρ'–ORTHOGONALITY IN BANACH SPACES

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Abstract. In this article, we introduce two new moduli of convexity $\delta_\perp(\rho,X)$ and $\delta_\perp(\rho',X)$ related to ρ-orthogonality and ρ'-orthogonality, which are connected with the modulus of convexity $\delta_X(\varepsilon)$. The connections between these two parameters and other well-known constants are built. In the meantime, this two new coefficients are calculated for X being some specific spaces. Moreover, we also provide a characterization of the Radon plane with affine-regular hexagonal unit sphere in terms of $\delta_\perp(\rho,X)$. To consider the moduli of smoothness related to ρ-orthogonality and ρ'-orthogonality, we also treat $\rho_\perp(\rho,X)$ and $\rho_\perp(\rho',X)$.

1. Introduction

We denote by X is a real Banach space with the norm $\|\cdot\|$ and the unit sphere S_X. Throughout this paper, we assume that the dimension of X is not less than two. For the case that X is a Hilbert space, an element $x \in X$ is said to be orthogonal to $y \in X$ (denoted by $x \perp y$) if the inner product $\langle x,y\rangle$ is zero. In the general setting of Banach spaces, many concepts of orthogonality have been introduced by means of propositions equivalent to the standard orthogonality in Hilbert spaces. For instance, Birkhoff [3] proposed the concept of Birkhoff orthogonality: for any two elements x and y, if $\|x+\lambda y\| \geqslant \|x\|$ for all $\lambda \in \mathbb{R}$, then x is said to be Birkhoff orthogonal to y (denoted by $x \perp_B y$). Besides the above orthogonality notion, one of the probable concepts of orthogonality is related to the norm derivatives, which is defined by

$$\rho_\pm(x,y) = \lim_{t\to 0^\pm} \frac{\|x+ty\|^2 - \|x\|^2}{2t} = \|x\| \lim_{t\to 0^\pm} \frac{\|x+ty\| - \|x\|}{t}.$$

For more details about this, the reader can consult [1, 4, 6, 19] and the references therein. Furthermore, Miličić [15] introduced the mapping $\langle\cdot,\cdot\rangle_g : X \times X \to \mathbb{R}$ as follows:

$$\rho(x,y) = \langle y,x\rangle_g = \frac{\rho_-(x,y) + \rho_+(x,y)}{2}.$$

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In fact, this functional is also a generalized form of inner product in Hilbert spaces. In particular, the orthogonality relation associated with the above functional is defined by

$$x \perp_\rho y \Leftrightarrow \rho(x,y) = 0.$$

By use of the functional ρ, Miličić [17] also offered another orthogonality relation:

$$x \perp_{\rho'} y \iff \|x\|^2\rho(x,y) + \|y\|^2\rho(y,x) = 0.$$

Note that for all Banach spaces X, we derive that for any $x,y \in X$,

$$\rho_-(x,y) \leqslant 0 \leqslant \rho_+(x,y) \Leftrightarrow x \perp_B y \tag{1}$$

by Theorem 50 in [6]. From (1), we can obtain that $\perp_\rho \subset \perp_B$. Moreover, combining (1) and Proposition 2.2.4 in [1], we derive that $\perp_\rho = \perp_B$ if and only if X is smooth. However, one can deduce that the relation $\perp_B \subset \perp_\rho$ may not hold from Example 5 in [4], in which the space X is not smooth. Hence, the orthogonalities $\perp_\rho$ and $\perp_B$ could not coincide unless X is smooth. Moreover, the orthogonalities $\perp_{\rho'}$ and $\perp_B$, $\perp_\rho$ also may not coincide. The following example illustrates this fact.

EXAMPLE 1. Consider the Banach space $X = \mathbb{R}^2$ endowed with the norm

$$\|x\| = \|(x_1,x_2)\| = \max\{|x_1|,|x_2|\}.$$

(1) Let $x = (1,1)$ and $y = (1,0)$. It is clear that $x,y \in S_X$ such that $x \perp_B y$. And it follows from $x+ty = (1+t,1)$ that we have

$$\rho_-(x,y) = \|x\| \lim_{t\to 0^-} \frac{\|x+ty\|-\|x\|}{t} = \lim_{t\to 0^-} \frac{1-1}{t} = 0,$$

$$\rho_+(x,y) = \|x\| \lim_{t\to 0^+} \frac{\|x+ty\|-\|x\|}{t} = \lim_{t\to 0^+} \frac{1+t-1}{t} = 1.$$

Hence

$$\rho(x,y) = \frac{\rho_-(x,y)+\rho_+(x,y)}{2} = \frac{1}{2}.$$

Similarly, from $y+tx = (1+t,t)$, we derive that

$$\rho(y,x) = \rho_-(y,x) = \rho_+(y,x) = 1.$$

Then

$$\|x\|^2\rho(x,y) + \|y\|^2\rho(y,x) = \frac{1}{4}+1 = \frac{5}{4},$$

whence $x \not\perp_{\rho'} y$. Therefore $\perp_B \not\subset \perp_{\rho'}$.

(2) Let $x = \left(\frac{1}{3},1\right)$ and $y = \left(1,-\frac{1}{3}\right)$. It is obvious that $x,y \in S_X$ and $x+ty = \left(\frac{1}{3}+t, 1-\frac{t}{3}\right)$. Then $x \not\perp_B y$. Indeed, taking $t = \frac{1}{2}$, we have

$$\left\|x+\frac{1}{2}y\right\| = \frac{5}{6} < 1 = \|x\|.$$

Moreover, it is not hard to compute

$$\rho(x,y)=\rho_-(x,y)=\rho_+(x,y)=-\frac{1}{3},$$

whence $x \not\perp_\rho y$. Similarly, it follows from $y+tx=\left(1+\frac{t}{3},-\frac{1}{3}+t\right)$ that we have

$$\rho(y,x)=\rho_-(y,x)=\rho_+(y,x)=\frac{1}{3}.$$

Thus we derive

$$\|x\|^2\rho(x,y)+\|y\|^2\rho(y,x)=-\frac{1}{3}+\frac{1}{3}=0,$$

which implies that $x\perp_{\rho'} y$. Therefore $\perp_{\rho'}\not\subset\perp_B$ and $\perp_{\rho'}\not\subset\perp_\rho$.

