MOVING AROUND THE SUMS OF ORTHOGONAL UNIT VECTORS

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Abstract. We discuss the extreme values of the sum of two Birkhoff orthogonal unit vectors in a normed space. In addition, we obtain some relationships between these values with some moduli of convexity and smoothness, as well as with the notions of uniform convexity or uniform non-squareness. Finally, we present some illustrative examples.

1. Introduction

Let $(X, \| \cdot \|)$ be a real Banach space with unit ball $B$ and sphere $S$. If $X$ is Hilbert we shall denote it by $H$. Given $u, v \in X$, we say $u$ is Birkhoff orthogonal to $v$ (denoted by $u \perp v$) if $\|u\| \leq \|u + \lambda v\|$ for every $\lambda \in \mathbb{R}$. If $S$ is a regular $n$-gon with $n = 4$ or $n = 6$, we can easily find pairs of unit orthogonal vectors such that $u + v$ or $u + v^2$ belong to $S$. Along this paper we study the relationship between the range of $u + v$ (where $u$ and $v$ are orthogonal unit vectors) and concepts like modulus of convexity, uniform convexity, uniform non-squareness or modulus of smoothness.

$X$ is strictly convex (SC for short) if $\|x + y\| < 2$ for all different unit vectors $x, y$. $X$ is uniformly convex (UC for short) if for every $0 < \epsilon < 2$ there exists $\delta > 0$ such that $\|x + y\| \leq 2(1 - \delta)$ for all $x, y \in B$ with $\|x - y\| \geq \epsilon$. $X$ is uniformly non-square (UNS for short) if there exists $\delta > 0$ such that $\|x + y\| \leq 2(1 - \delta)$ for all $x, y \in B$ with $\|x - y\| > 2(1 - \delta)$. Uniform convexity implies strict convexity and uniform non-squareness. Uniform convexity and strict convexity are equivalent concepts if $X$ is finite dimensional, but there exist infinite dimensional normed spaces that are SC but not UC. $\mathbb{R}^2$ endowed with the hexagonal norm is UNS but not SC. If $X$ is UC (or also UNS) then $X$ is reflexive. See [14] or [4] for details.

For $\epsilon \in [0, 2]$, the modulus of convexity $\delta(\epsilon)$, the coefficient of convexity $\epsilon_0$, and the modulus of smoothness $\rho(\epsilon)$ of $X$ are defined in the following way:

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S; \|x - y\| = \epsilon\right\}$$

$$= \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1; \|y\| \leq 1; \|x - y\| \geq \epsilon\right\},$$

$$\epsilon_0 = \sup\{\epsilon \in [0, 2] : \delta(\epsilon) = 0\},$$


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\[ \rho(\varepsilon) = \sup \left\{ 1 - \frac{\|x+y\|}{2} : x, y \in S; \|x-y\| = \varepsilon \right\} \]

\[ = \sup \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \geq 1; \|y\| \geq 1; \|x-y\| \leq \varepsilon \right\}. \]

These are some properties for \( \delta(\varepsilon), \varepsilon_0, \) and \( \rho(\varepsilon) \) (see [14], [2], and [4]):

**Proposition 1.** Let \( X \) be a normed space. Then:

1. \( X \) is UC \( \iff \delta(\varepsilon) > 0 \ \forall \varepsilon \in (0, 2] \iff \varepsilon_0 = 0. \)
2. \( X \) is SC \( \iff \delta(2) = 1. \)
3. \( X \) is not UNS iff one of the following conditions holds:
   - \( (a) \) \( \delta(\varepsilon) = 0 \) for all \( \varepsilon \in [0, 2) \)
   - \( (b) \) \( \varepsilon_0 = 2 \)
   - \( (c) \) \( \rho(\varepsilon) = \frac{\varepsilon}{2} \) for some \( \varepsilon \in (0, 2) \)
   - \( (d) \) \( \rho(\varepsilon) = \frac{\varepsilon}{2} \) for all \( \varepsilon \in [0, 2] \)
4. \( \delta(\varepsilon) \) and \( \frac{\delta(\varepsilon)}{\varepsilon} \) are non-decreasing. Besides, \( \delta(\varepsilon) \) is strictly increasing for \( \varepsilon \in [\varepsilon_0, 2]. \) \( \delta(\varepsilon) \) is continuous in \( [0, 2) \) and \( \lim_{\varepsilon \to 2^-} \delta(\varepsilon) = 1 - \frac{\varepsilon_0}{2}. \) In general, \( \delta(\varepsilon) \) is not convex.
5. \( \rho(\varepsilon) \) is strictly increasing, continuous and convex in \( [0, 2]. \) \( \frac{\rho(\varepsilon)}{\varepsilon} \) is non-decreasing in \( (0, 2]. \)
6. \( \delta(\varepsilon) \leq 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \leq \rho(\varepsilon) \) for every \( \varepsilon \in [0, 2], \) and equalities hold (for all \( \varepsilon \)) if and only if \( X \) is Hilbert. Besides, \( \rho(\varepsilon) \leq \frac{\varepsilon}{2} \).