(3) Let $x=\left(\frac{1}{n},1\right)$ and $y=(1,0)$ with $n\geqslant 1$. Then it is evident that $x,y\in S_X$ and $x+ty=\left(\frac{1}{n}+t,1\right)$. In addition, it is easily seen that

$$\rho(x,y)=\rho_-(x,y)=\rho_+(x,y)=0,$$

which means that $x\perp_\rho y$. Similarly, from $y+tx=\left(1+\frac{t}{n},t\right)$, we can deduce that

$$\rho(y,x)=\rho_-(y,x)=\rho_+(y,x)=\frac{1}{n}.$$

Hence

$$\|x\|^2\rho(x,y)+\|y\|^2\rho(y,x)=\frac{1}{n},$$

then $x\not\perp_{\rho'} y$. Therefore $\perp_\rho\not\subset\perp_{\rho'}$.

Recall that the modulus of convexity of X is defined in [5] by

$$\delta_X(\varepsilon)=\inf\left\{1-\frac{\|x+y\|}{2}:x,y\in S_X,\|x-y\|=\varepsilon\right\},\ (0\leqslant\varepsilon\leqslant 2).$$

The space X is said to be uniformly convex if $\delta_X(\varepsilon)>0$ for any $\varepsilon\in(0,2]$.

Accordingly, Banaś [2] considered the modulus of smoothness of X as follows:

$$\rho_X(\varepsilon)=\sup\left\{1-\frac{\|x+y\|}{2}:x,y\in S_X,\|x-y\|=\varepsilon\right\},\ (0\leqslant\varepsilon\leqslant 2).$$

Notice that the space X is uniformly smooth if and only if $\lim_{\varepsilon\to 0^+}\frac{\rho_X(\varepsilon)}{\varepsilon}=0$.

When studying geometric properties of Banach spaces, the modulus of convexity has been widely studied and played an important role for several decades. Be similar to the modulus of convexity, the modulus of smoothness has also been considered in the literature, as far as the smoothness of a space is concerned. Based on the fact, we have considered the modulus of convexity and the modulus of smoothness related to Birkhoff orthogonality in [7]:

$$\delta_B(X)=\inf\left\{1-\frac{\|x+y\|}{2}:x,y\in S_X,x\perp_B y\right\},$$

$$\rho_B(X) = \sup\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, x \perp_B y\right\}.$$

Here are some conclusions about $\delta_B(X)$ and $\rho_B(X)$ that we will use in this paper:

(1) $0 \leqslant \delta_B(X) \leqslant 1 - \frac{\sqrt{2}}{2} \leqslant \rho_B(X) \leqslant \frac{1}{2}$, for any Banach spaces X (see [7], Proposition 2.3, Corollary 3.2);

(2) If X is uniformly convex, then $\delta_B(X) > 0$ (see [7], Corollary 2.2);

(3) If $\delta_B(X) > 0$ or $\rho_B(X) < \frac{1}{2}$, then X is uniformly non-square (see [7], Proposition 2.14, Proposition 3.6);

(4) X is a Hilbert space if and only if $\delta_B(X) = \rho_B(X) = 1 - \frac{\sqrt{2}}{2}$ (see [7], Theorem 2.13, Theorem 3.4);

(5) Let X be a Radon plane. Then $\delta_B(X) = 0$ if and only if its unit sphere is an affine-regular hexagon (see [7], Theorem 2.18).

On account of the fact that they may not coincide in terms of ρ-orthgonality and Birkhoff orthogonality, unless X is smooth. Moreover, it follows from Example 1 that ρ'-orthogonality and ρ-orthogonality, Birkhoff orthogonality could not coincide. Now it is natural for us to consider the following four parameters:

$$\delta_\perp(\rho, X) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, x \perp_\rho y\right\},$$

$$\delta_\perp(\rho', X) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, x \perp_{\rho'} y\right\},$$

$$\rho_\perp(\rho, X) = \sup\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, x \perp_\rho y\right\},$$

$$\rho_\perp(\rho', X) = \sup\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, x \perp_{\rho'} y\right\},$$

which can be regard as discussing the moduli of convexity and the moduli of smoothness related to ρ-orthogonality and ρ'-orthogonality. The article is planned as follows:

In Section 2, we consider the moduli of convexity $\delta_\perp(\rho, X)$ and $\delta_\perp(\rho', X)$ related to ρ-orthogonality and ρ'-orthogonality. First of all, the connection between $\delta_\perp(\rho, X)$ and $\delta_B(X)$ is built by us. Meanwhile, the exact values of the constant $\delta_\perp(\rho, X)$ in some concrete spaces were also calculated. As an application, we use the lower bound of $\delta_\perp(\rho, X)$ in Radon planes to characterize the Radon plane with affine-regular hexagonal unit sphere. Besides, we also build the connection between $\delta_\perp(\rho', X)$ and other well-known geometric constants. In addition, the exact values of the parameter $\delta_\perp(\rho', X)$ in some specific spaces were also computed by us.

In Section 3, we will be concerned with the moduli of smoothness $\rho_\perp(\rho, X)$ and $\rho_\perp(\rho', X)$ related to ρ-orthogonality and ρ'-orthogonality. First of all, the connection between $\rho_\perp(\rho, X)$ and $\rho_B(X)$ is established by us. Meanwhile, we figure out some accurate values of $\rho_\perp(\rho, X)$ in some specific spaces. In the meantime, we also investigate the relation between $\rho_\perp(\rho', X)$ and some geometric constants. Moreover, the exact values of the parameter $\rho_\perp(\rho', X)$ in some concrete spaces were also figured out by us.

2. The moduli of convexity related to ρ-orthogonality and ρ'-orthogonality

In this section, we shall study the two moduli of convexity related to ρ-orthogonality and ρ'-orthogonality on account of the definition of the classical modulus of convexity, as well as the concepts of ρ-orthogonality and ρ'-orthogonality.

2.1. The modulus of convexity related to ρ-orthogonality

2.1.1. Some basic properties of the parameter $\delta_{\perp}(\rho,X)$

Firstly, by applying $\perp_{\rho}\subset\perp_{B}$ for any Banach spaces X, one can directly obtain the following relation between $\delta_{\perp}(\rho,X)$ and $\delta_{B}(X)$.

LEMMA 1. *Let X be a Banach space. Then $\delta_{B}(X)\leqslant\delta_{\perp}(\rho,X)$.*

Combining the above lemma and the fact that $\delta_{B}(X)\geqslant 0$ for any Banach spaces X, the following conclusion holds:

COROLLARY 1. *Let X be a Banach space. Then $\delta_{\perp}(\rho,X)\geqslant 0$.*

From Lemma 1 and the fact that $\delta_{B}(X)>0$ if X is uniformly convex. Then we derive the following corollary:

COROLLARY 2. *Suppose that X is a Banach space with $\delta_{\perp}(\rho,X)=0$, then X is not uniformly convex.*

In the sequel, we provide two examples to illustrate that the lower bound of $\delta_{\perp}(\rho,X)$ can be attained.

EXAMPLE 2. Let X be the space $\mathbb{R}^2$ with the norm $\|(x_1,x_2)\|=\max\{|x_1|,|x_2|\}$. Then $\delta_{\perp}(\rho,X)=0$.

Proof. Let $x=(1,1)$ and $y=(1,-1)$. Then we have $x,y\in S_X$ and $\|x+y\|=2$. Now it follows from $x+ty=(1+t,1-t)$ that we derive that $\rho_{-}(x,y)=-1$ and $\rho_{+}(x,y)=1$. Then we have $\rho(x,y)=0$ by a direct computation, which implies that $x\perp_{\rho}y$. Thus, according to the definition of $\delta_{\perp}(\rho,X)$ and Corollary 1, we deduce that

$$0\leqslant\delta_{\perp}(\rho,X)\leqslant 1-\frac{\|x+y\|}{2}=0,$$

which completes the proof. □

EXAMPLE 3. Let X be the space $\mathbb{R}^2$ with the norm $\|(x_1,x_2)\|=|x_1|+|x_2|$. Then $\delta_{\perp}(\rho,X)=0$.

Proof. Let $x=(1,0)$ and $y=(0,1)$. Then it is evident that $x,y\in S_X$ and $\|x+y\|=2$. Further, since $x+ty=(1,t)$, we have $x\perp_\rho y$. Now, by combining the definition of $\delta_\perp(\rho,X)$ and Corollary 1, we derive

$$0\leqslant \delta_\perp(\rho,X)\leqslant 1-\frac{\|x+y\|}{2}=0.$$

This completes the proof. $\square$

Notice that we have considered the conditions of $\delta_\perp(\rho,X)=0$ (see Example 2, Example 3), in which the spaces are not uniformly non-square Banach spaces. In fact, one can also deduce that there exists a uniformly non-square Banach space with $\delta_\perp(\rho,X)=0$. Recall that the Banach space X is called uniformly non-square provided that there exists $\delta\in(0,1)$ such that $\left\|\frac{x+y}{2}\right\|\leqslant 1-\delta$ or $\left\|\frac{x-y}{2}\right\|\leqslant 1-\delta$ for all $x,y\in S_X$ (see [11]).

EXAMPLE 4. Let X be the space $\mathbb{R}^2$ with the norm defined by

$$\|(x_1,x_2)\|=\begin{cases}\|(x_1,x_2)\|_2, & (x_1x_2\geqslant 0),\\ \|(x_1,x_2)\|_\infty, & (x_1x_2\leqslant 0).\end{cases}$$

Then $\delta_\perp(\rho,X)=0$.

Proof. Let $x=(-1,1-\frac{1}{n})$ and $y=(0,1)$ with $n\geqslant 1$. It is clear that $x,y\in S_X$ and $\|x+y\|=2-\frac{1}{n}$. And it is easy to check that $x+ty=(-1,1-\frac{1}{n}+t)$. In this situation, one has $x\perp_\rho y$. Moreover, it follows from the definition of $\delta_\perp(\rho,X)$ that we have

$$\delta_\perp(\rho,X)\leqslant 1-\frac{\|x+y\|}{2}=\frac{1}{2n}$$

for all $n\geqslant 1$ and therefore $\delta_\perp(\rho,X)=0$ by Corollary 1 and the above inequality. $\square$

In order to characterize the Hilbert space in terms of $\delta_\perp(\rho,X)$ on smooth Banach spaces, we present the following result by the fact that $\perp_\rho=\perp_B$ if and only if X is smooth.

LEMMA 2. *Let X be a smooth Banach space. Then $\delta_\perp(\rho,X)=\delta_B(X)$.*

From the fact that $\delta_B(X)\leqslant 1-\frac{\sqrt{2}}{2}$ for any Banach spaces X and the above lemma, the following conclusion holds:

COROLLARY 3. *Let X be a smooth Banach space. Then $\delta_\perp(\rho,X)\leqslant 1-\frac{\sqrt{2}}{2}$.*

It is evident that the equality

$$\|x+y\|^4-\|x-y\|^4=8(\|x\|^2\langle x,y\rangle)+\|y\|^2\langle y,x\rangle),(x,y\in X)$$

holds in Hilbert spaces. So as to generalize the above equality, Miličić [17] further introduced the following equality:

$$\|x+y\|^4-\|x-y\|^4=8(\|x\|^2\rho(x,y)+\|y\|^2\rho(y,x)),(x,y\in X). \tag{2}$$

The author in [17] called the space which satisfies the equality (2) as quasi-inner-product space and deduced that the quasi-inner-product space is smooth.