Our discussion in the next sections will be deeply concerned with the following not yet well studied conjugate parameters:

\[ h^-_X = \inf \{\|x+y\| : x, y \in S; x \perp y\} \]

\[ h^+_X = \sup \{\|x+y\| : x, y \in S; x \perp y\}. \]

We shall write simply \( h^- \) and \( h^+ \) when no confusion can arise. Clearly \( 1 \leq h^- \leq h^+ \leq 2 \) by the properties of the norm and the definition of Birkhoff orthogonality, but also (see Proposition 3 in Section 3) the following is true:

\[ 1 \leq h^-_X \leq h^-_H = \sqrt{2} = h^+_H \leq h^+_X \leq 2. \]
2. The starting point

Our paper was partly suggested by the following result proved in [13] (p. 590):

**Proposition 2.** Let $X$ be a UC-normed space and $z = u + v$, with $\|u\| = \|v\|$ and $u \perp v$. Then there exists $\varepsilon > 0$ such that:

1. If $z \in S$, then $\|u\| = \|v\| \leq 1 - \varepsilon$.
2. If $u, v \in S$, then $\|z\| \geq 1 + \varepsilon$.

**Proof.** (1) is proved in [13]. For (2), apply (1) to $z' = \frac{z}{\|z\|}, u' = \frac{u}{\|z\|}$ and $v' = \frac{v}{\|z\|}$. □

We will show that the above result can be generalized, with rather simple proofs, in several ways.

**Theorem 1.** Let $X$ be a normed space. Then for all $a \in \left[ \frac{1}{\sqrt{5}}, \frac{1}{h} \right]$ we have

$$\delta(2a - 1) \leq \frac{1 - a}{2}. \quad (i)$$

**Proof.** Let $u, v \in S$, $u \perp v$, and $z = u + v$. Set $a = 1/\|z\| \in \left( \frac{1}{\sqrt{5}}, \frac{1}{h} \right)$ and $z' = az \in S$. We have

$$\|z' - u\| = \|a(u + v) - u\| = \|av - (1 - a)u\| \geq a - (1 - a) = 2a - 1.$$

Thus $\left\| \frac{z' + u}{2} \right\| \leq 1 - \delta(2a - 1)$. On the other hand,

$$\left\| \frac{z' + u}{2} \right\| = \frac{\|a(u + v) + u\|}{2} = \frac{\left\| (1 + a)u + av \right\|}{2} = \frac{(1 + a)}{2} \left\| u + \frac{a}{(1 + a)} v \right\| \geq \frac{1 + a}{2}.$$

Therefore $\frac{1 + a}{2} \leq 1 - \delta(2a - 1)$, and $\delta(2a - 1) \leq \frac{1 - a}{2}$ for all $a \in \left( \frac{1}{\sqrt{5}}, \frac{1}{h} \right)$. Since $\delta(2a - 1)$ and $\frac{1 - a}{2}$ are continuous functions, we obtain the thesis. □

**Remark 1.** Since $\left[ \frac{1}{\sqrt{5}}, \frac{1}{h} \right] \subset \left[ \frac{1}{2}, 1 \right]$, Theorem 1 can be applied at most for $a \in \left[ \frac{1}{2}, 1 \right]$. Besides, from (6) in Proposition 1 we have

$$\delta(2a - 1) \leq 1 - \sqrt{1 - \frac{(2a - 1)^2}{4}} \leq \frac{1 - a}{2}$$

for all $a \in \left[ \frac{1}{2}, \frac{\sqrt{11} + 1}{5} \right]$. Thus Theorem 1 has some interest only for $a \in (\frac{\sqrt{11} + 1}{5}, 1]$. 


REMARK 2. A reason for (i) in Theorem 1 being not sharp: If $X$ is Hilbert we have always $\frac{1}{2} = \|u+v\| = \sqrt{2} \approx 1.414$ for any orthogonal vectors $u, v \in S$. Formula (i) only gives $\frac{1}{a} \geq \frac{5}{\sqrt{11}+1} \approx 1.158$; in proving Theorem 1 we use $\|u(1+a) + av\| \geq 1 + a$, while in Hilbert spaces we have $\|u(1+a) + av\| = \sqrt{(1+a)^2 + a^2} > 1 + a$, and $\|u(a-1) + av\| \geq 2a - 1$, while in Hilbert spaces $\|u(a-1) + av\| = \sqrt{(a-1)^2 + a^2} > 2a - 1$. 

For $X$ finite dimensional, $h^- = 1$ or $h^+ = 2$ implies $\varepsilon_0 \geq 1$. This can be justified as follows. If $h^- = 1$ there exist $u, v \in S$ with $u \perp v$ such that $\|u + v\| = 1$, and then the segment joining $u$ and $u + v$ (of length 1) must be on $S$. Therefore, $\delta(1) = 0$. If $h^+ = 2$, there exist $u, v \in S$, with $u \perp v$ such that $\|\frac{u+v}{2}\| = 1$, and then the whole segment joining $u$ and $v$ (of length at least 1) must be on $S$. Thus $\delta(1) = 0$. Theorem 2 generalizes all the above results for any normed space.