On account of Lemma 2 and the fact that the quasi-inner-product space is smooth, the following corollary is valid:

COROLLARY 4. *Suppose that X is a quasi-inner-product space, then $\delta_{\perp}(\rho,X)=\delta_B(X)$.*

Taking into account of Example 4, it is natural to ask whether there exists a relation between $\delta_{\perp}(\rho,X)$ and uniform non-squareness, but we have yet the more general conclusion about this. However, by Lemma 2 and the fact that X is uniformly non-square if $\delta_B(X)>0$, the following result is true.

COROLLARY 5. *Assume that X is a smooth Banach space with $\delta_{\perp}(\rho,X)>0$, then X is uniformly non-square.*

REMARK 1. Notice that the converse of Lemma 2 is not valid. Actually, if $X=(\mathbb{R}^2,\|\cdot\|_\infty)$, then $\delta_{\perp}(\rho,X)=0$ by Example 2. Moreover, we can deduce that $\delta_B(X)=0$ from Example 2.1 in [7]. And it is clear that $\delta_{\perp}(\rho,X)=\delta_B(X)$. Thus it is clear that the converse is not true since the corresponding space is not smooth.

Next, we give a characterisation of the Hilbert space in terms of the coefficient $\delta_{\perp}(\rho,X)$ on smooth Banach spaces.

THEOREM 1. *Assume that X is a smooth Banach space, then X is a Hilbert space if and only if $\delta_{\perp}(\rho,X)=1-\frac{\sqrt{2}}{2}$.*

Proof. According to Lemma 2 and the fact that X is a Hilbert space if and only if $\delta_B(X)=1-\frac{\sqrt{2}}{2}$, then we can derive that the conclusion holds. □

2.1.2. The applicatin of the coefficient $\delta_{\perp}(\rho,X)$

Notice that an orthogonality "$\perp$" is said to be symmetric, if $x\perp y$ implies $y\perp x$. It is obvious that the usual orthogonality in Hilbert spaces is symmetric. However, the Birkhoff orthogonality is not symmetric in general. But, James showed the following conclusion in [10]:

LEMMA 3. [10] *A Banach space X whose dimension is at least three is a Hilbert space if and only if Birkhoff orthogonality is symmetric in X.*

The assumption on the dimension on the space in the above lemma could not be removed. Actually, if a two-dimensional Banach space is symmetric, then the space is called the Radon plane. For a survey on Radon planes, including further results, can be found in [14].

Since Radon planes are Banach spaces, now we present the following result from Corollary 1:

COROLLARY 6. *Let X be a Radon plane. Then $\delta_{\perp}(\rho,X) \geqslant 0$.*

By an affine-regular hexagon we mean any non-degenerate affine image of the regular hexagon (see [13]). Note that if we take X as $\mathbb{R}^2$ endowed with the norm $\ell_\infty - \ell_1$, then the space X is a Radon plane (see [10]) such that its unit sphere S_X is an affine-regular hexagon (see Figure 1). The following example indicates that the lower bound shown in the above corollary is sharp.

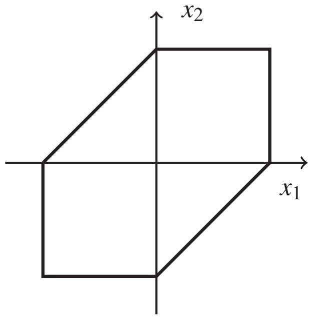

Figure 1: *The unit sphere of* $(\mathbb{R}^2, \ell_\infty - \ell_1)$

EXAMPLE 5. Let X be the space $\mathbb{R}^2$ endowed with the norm defined by

$$\|(x_1,x_2)\| = \begin{cases} \|(x_1,x_2)\|_\infty, & (x_1x_2 \geqslant 0), \\ \|(x_1,x_2)\|_1, & (x_1x_2 \leqslant 0). \end{cases}$$

Then $\delta_{\perp}(\rho,X) = 0$.

Proof. Suppose that $x = (1-\frac{1}{n},1)$ and $y = (1,0)$ with $n \geqslant 1$. Then we derive $x,y \in S_X$ and $\|x+y\| = 2-\frac{1}{n}$. It follows from $x+ty = \left(1-\frac{1}{n}+t,1\right)$ that we obtain $x \perp_\rho y$ by a direct computation. Moreover, we have

$$\delta_{\perp}(\rho,X) \leqslant 1 - \frac{\|x+y\|}{2} = \frac{1}{2n}$$

for any $n \geqslant 1$ by the definition of $\delta_{\perp}(\rho,X)$ and hence $\delta_{\perp}(\rho,X) = 0$ from Corollary 6 and the above inequality. □

In [7], we have derived that there exists an equivalent characterization between the lower bound of $\delta_B(X)$ and that the unit sphere is an affine-regular hexagon under the assumption the space X is a Radon plane. It is natural to ask whether there exists the similar result about $\delta_\perp(\rho,X)$. Now, we provide a positive answer about this.

THEOREM 2. *Let X be a Radon plane. Then $\delta_\perp(\rho,X)=0$ if and only if the unit sphere S_X is an affine-regular hexagon.*

Proof. Assume that $\delta_\perp(\rho,X)=0$, then it follows from the inequality $0\leqslant\delta_B(X)\leqslant\delta_\perp(\rho,X)$ and the fact that $\delta_B(X)=0$ if and only if its unit sphere is an affine-regular hexagon. Therefore, the unit sphere S_X is an affine-regular hexagon.

For the converse, suppose that S_X is an affine-regular hexagon. Then there exist the vectors $u,v\in S_X$ such that $\pm u$, $\pm v$, $\pm(u+v)$ are the vertices of S_X (see Figure 2).

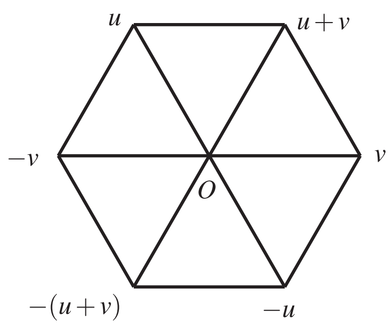

Figure 2: *Affine-regular hexagonal unit sphere.*

Now let $x=(1-\frac{1}{n})(u+v)+\frac{1}{n}v\in S_X$ and $y=u\in S_X$ with $n\geqslant 1$. Then

$$x+ty=\left(1-\frac{1}{n}+t\right)(u+v)+\left(\frac{1}{n}-t\right)v,$$

which means that $\|x+ty\|=1$ whenever $t\to 0^+$ or $t\to 0^-$. Thus it is easy to see that $x\perp_\rho y$. Moreover, we can also derive that

$$\begin{aligned}\|x+y\| &= \left\|\left(2-\frac{1}{n}\right)u+v\right\| = \left(2-\frac{1}{n}\right)\left\|u+\frac{n}{2n-1}v\right\| \\ &= \left(2-\frac{1}{n}\right)\left\|\frac{n-1}{2n-1}u+\frac{n}{2n-1}(u+v)\right\| = 2-\frac{1}{n}.\end{aligned}$$

Then we can obtain

$$\delta_\perp(\rho,X)\leqslant 1-\frac{1}{2}\|x+y\|=\frac{1}{2n}$$

for all $n\geqslant 1$ by the definition of $\delta_\perp(\rho,X)$ and hence $\delta_\perp(\rho,X)=0$ from Corollary 6 and the above inequality. This proves the theorem. □

2.2. The modulus of convexity related to ρ'-orthogonality

Firstly, in the paper [18], it is noted that the modulus of convexity $\delta_X(\varepsilon)$ is reformulated as:

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \|x-y\| \geqslant \varepsilon\right\}.$$

In order to discuss the relationship between $\delta_\perp(\rho', X)$ and $\delta_X(\varepsilon)$, we present the following lemma:

LEMMA 4. ([16]) *Assume that X is a Banach space. If $x, y \in X$, then*
(1) $\rho(x,x) = \|x\|^2$;
(2) $\rho(\alpha x, y) = \rho(x, \alpha y) = \alpha\rho(x,y)$ for any $\alpha \in \mathbb{R}$;
(3) $\rho(x,y) \leqslant \|x\|\|y\|$;
(4) $\rho(x, \alpha x + y) = \alpha\|x\|^2 + \rho(x,y)$ for any $\alpha \in \mathbb{R}$.

LEMMA 5. *Let X be a Banach space. Then $\delta_\perp(\rho', X) \geqslant \delta_X(1)$.*

Proof. For any $x, y \in S_X$ with $x \perp_{\rho'} y$, it follows from Lemma 4 that we have

$$\begin{aligned} 2 &= \|x\|^2 - \|x\|^2\rho(x,y) + \|y\|^2 - \|y\|^2\rho(y,x) \\ &= \|x\|^2 - \rho(x,y) + \|y\|^2 - \rho(y,x) \\ &= \rho(x, x-y) + \rho(y, y-x) \\ &\leqslant 2\|x-y\|, \end{aligned}$$

which implies that $\|x-y\| \geqslant 1$, Thus we can obtain

$$\begin{aligned} \delta_\perp(\rho', X) &= \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, x \perp_{\rho'} y\right\} \\ &\geqslant \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \|x-y\| \geqslant 1\right\} \\ &= \delta_X(1). \end{aligned}$$

This completes the proof. □

COROLLARY 7. *Let X be Banach space. Then $\delta_\perp(\rho', X) \geqslant 0$.*

COROLLARY 8. *Let X be a Banach space. If $\delta_\perp(\rho', X) = 0$, then X is not uniformly convex.*

Proof. Suppose that $\delta_\perp(\rho', X) = 0$, then we have $\delta_X(1) = 0$ by Lemma 5. Hence the corollary holds. □

Now, we provide the following example to indicate that the lower bound of $\delta_\perp(\rho', X)$ can be attained.

EXAMPLE 6. Let X be the space $\mathbb{R}^2$ with the norm $\|(x_1,x_2)\| = \max\{|x_1|,|x_2|\}$. Then $\delta_{\perp}(\rho',X) = 0$.

Proof. Suppose that $x = (1,1)$ and $y = (1,-1)$. It is obvious that $x,y \in S_X$ and $\|x+y\| = 2$. Then from the proof of Example 2, we derive that $\rho(x,y) = 0$. Similarly, it is easily seen that $\rho(y,x) = 0$. Hence $x \perp_{\rho'} y$. Then it follows from the definition of $\delta_{\perp}(\rho',X)$ and Corollary 7 that we can deduce

$$0 \leqslant \delta_{\perp}(\rho',X) \leqslant 1 - \frac{\|x+y\|}{2} = 0,$$

which implies that the desired result. □

The James or non-square constant was defined in [8] as follows:

$$J(X) = \sup\{\min\{\|x+y\|,\|x-y\|\} : x,y \in S_X\}.$$

Later, the equivalent forms of the James constant

$$J(X) = \sup\{\|x+y\| : x,y \in S_X, \|x+y\| = \|x-y\|\}$$

was introduced by some scholars (see Proposition 4.6 in [12] and Theorem 5 in [9]).

Some basic properties of $J(X)$ are as follows:

(i) $\sqrt{2} \leqslant J(X) \leqslant 2$ (see [8], Theorem 2.5).

(ii) $J(X) < 2$ if and only if X is uniformly non-square (see [8], Theorem 3.4).

From the property (i), the best value of James constant is $\sqrt{2}$. It is natural to have the following implication by the parallelogram law:

(iii) If X is a Hilbert space, then $J(X) = \sqrt{2}$.

In the sequel, we shall build the connection between $\delta_{\perp}(\rho',X)$ and $J(X)$ in quasi-inner-product spaces.

LEMMA 6. *Assume that X is a quasi-inner-product space, then* $\delta_{\perp}(\rho',X) = 1 - \frac{1}{2}J(X)$.