THEOREM 2. Let $X$ be a normed space. Then $\varepsilon_0 < 1 \implies h^- > 1$ and $h^+ < 2$.

Proof. The function $f(x) := \delta(2x - 1)$ is non decreasing respect to $x$, while $g(x) := \frac{1}{\sqrt{2x}}$ is strictly decreasing. Both functions are continuous for $x \in [\frac{1}{2}, \frac{3}{2})$.

For $x = \frac{\varepsilon_0 + 1}{2} < 1$ we have $f\left(\frac{\varepsilon_0 + 1}{2}\right) = \delta(\varepsilon_0) = 0 < g\left(\frac{\varepsilon_0 + 1}{2}\right) = \frac{1}{\sqrt{2\varepsilon_0}}.

For $x = 1$ we have $f(1) = \delta(1) > g(1) = 0.

Thus, there exists a unique $\tilde{a} \in (\frac{\varepsilon_0 + 1}{2}, 1)$ such that $\delta(2\tilde{a} - 1) = \frac{1 - \tilde{a}}{2}$, and $\delta(2x - 1) \leq \frac{1 - x}{2}$ only for $x \leq \tilde{a}$.

Let $z = u + v$ with $u, v \in S$ and $u \perp v$. Denote $a = \frac{1}{\|z\|}$. According to Theorem 1, $\delta(2a - 1) \leq \frac{1 - a}{2}$ and then $a \leq \tilde{a}$. Therefore $\|z\| = \frac{1}{a} \geq \frac{1}{\tilde{a}} > 1$, and $h^- \geq \frac{1}{\tilde{a}} > 1$.

Now let $\tilde{\delta}(1) > 0$. Let us consider any pair $u, v \in S$ such that $u \perp v$. Since $\|u - v\| \geq 1$, we have $\frac{\|u + v\|}{2} \leq 1 - \tilde{\delta}(1)$ and $\|u + v\| \leq 2 - 2\tilde{\delta}(1)$. Thus, we conclude $h^+ \leq 2 - 2\tilde{\delta}(1) < 2$. \(\square\)

COROLLARY 1. Let $X$ be a normed space. Then $\varepsilon_0 < 1 \implies \delta(2/h^- - 1) \leq \frac{1}{2} - \frac{1}{2h^-}$.

Proof. From the proof of Theorem 2 we know that $\tilde{a} \geq \frac{1}{h^-}$, where $\tilde{a}$ verifies $\delta(2\tilde{a} - 1) = \frac{1 - \tilde{a}}{2}$. Since $\delta(2x - 1)$ is non decreasing and $\frac{1 - x}{2}$ is strictly decreasing we have $\delta(2/h^- - 1) \leq \delta(2\tilde{a} - 1) = \frac{1 - \tilde{a}}{2} \leq \frac{1 - \frac{1}{h^-}}{2} = \frac{1}{2} - \frac{1}{2h^-}$. \(\square\)

Due to (6) in Proposition 1 (see Remark 1), the value denoted $\tilde{a}$ in the above proof belongs to $[\sqrt{\frac{\|u\|+1}{2}, 1}]$.

Concerning the reverse implication of Theorem 2, see the discussion in the last section.
3. Some history and known results about $h^-$ and $h^+$

These two numbers, or at least the first one, have been considered in a few papers during the years starting from [6]. In that paper the inequality $J \leq \frac{2}{h^-}$ between $h^-$ and the Jung constant $J$ was indicated. A short history of $h^-$ up to 1989 and some connections with other parameters was traced in [12], a not easily available paper. But a detailed study of $h^-$ and $h^+$ seems to be still lacking. For a recent attempt for generalizing them, see [10], Section 3. The following result is known.

**Proposition 3.** Let $X$ be a normed space. Then

$$1 \leq h^- \leq h^+_H = \sqrt{2} \leq h^+_R \leq 2.$$

*Proof.* Let $X_2$ be a 2-dimensional subspace of $X$ and $S_2 = S \cap X_2$. For $x, y \in S_2$, let $[x, y] = \sigma$ denote the orientation of $x, y$ respect to a fixed orientation $\sigma$ in $S_2$. Let us consider this set:

$$S_B = \{x + y : x, y \in S_2; x \perp y; [x, y] = \sigma\}.$$

Joly ([11]) (see also [1]) proves that $S_B$ is a rectifiable Jordan curve that is symmetric respect to the origin and delimits an area in $X_2$ equal to two times the area delimited by $S_2$. Since $\sqrt{2}S$ has the same properties, $S_B$ and $\sqrt{2}S$ must share at least two common points. Therefore, for every normed space $X$ there exists a vector $x + y \in S_B \cap \sqrt{2}S$, and $h^- \leq \sqrt{2} \leq h^+$. The rest of the equalities and inequalities are obvious consequences of the properties of the norms and the definition of Birkhoff orthogonality. □

From Proposition 3 we obtain the following (characterizations (10.3) and (10.3') in [3], p. 79).