Proof. Suppose that X is a quasi-inner-product space, then from (2) we can deduce that

$$x \perp_{\rho'} y \Leftrightarrow \|x+y\| = \|x-y\|$$

for any $x,y \in X$. Hence we derive

$$\begin{aligned}
\delta_{\perp}(\rho',X) &= \inf\left\{1 - \frac{\|x+y\|}{2} : x,y \in S_X, x \perp_{\rho'} y\right\} \\
&= \inf\left\{1 - \frac{\|x+y\|}{2} : x,y \in S_X, \|x+y\| = \|x-y\|\right\} \\
&= 1 - \frac{1}{2}\sup\{\|x+y\| : x,y \in S_X, \|x+y\| = \|x-y\|\} \\
&= 1 - \frac{1}{2}J(X),
\end{aligned}$$

which completes the proof. □

REMARK 2. In fact, the converse of the above lemma is not true. Actually, if $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, we can deduce $\delta_\perp(\rho', X) = 0$ from Example 6, in which X is not a quasi-inner-product space. Moreover, one can also obtain $J(X) = 2$ since X is not uniformly non-square. Then it is obvious that $\delta_\perp(\rho', X) = 1 - \frac{1}{2}J(X)$. Thus the converse does not hold.

It is well known that ℓ^4 is a quasi-inner-product space by Lemma 1 in [17]. Hence it follows from Lemma 6 that the following conclusion holds:

EXAMPLE 7. Let $X = \ell^4$. Then $\delta_\perp(\rho', X) = 1 - 2^{-\frac{1}{4}}$.

Proof. From Theorem 3.1 in [8], we have $J(X) = 2^{\frac{3}{4}}$. Then the conclusion is true by Lemma 6. □

Next, we study the relationship between $\delta_\perp(\rho', X)$ and $\delta_\perp(\rho, X)$ on the smooth Radon plane.

LEMMA 7. *Let X be a smooth Radon plane. Then $\delta_\perp(\rho', X) \leqslant \delta_\perp(\rho, X)$.*

Proof. For any $x, y \in S_X$ such that $x \perp_B y$, we can conclude that $x \perp_\rho y$ since X is smooth. Hence it is obvious that $\rho(x, y) = 0$. And since X is a Radon plane, then $y \perp_B x$. Hence it is clear that $y \perp_\rho x$. This implies that $\rho(y, x) = 0$. Furthermore, it is evident that $x \perp_{\rho'} y$. Then, due to Lemma 2, we have

$$\begin{aligned}
\delta_\perp(\rho', X) &= \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, x \perp_{\rho'} y\right\} \\
&\leqslant \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, x \perp_B y\right\} \\
&= \delta_B(X) = \delta_\perp(\rho, X).
\end{aligned}$$

This completes the proof. □

In the end, we give the main result in this subsection.

THEOREM 3. *Suppose that X is a smooth Radon plane, then $\delta_\perp(\rho', X) = 1 - \frac{\sqrt{2}}{2}$ if and only if X is a Hilbert space.*

Proof. Let $\delta_\perp(\rho', X) = 1 - \frac{\sqrt{2}}{2}$. Then $\delta_\perp(\rho, X) = 1 - \frac{\sqrt{2}}{2}$ by virtue of Lemma 7 and Corollary 3. Thus the space X is a Hilbert space by Theorem 1.

Conversely, suppose that X is a Hilbert space, then one has $J(X) = \sqrt{2}$. Therefore we derive $\delta_\perp(\rho', X) = 1 - \frac{\sqrt{2}}{2}$ by Lemma 6. This proves the theorem. □

3. The moduli of smoothness related to ρ-orthogonality and ρ'-orthogonality

In this section, we shall consider the moduli of smoothness related to ρ-orthogonality and ρ'-orthogonality based on the definition of the modulus of smoothness as well as the notions of ρ-orthogonality and ρ'-orthogonality.

3.1. The modulus of smoothness related to ρ-orthogonality

Similar to Lemma 2, the following relation between $\rho_\perp(\rho,X)$ and $\rho_B(X)$ holds.

LEMMA 8. *Let X be a Banach space. Then $\rho_\perp(\rho,X) \leqslant \rho_B(X)$.*

By the above lemma and the fact that $\rho_B(X) \leqslant \frac{1}{2}$ for any Banach spaces X, one can derive the following corollary:

COROLLARY 9. *Let X be a Banach space. Then $\rho_\perp(\rho,X) \leqslant \frac{1}{2}$.*

Now we provide the following two examples to illustrate that the upper bound of $\rho_\perp(\rho,X)$ can be attained.

EXAMPLE 8. Let X be the space $\mathbb{R}^2$ with the norm $\|(x_1,x_2)\| = \max\{|x_1|,|x_2|\}$. Then $\rho_\perp(\rho,X) = \frac{1}{2}$.

Proof. Suppose that $x = (1,0)$ and $y = (0,1)$. It is clear that $x,y \in S_X$ and $\|x+y\| = 1$. From $x+ty = (1,t)$, it is not hard to calculate

$$\rho(x,y) = \rho_-(x,y) = \rho_+(x,y) = 0,$$

which means that $x \perp_\rho y$. Thus it follows from the definition of $\rho_\perp(\rho,X)$ and Corollary 9 that we have

$$\frac{1}{2} = 1 - \frac{\|x+y\|}{2} \leqslant \rho_\perp(\rho,X) \leqslant \frac{1}{2},$$

which implies that $\rho_\perp(\rho,X) = \frac{1}{2}$. $\square$

EXAMPLE 9. Let X be the space $\mathbb{R}^2$ with the norm $\|(x_1,x_2)\| = |x_1| + |x_2|$. Then $\rho_\perp(\rho,X) = \frac{1}{2}$.