**Theorem 3.** Let $X$ be a normed space. The following are equivalent:

1. $X$ is Hilbert.
2. $h^- \geq \sqrt{2}$ (equivalently, $h^- = \sqrt{2}$).
3. $h^+ \leq \sqrt{2}$ (equivalently, $h^+ = \sqrt{2}$).
4. $x, y \in S$, $x \perp y$ imply $\|x - y\| = \sqrt{2}$.
5. $h^- = h^+$.

Therefore, if $X$ is not Hilbert, there exist $x_1, y_1, x_2, y_2 \in S$ such that $x_1 \perp y_1$, $x_2 \perp y_2$ with $\|x_1 + y_1\| < \sqrt{2}$ and $\|x_2 + y_2\| > \sqrt{2}$.

We recall the following result, given in [6], with an awkward proof (based also on a result concerning $h^-$ and the Jung’s constant of $X$).

**Proposition 4.** Let $X$ be a normed space and $t_0 \in \mathbb{R}$ such that $t_0 + 2\delta(t_0) = 1$. Then $h^- \geq \frac{1}{t_0}$.
If \( \varepsilon_0 \geq 1 \) then \( t_0 = 1 \) and we only obtain \( h^- \geq 1 \). If \( \delta(1) > 0 \) then \( t_0 < 1 \) and we obtain \( h^- > 1 \), which is one of the statements of Theorem 2.

**Remark 3.** Note that the proof of Theorem 2 gives the estimate \( h^- \geq \frac{1}{a} \) when \( \varepsilon_0 < 1 \), where \( a \in \left(\sqrt{\frac{3}{5}}, 1\right) \) verifies \( \delta(2a - 1) = \frac{1-a}{2} \). Since \( 2a - 1 < a \) and \( \delta(.) \) is non decreasing, we have \( 1 = t_0 + 2\delta(t_0) = a + 2\delta(2a - 1) \leq a + 2\delta(a) \).

Therefore, for \( \varepsilon_0 < 1 \) we conclude that \( t_0 \leq a \in \left(\sqrt{\frac{3}{5}}, 1\right) \), and \( h^- \geq \frac{1}{t_0} \geq \frac{1}{a} \).

**Remark 4.** Also the estimation provided by Proposition 4 is not sharp. If \( t_0 + 2\delta(t_0) = 1 \) then

\[
t_0 + 2 \left(1 - \sqrt{1 - \frac{t_0^2}{4}}\right) \geq 1 \iff 2t_0^2 + 2t_0 - 3 \geq 0 \iff t_0 \geq \frac{\sqrt{7} - 1}{2}
\]

and the best lower bound from the above proposition would be \( h^- \geq \frac{1}{t_0} \geq \frac{2}{\sqrt{7} - 1} \approx 1.215 \), which would be the estimation for a Hilbert space. Instead \( h_H = \sqrt{2} \approx 1.414 \).

### 4. Our constants and moduli

In this section we focus on the modulus of smoothness \( \rho(\varepsilon) \) and its connections with \( \delta(\varepsilon) \), \( h^- \), and \( h^+ \).

**Theorem 4.** Let \( X \) be a normed space. Then the following inequalities hold:

1. \( 2(1 - \rho(h^-)) \leq h^+ \)
2. \( h^+ \leq 2(1 - \delta(h^-)) \)
3. \( 2(1 - \rho(h^+)) \leq h^- \)
4. \( h^- \leq 2(1 - \delta(h^+)) \) if \( h^+ < 2 \).
   
   For finite dimensional spaces, the inequality holds also if \( h^+ = 2 \).
5. If \( h^+ = 2 \), then \( h^- \leq 2 - \lim_{\varepsilon \to 2^-} \delta(\varepsilon) = \varepsilon_0 \); thus, \( 1 \leq h^- \leq \varepsilon_0 \) and \( \delta(1) = 0 \).

**Proof.** (1). Given \( \sigma > 0 \), we can find \( x, y \in S, x \perp y, \) such that \( ||x - y|| < h^- + \sigma \). This implies

\[
1 - \rho(h^- + \sigma) \leq \frac{||x + y||}{2},
\]

and \( 2(1 - \rho(h^- + \sigma)) \leq h^+ \). Since \( \rho(.) \) is continuous we obtain \( 2(1 - \rho(h^-)) \leq h^+ \).

(2). Take \( x, y \in S \) with \( x \perp y \). We have \( ||x - y|| \geq h^- \), so \( \frac{||x + y||}{2} \leq 1 - \delta(h^-) \). Thus \( h^- \leq 2(1 - \delta(h^-)) \).
(3). Given $\sigma > 0$, we can take $x, y \in S$, $x \perp y$ such that $\|x + y\| < h^- + \sigma$. We have

$$1 - \rho(h^+) \leq 1 - \rho(\|x - y\|) \leq \frac{\|x + y\|}{2} < \frac{h^- + \sigma}{2}.$$ 

Since $\sigma$ is arbitrary, we obtain $2(1 - \rho(h^+)) \leq h^-.$

(4). Given $\sigma > 0$, we can take $x, y \in S$ such that $x \perp y$ and $\|x - y\| > h^+ - \sigma$. By the definition of $\delta(.)$, we have

$$\frac{\|x + y\|}{2} \leq 1 - \delta(h^+ - \sigma)$$

and $h^- \leq 2(1 - \delta(h^+ - \sigma))$. Since $\delta(.)$ is continuous in $[0, 2]$, if $h^+ < 2$ we obtain $h^- \leq 2(1 - \delta(h^+))$.

If $X$ is finite dimensional and $h^+ = 2$, there exist $x, y \in S$ with $x \perp y$ and $\|x - y\| = 2$. Then $\frac{\|x + y\|}{2} \leq 1 - \delta(2)$ and $h^- \leq 2(1 - \delta(h^+))$ also holds for $h^+ = 2$.

(5). If $h^+ = 2$, given $\sigma > 0$ we conclude that $h^- \leq 2(1 - \delta(h^+ - \sigma))$ similarly to (4). Since $\lim_{\varepsilon \to 2} \delta(\varepsilon) = 1 - \frac{\varepsilon_0}{2}$ (see Proposition 1), we obtain

$$h^- \leq \lim_{\sigma \to 0} 2(1 - \delta(h^+ - \sigma)) = \lim_{\varepsilon \to 2^-} 2(1 - \delta(\varepsilon)) = \varepsilon_0,$$

so $1 \leq h^- \leq \varepsilon_0$ and $\delta(1) = 0$. □

**Remark 5.** We recall that the following equalities are true:

(*) $\rho(2(1 - \rho(\varepsilon))) = 1 - \frac{\varepsilon}{2}$ (see (4.3) in [4]).

(**) $\delta(2(1 - \delta(\varepsilon))) = 1 - \frac{\varepsilon}{2}$ if $\varepsilon \geq \varepsilon_0$ (see [9], p. 56).

By means of (*), the inequalities (1) and (3) of Theorem 4 are equivalent. In fact, applying (*) to inequality (1) we obtain

$$\frac{1 - h^-}{2} \leq \rho(h^+),$$

that is (3); and applying (*) to inequality (3) we deduce

$$\frac{1 - h^+}{2} \leq \rho(h^-),$$

that is (1).

Besides, (2) and (4) are almost equivalent by (**): if $h^- \geq \varepsilon_0$ applying (**) to (2) we have

$$\delta(h^+) \leq 1 - \frac{h^-}{2},$$

that is (4); if $h^+ \geq \varepsilon_0$, similarly (4) implies

$$\delta(h^-) \leq 1 - \frac{h^+}{2},$$

that is (2).
We briefly discuss in which range of values the inequalities in Theorem 4 are interesting.

(1) is useful only when $2(1 - \rho(h^-)) > \sqrt{2}$, and the greatest value of $2(1 - \rho(h^-))$ is achieved for the minimum value of $h^-$. Since $h^+ \geq \sqrt{2}$ and

$$2(1 - \rho(h^-)) \leq 2(1 - \rho(1)) \leq 2 \left(1 - \left(1 - \sqrt{1 - \frac{1}{4}}\right)\right) = \sqrt{3},$$

inequality (1) is interesting when $2(1 - \rho(h^-)) \in (\sqrt{2}, \sqrt{3}]$. The best lower bound deduced from (1) would be $h^+ \geq \sqrt{3}$. If $h^- = \sqrt{2}$, then $X$ is Hilbert, $h^+ = \sqrt{2}$, and (1) is sharp:

$$2(1 - \rho(\sqrt{2})) = 2 \left(1 - \left(1 - \sqrt{1 - \frac{1}{4}}\right)\right) = \sqrt{2}.$$

(2) is always non-trivial if $\delta(1) > 0$ because $h^- \geq 1$. The range of values of interest for $2(1 - \delta(h^-))$ is $[\sqrt{2}, 2]$. If $h^- = 1$, then

$$2(1 - \delta(1)) \geq 2 \left(1 - \left(1 - \sqrt{1 - \frac{1}{4}}\right)\right) = \sqrt{3},$$

and $2(1 - \delta(h^-)) \in [\sqrt{3}, 2]$. We present examples of spaces (see Example 4 in Section 5) with $h^- = 1$ and $h^+$ slightly greater than $\sqrt{3}$, and also $h^-$ slightly greater than 1 and $h^+$ smaller than $\sqrt{3}$. From (2) we deduce that $h^+ = 2$ implies $\delta(h^-) = 0$ and $1 \leq h^- \leq \varepsilon_0$, which is (5).

From (3) we obtain a non-trivial estimate when $2(1 - \rho(h^+)) > 1$. In that case

$$\frac{1}{2} > \rho(h^+) \geq 1 - \sqrt{1 - \frac{(h^+)^2}{4}} \iff h^+ < \sqrt{3}.$$ 

The range of values of interest of $2(1 - \rho(h^+))$ in (3) is $[1, \sqrt{2}]$.

(4) is non-trivial if $2(1 - \delta(h^+)) \leq \sqrt{2}$, and the range of values of interest for $2(1 - \delta(h^+))$ in (4) is $[1, \sqrt{2}]$.