Proof. Assume that $x = \left(-\frac{1}{2},-\frac{1}{2}\right)$ and $y = \left(\frac{1}{2},-\frac{1}{2}\right)$. It is obvious that $x,y \in S_X$ and $\|x+y\| = 1$. Due to the fact that $x+ty = \left(-\frac{1}{2}+\frac{1}{2}t,-\frac{1}{2}-\frac{1}{2}t\right)$, we have

$$\rho(x,y) = \rho_-(x,y) = \rho_+(x,y) = 0,$$

which implies that $x \perp_\rho y$. Consequently, we obtain

$$\frac{1}{2} = 1 - \frac{\|x+y\|}{2} \leqslant \rho_\perp(\rho,X) \leqslant \frac{1}{2},$$

from the definition of $\rho_\perp(\rho,X)$ and Corollary 9. This completes the proof. $\square$

Note that we have considered the conditions of $\rho_\perp(\rho,X) = \frac{1}{2}$ by Example 8 and Example 9, in which the spaces are not uniformly non-square Banach spaces. In fact, one can also conclude that there exists a uniformly non-square Banach space with $\rho_\perp(\rho,X) = \frac{1}{2}$.

EXAMPLE 10. Let X be the space $\mathbb{R}^2$ with the norm defined by

$$\|x\| = \|(x_1,x_2)\| = \begin{cases} \|(x_1,x_2)\|_\infty, & (x_1x_2 \geqslant 0), \\ \|(x_1,x_2)\|_1, & (x_1x_2 \leqslant 0). \end{cases}$$

Then $\rho_\perp(\rho,X) = \frac{1}{2}$.

Proof. Let $x = \left(\frac{1}{n},1\right)$ and $y = (1,0)$ with $n \geqslant 1$. It is clear that $x,y \in S_X$ and $\|x+y\| = 1+\frac{1}{n}$. And it follows from $x+ty = \left(\frac{1}{n}+t,1\right)$ that we have

$$\rho(x,y) = \rho_-(x,y) = \rho_+(x,y) = 0,$$

which indicates that $x \perp_\rho y$. Thus we conclude

$$\frac{1}{2} - \frac{1}{2n} = 1 - \frac{\|x+y\|}{2} \leqslant \rho_\perp(\rho,X)$$

for any $n \geqslant 1$ by the definition of $\rho_\perp(\rho,X)$ and thus $\rho_\perp(\rho,X) = \frac{1}{2}$ by Corollary 9 and the above inequality. $\square$

So as to characterize the Hilbert space in terms of $\rho_\perp(\rho,X)$ on smooth Banach spaces, we give the following conclusion by the fact that $\perp_\rho = \perp_B$ if and only if X is smooth.

LEMMA 9. *Let X be a smooth Banach space. Then $\rho_\perp(\rho,X) = \rho_B(X)$.*

From the above lemma and the fact that $1-\frac{\sqrt{2}}{2} \leqslant \rho_B(X) \leqslant \frac{1}{2}$ for all Banach spaces X, the following conclusion is clearly valid.

COROLLARY 10. *If X is a smooth Banach space, then $1-\frac{\sqrt{2}}{2} \leqslant \rho_\perp(\rho,X) \leqslant \frac{1}{2}$.*

Combining Lemma 9 and the fact that X is uniformly non-square if $\rho_B(X) < \frac{1}{2}$, one can also obtain the following conclusion:

COROLLARY 11. *Let X be a smooth Banach space with $\rho_\perp(\rho,X) < \frac{1}{2}$, then X is uniformly non-square.*

REMARK 3. In fact, the converse of Lemma 9 is not valid. Actually, when $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, one can deduce $\rho_\perp(\rho,X) = \frac{1}{2}$ from Example 8. Moreover, it follows from Example 3.1 in [7] that $\rho_B(X) = \frac{1}{2}$. Hence $\rho_\perp(\rho,X) = \rho_B(X)$, but the corresponding space X is not smooth. Thus the converse is not true.

Similar to Theorem 1, we also derive the following equivalent characterization of Hilbert spaces in terms of $\rho_\perp(\rho,X)$ on smooth Banach spaces.

THEOREM 4. *Let X be a smooth Banach space. Then $\rho_\perp(\rho,X) = 1-\frac{\sqrt{2}}{2}$ if and only if X is a Hilbert space.*

3.2. The modulus of smoothness related to ρ'-orthogonality

First , we give the upper bound of $\rho_{\perp}(\rho',X)$.

LEMMA 10. *Let X be a Banach space. Then $\rho_{\perp}(\rho',X) \leqslant \frac{1}{2}$.*

Proof. For all $x,y \in S_X$ satisfying $x \perp_{\rho'} y$, from Lemma 4, we can derive that

$$\begin{aligned} 2 &= \|x\|^2 + \|x\|^2\rho(x,y) + \|y\|^2 + \|y\|^2\rho(y,x) \\ &= \|x\|^2 + \rho(x,y) + \|y\|^2 + \rho(y,x) \\ &= \rho(x,x+y) + \rho(y,x+y) \\ &\leqslant 2\|x+y\|, \end{aligned}$$

which means that $\|x+y\| \geqslant 1$. Thus we can obtain

$$1 - \frac{\|x+y\|}{2} \leqslant 1 - \frac{1}{2} = \frac{1}{2},$$

which completes the proof. □

Next, we provide the following example to illustrate that the upper bound of $\rho_{\perp}(\rho',X)$ can be attained.

EXAMPLE 11. Let X be the space $\mathbb{R}^2$ with the norm $\|(x_1,x_2)\| = |x_1| + |x_2|$. Then $\rho_{\perp}(\rho',X) = \frac{1}{2}$.

Proof. Assume that $x = \left(-\frac{1}{2}, -\frac{1}{2}\right)$ and $y = \left(\frac{1}{2}, -\frac{1}{2}\right)$, it is obvious that $x,y \in S_X$ and $\|x+y\| = 1$. It is easily seen that $\rho(x,y) = \rho(y,x) = 0$, then we can deduce that $x \perp_{\rho'} y$. Moreover, we can conclude

$$\frac{1}{2} = 1 - \frac{\|x+y\|}{2} \leqslant \rho_{\perp}(\rho',X) \leqslant \frac{1}{2}$$

from the definition of $\rho_{\perp}(\rho',X)$ and Lemma 10. Thus we obtain $\rho_{\perp}(\rho',X) = \frac{1}{2}$. □

The Schäffer constant $S(X)$ was studied in [8] as follows:

$$S(X) = \inf\{\max\{\|x+y\|, \|x-y\|\} : x,y \in S_X\}.$$

Later, the equivalent forms of the Schäffer constant

$$S(X) = \inf\{\|x+y\| : x,y \in S_X, \|x+y\| = \|x-y\|\}$$

was introduced by some scholars (see Proposition 4.6 in [12] and Theorem 5 in [9]).