Inequality (4) is not always true when $h^+ = 2$: there are spaces with $\delta(2) = 1$, $h^+ = 2$, and obviously $h^- \geq 1$ (see Example 1 in Section 5).

We indicate two consequences of Theorem 4. Since $\delta(\sqrt{3}) \leq 1 - \sqrt{1 - \frac{(\sqrt{3})^2}{4}} = \frac{1}{2}$, we have that $\delta(h^+)$ could be greater than $\frac{1}{2}$ for $h^+ > \sqrt{3}$, and the next result is non-trivial when $h^+ > \sqrt{3}$.

**Corollary 2.** Let $X$ be a normed space. If one of the following conditions hold:

1. $h^+ < 2$,
2. $X$ is finite dimensional,
(3) \( h^- \geq \varepsilon_0 \),
then \( \delta(h^+) \leq 1 - \frac{h^-}{2} \leq \frac{1}{2} \).

**Proof.** If \( h^+ < 2 \) or \( X \) is finite dimensional, then this is (4) of Theorem 4. If \( h^+ \geq \varepsilon_0 \), then we can apply (***) as explained in Remark 5. \( \Box \)

Therefore \( \delta(h^+) \leq \frac{1}{2} \) if any of the conditions of Corollary 2 holds. The inequality \( \delta(h^+) \leq 1 - \frac{h^-}{2} \) is not always true (see again Example 1 in Section 5).

**COROLLARY 3.** Let \( X \) be a normed space. We always have:

(1) \( 2 \left( 1 - \frac{\sqrt{2}}{2} \right) \leq 2(1 - \rho(\sqrt{2})) \leq 2(1 - \rho(h^-)) \leq h^- \leq 2(1 - \delta(h^-)) \leq 2(1 - \delta(1)). \)

(2) \( 0 \leq 2(1 - \rho(h^+)) \leq h^- \leq \begin{cases} 2(1 - \delta(h^+)) & \text{if dim}(X) < \infty \text{ or } h^+ < 2 \\ 2(1 - \lim_{\varepsilon \to 0} \delta(\varepsilon)) & \text{otherwise.} \end{cases} \)

**Proof.** Since \( \delta(.) \) and \( \rho(.) \) are non-decreasing functions, (1) and (2) of Theorem 4 joint with Proposition 1 imply the first chain of inequalities. The second one is consequence of (3), (4), and (5) of Theorem 4. \( \Box \)

**REMARK 6.** The inequalities of Theorem 4 are mostly sharp in the sense that equality is possible (in Hilbert spaces or when \( h^- \) and/or \( h^+ \) has one of its extreme values). All the inequalities are not equalities in general.

New result is connected with \( \varepsilon_0 \).

**THEOREM 5.** Let \( X \) a finite dimensional normed space and \( \alpha = \min \{ \varepsilon_0, 1 \} \). Then \( h^- \leq 2 - \alpha \).

In particular, \( \varepsilon_0 \geq 1 \) implies \( h^- = 1 \).

**Proof.** Note that \( \alpha \leq \varepsilon_0 \), so \( \delta(\alpha) = 0 \).

For every \( n \in \mathbb{N} \) we can find \( x_n, y_n \in S \) with \( \|x_n - y_n\| = \alpha \) and \( \|\frac{x_n + y_n}{2}\| > 1 - \frac{1}{n} \). By compactness, we can find \( x, y \in S \) such that \( \|x - y\| = \alpha \) and \( \|\frac{x + y}{2}\| \geq 1 \), so \( \|\frac{x + y}{2}\| = 1 \).

This implies that \( S \) contains \( x, y, \) and \( \frac{x + y}{2} \). If we set \( f(\lambda) = \|x + \lambda(y - x)\| \), we have \( f(0) = f(1) = f(\frac{1}{2}) \), and then \( 1 = \min \{ \|x + \lambda(y - x)\| : \lambda \in \mathbb{R} \} \) by convexity.

Therefore, \( x \perp y - x \).

Set \( z = \frac{y - x}{\alpha} \in S \). Then

\[
\|x + z\| \leq \|x + y - x\| + \|z - (y - x)\|
\]

\[
= 1 + \left\| \frac{y - x}{\alpha} + (x - y) \right\| = 1 + \left( \frac{1 - \alpha}{\alpha} \right) \|x - y\| = 2 - \alpha. \hspace{1em} \Box
\]
Remark 7. Clearly the above inequality is not sharp. Indeed, since \( h^- \leq \sqrt{2} \) for every \( X \) (in particular whenever \( \varepsilon_0 > 0 \)), the above estimate is trivial for \( \alpha \leq 2 - \sqrt{2} \).

Remark 8. Theorem 5 and inequality (4) of Theorem 4 are formally independent. Let us compare both for finite dimensional spaces. If \( \varepsilon_0 = 0 \), then \( 2\delta(h^+) \) can be larger than \( \alpha = 0 \). If \( \varepsilon_0 = 1.1 \), then \( \alpha = 1 \) can be larger than \( 2\delta(h^+) \): in fact, \( \lim_{\varepsilon \to 2^-} \delta(\varepsilon) = 1 - \frac{\varepsilon_0}{2} \) always holds (see Proposition 1), thus, if \( h^+ < 2 \) we obtain

\[
2\delta(h^+) \leq 2 \lim_{\varepsilon \to 2^-} \delta(\varepsilon) = 2 \left( 1 - \frac{1.1}{2} \right) = 0.9 < \alpha.
\]

Corollary 4. Let \( X \) be a normed space and \( \varepsilon_0 \geq 1 \). If \( X \) is finite dimensional or \( h^+ = 2 \), then \( h^- \leq \varepsilon_0 \).

Proof. This is consequence of Theorem 5 and (5) of Theorem 4. \( \square \)

Remark 9. We do not know if the thesis of Theorem 5 is true for infinite dimensional spaces. We also proved some other results under the assumption \( \dim(X) < \infty \).

In finite dimensional spaces the uniform convexity (\( \varepsilon_0 = 0 \)) is equivalent to strict convexity (\( \delta(2) = 1 \), see Proposition 1). Nevertheless, in this context passing to infinite dimension is dangerous. In fact, there exist spaces that are SC but are not UNS (\( \varepsilon_0 = 2 \)). We present Example 1 in Section 5 (see also [14], p. 11).

5. Some examples

Finally we present some examples to show the potential of the results in the above sections.

Example 1. \( l^p(\mathbb{R}^2) \): \( 1 = h^- < \varepsilon_0 = h^+ = 2 \).

Given \( p > 1 \), let \( l^p(\mathbb{R}^2) = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2 \ldots \oplus \mathbb{R}^2 \) be the space of sequences \( X = (x_2, x_3, \ldots, x_n, \ldots) \), where \( \mathbb{R}^2_n \) denotes the space \( \mathbb{R}^2 \) with the norm \( \| x_n \| = \| (x_{n_1}, x_{n_2}) \| = (|x_{n_1}|^n + |x_{n_2}|^n)^{1/n} \), and

\[
\| (x_2, x_3, \ldots, x_n, \ldots) \| = \left( \sum_{n=2}^{\infty} \| x_n \|^p \right)^{1/p}.
\]

\( S \) contains curves nearer and nearer to segments of length 1, so \( \delta(1) = 0 \). Let \( X_n, Y_n, U_n, V_n \), for \( n \in \mathbb{N} \) be as follows:

\[
X_n = \| (x_2, x_3, \ldots) \| \in l^p(\mathbb{R}^2) \text{ such that } x_i = (0, 0) \text{ for } i \neq n, \text{ and } x_n = \frac{(1,1)}{\| (1,1) \|};
\]

\[
Y_n = \| (y_2, y_3, \ldots) \| \in l^p(\mathbb{R}^2) \text{ such that } y_i = (0, 0) \text{ for } i \neq n, \text{ and } y_n = \frac{(-1,1)}{\| (-1,1) \|};
\]

\[
U_n = \| (u_2, u_3, \ldots) \| \in l^p(\mathbb{R}^2) \text{ such that } u_i = (0, 0) \text{ for } i \neq n, \text{ and } u_n = \frac{(1,0)}{\| (1,0) \|};
\]

\[
V_n = \| (v_2, v_3, \ldots) \| \in l^p(\mathbb{R}^2) \text{ such that } v_i = (0, 0) \text{ for } i \neq n, \text{ and } v_n = \frac{(0,1)}{\| (0,1) \|}.
\]
We have that $X, Y, U, V \in S$; $U \perp V$, and $\|U + V\|$ tends to 1 when $n$ tends to infinity, so $h^- = 1$. $X \perp Y$, and $\|X \pm Y\|$ tends to 2 when $n$ tends to infinity, therefore $h^+ = 2$, and given $\varepsilon > 0$ we can find $X, Y \in S$, $X \perp Y$, with $\|X \pm Y\| > 2 - \varepsilon$. Consequently, $\ell^p(\mathbb{R}^2)$ is not UNS ($\varepsilon_0 = 2$), it is SC ($\delta(2) = 1$) but it is not UC (see [7]; [14], p. 10).

**EXAMPLE 2.** ($\mathbb{R}^2, \|\cdot\|_\infty$) or ($\mathbb{R}^2, \|\cdot\|_1$): $1 = h^- < \varepsilon_0 = h^+ = 2$.

Let us consider $\mathbb{R}^2$ endowed with the norm $\|x\|_\infty = \|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$ and $\|x\|_1 = \|(x_1, x_2)\|_1 = |x_1| + |x_2|$ for any $x = (x_1, x_2) \in \mathbb{R}^2$.

Let us consider $x = (1, 0), y = (0, 1), u = (1, 1), v = (1, -1) \in \mathbb{R}^2$. We have that $\|x\|_\infty = \|y\|_\infty = \|u\|_\infty = \|v\|_\infty = 1$; $x \perp y$, and $\|x + y\|_\infty = 1$, so $h^- = 1$. $u \perp v$, and $\|u + v\|_\infty = 2$, so $h^+ = 2$.