Some elementary properties of $S(X)$ are as follows:

(I) $1 \leqslant S(X) \leqslant \sqrt{2}$ (see [8], Theorem 2.5).

(II) $J(X)S(X) = 2$ (see [8], Theorem 2.5).

From the properties (ii), (iii) and (II), the following conclusions hold:

(III) $S(X)>1$ if and only if X is uniformly non-square.

(IV) If X is a Hilbert space, then $S(X)=\sqrt{2}$.

Now, we study the relation between $\rho_\perp(\rho',X)$ and $S(X)$ in quasi-inner- product spaces.

LEMMA 11. *Let X be a quasi-inner-product space. Then $\rho_\perp(\rho',X)=1-\frac{1}{2}S(X)$.*

Proof. Suppose that X is a quasi-inner-product space, then it follows from (2) that we have

$$x\perp_{\rho'}y \Leftrightarrow \|x+y\|=\|x-y\|$$

for all $x,y\in X$. Thus we can obtain

$$\begin{aligned}
\rho_\perp(\rho',X) &= \sup\left\{1-\frac{\|x+y\|}{2}:x,y\in S_X, x\perp_{\rho'}y\right\}\\
&= \sup\left\{1-\frac{\|x+y\|}{2}:x,y\in S_X, \|x+y\|=\|x-y\|\right\}\\
&= 1-\frac{1}{2}\inf\{\|x+y\|:x,y\in S_X, \|x+y\|=\|x-y\|\}\\
&= 1-\frac{1}{2}S(X).
\end{aligned}$$

This completes the proof. □

REMARK 4. Note that the converse of the above lemma does not hold. In fact, if $X=(\mathbb{R}^2,\|\cdot\|_1)$, we can conclude $\rho_\perp(\rho',X)=\frac{1}{2}$ from Example 11. In addition, we derive that $S(X)=1$ since X is not uniformly non-square. And it is clear that $\rho_\perp(\rho',X)=1-\frac{1}{2}S(X)$. Then the converse is not true since the corresponding space X is not a quasi-inner-product space.

Similar to Example 7, the following result holds:

EXAMPLE 12. Let $X=\ell^4$. Then $\rho_\perp(\rho',X)=1-2^{-\frac{3}{4}}$.

Proof. From Theorem 3.1 in [8], one can obtain $S(X)=2^{\frac{1}{4}}$. Hence we can conclude that $\rho_\perp(\rho',X)=1-2^{-\frac{3}{4}}$ from the above lemma. □

Next, we also discuss the relation between $\rho_\perp(\rho',X)$ and $\rho_\perp(\rho,X)$ on the smooth Radon plane.

LEMMA 12. *If X is a smooth Radon plane, then $\rho_\perp(\rho',X)\geqslant\rho_\perp(\rho,X)$.*

Proof. From the proof of Lemma 7, we can derive that $\perp_B \subset \perp_{\rho'}$. Hence it follows from Lemma 9 and the definition of $\rho_\perp(\rho',X)$ that we have

$$\begin{aligned}\rho_\perp(\rho',X) &= \sup\left\{1-\frac{\|x+y\|}{2} : x,y\in S_X, x\perp_{\rho'} y\right\}\\ &\geqslant \sup\left\{1-\frac{\|x+y\|}{2} : x,y\in S_X, x\perp_B y\right\}\\ &= \rho_B(X) = \rho_\perp(\rho,X),\end{aligned}$$

which finishes the proof. □

Now, by virtue of the above lemma and Corollary 10, one can deduce the following result:

COROLLARY 12. *Let X be a smooth Radon plane. Then $\rho_\perp(\rho',X)\geqslant 1-\frac{\sqrt{2}}{2}$.*

In the end, we will show that the property $\rho_\perp(\rho',X) = 1-\frac{\sqrt{2}}{2}$ characterizes the Hilbert space on smooth Radon planes.

THEOREM 5. *Let X be a smooth Radon plane. Then $\rho_\perp(\rho',X) = 1-\frac{\sqrt{2}}{2}$ if and only if X is a Hilbert space.*

Proof. Suppose that $\rho_\perp(\rho',X) = 1-\frac{\sqrt{2}}{2}$, then it follows from Lemma 12 and Corollary 10 that we derive $\rho_\perp(\rho,X) = 1-\frac{\sqrt{2}}{2}$. Thus the space X is a Hilbert space by Theorem 4.

Conversely, if X is a Hilbert space, then one has $S(X)=\sqrt{2}$. Hence we can deduce $\rho_\perp(\rho',X) = 1-\frac{\sqrt{2}}{2}$ from Lemma 11. □

Conclusions

In this paper, we introduce the two new moduli of convexity $\delta_\perp(\rho,X)$ and $\delta_\perp(\rho',X)$ related to ρ-orthogonality and ρ'-orthogonality. It is of interest to discuss the connections with other geometric constants and calculate the exact values of the new moduli on some specific spaces. Moreover, we offer an exploration of the parameter $\delta_\perp(\rho,X)$ on the Radon plane. Furthermore, the characterization of the Radon plane with affine-regular hexagonal unit sphere in terms of $\delta_\perp(\rho,X)$ is given by us. Similarly, we also study the two new moduli of smoothness $\rho_\perp(\rho,X)$ and $\rho_\perp(\rho',X)$ related to ρ-orthogonality and ρ'-orthogonality. However, there are still lots of pointless issues that await discussion. How can these four constants be used to characterize more geometric properties? How to computate the values of these constants for other classical Banach spaces? Therefore, more relevant results about these parameters will be left the reader who are interested in the theory of geometric constants in Banach spaces.

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