We can find easily a similar example that justifies $h^- = 1$ and $h^+ = 2$ for ($\mathbb{R}^2, \|\cdot\|_1$).

Both ($\mathbb{R}^2, \|\cdot\|_\infty$) and ($\mathbb{R}^2, \|\cdot\|_1$) are not SC ($\delta(2) < 1$), not UNS ($\varepsilon_0 = 2$), and not UC.

The above examples can be generalized as follows. In [5] these parameters were defined and studied:

$$g_\perp = \inf\{\max\{|x - y|, |x + y|\} : x, y \in S, x \perp y\}$$

and

$$J_\perp = \sup\{\min\{|x - y|, |x + y|\} : x, y \in S, x \perp y\}.$$  

In particular, (6.8) in [5] proves the following.

**PROPOSITION 5.** $X$ is not UNS $\iff J_\perp = 2$ $\iff g_\perp = 1$.

It is clear that $h^- \leq g_\perp \leq J_\perp \leq h^+$ always, thus we obtain the following.

**THEOREM 6.** If $X$ is not UNS, then $h^- = 1$ and $h^+ = 2$.

We can have $h^- < g_\perp$ and $h^+ > J_\perp$ as the following example shows. Another example where these strict inequalities hold will be Example 5.

**EXAMPLE 3.** ($\mathbb{R}^2, \|\cdot\|_{6n}$). $h^- = \varepsilon_0 = 1 < h^+ = 2$.

Let us consider $\mathbb{R}^2$ endowed with the norm

$$\|x\|_{6n} = \|(x_1, x_2)\|_{6n} = \max\{|x_1|, |x_2|\}, x_1x_2 \geq 0$$

so that the norm is determined by a regular hexagon whose vertices are $(\pm 1, 0)$, $(1, 1)$, $(0, \pm 1)$, $(-1, -1)$.

Let $x = (1, 1)$ and $y = (0, 1)$. We have that $\|x\|_{6n} = \|y\|_{6n} = 1$; $x \perp y$, $\|x + y\|_{6n} = 2$, and $\|x - y\|_{6n} = 1$, so $h^- = 1$ and $h^+ = 2$. Besides, $\varepsilon_0 = 1$ and $g_\perp = J_\perp = 3/2$ (see Example 5.1 in [5]).

($\mathbb{R}^2, \|\cdot\|_{6n}$) is not UC but UNS.
EXAMPLE 4. \((\mathbb{R}^2, \|\cdot\|_\circ)\). \(h^- = \varepsilon_0 = 1 < h^+ \approx 1.7602\).

Let us consider \(\mathbb{R}^2\) endowed with the norm \(\|\cdot\|_\circ\) whose unit sphere is the set

\[ S = \{(x, \pm 1) : |x| \leq 1\} \cup \left\{ \left( x, \pm \sqrt{1 - (x - 1)^2} \right) : 1 \leq x \leq 2 \right\} \]

\[ \cup \left\{ \left( x, \pm \sqrt{1 - (x + 1)^2} \right) : -2 \leq x \leq -1 \right\}, \]

and the vectors:

\[ x = \left( \frac{\sqrt{33} + 7}{8}, \frac{\sqrt{33} + 15}{\sqrt{32}} \right) \approx (1.5931, 0.8051), \]

\[ y = \left( -\frac{\sqrt{33} + 15}{16}, \frac{9 - \sqrt{33}(\sqrt{33} - 1) \sqrt{33} + 7}{\sqrt{32}} \right) \approx (-1.2965, 0.9550), \]

\[ u = (-1, 1), \text{ and } v = (2, 0). \]

We have that \(x, y, u, v \in S\); \(x \perp y\) and \(\|x + y\| \approx 1.7602\), which is the value of \(h^+\); \(u \perp v\) and \(\|u + v\| = 1\), which is the value of \(h^-\). In this example we also have \(h^- \cdot h^+ < 2\).

\((\mathbb{R}^2, \|\cdot\|_\circ)\) is not SC and it is UNS (\(\varepsilon_0 = 1\)).

We obtain \(h^- \approx 1.01\) and \(h^+ \approx 1.72\) if the unit ball of the space is slightly modified to the following:

\[ S = \{(x, \pm 1) : |x| \leq 0.8\} \cup \left\{ \left( x, \pm \sqrt{1 - (x - 0.8)^2} \right) : 0.8 \leq x \leq 1.8 \right\} \]

\[ \cup \left\{ \left( x, \pm \sqrt{1 - (x + 0.8)^2} \right) : -1.8 \leq x \leq -0.8 \right\}. \]

EXAMPLE 5. \((\mathbb{R}^2, \|\cdot\|_{8n})\). \(\varepsilon_0 \approx 0.83 < h^- \approx 1.1213 < h^+ \approx 1.7071\).

Let us consider \(\mathbb{R}^2\) endowed with the norm \(\|\cdot\|_{8n}\) whose unit ball is a regular octagon with vertices \((\pm 1, 0), (\pm \sqrt{2}, \sqrt{2}), (0, \pm 1), (\pm \sqrt{2}, -\sqrt{2})\). Let us consider \(x = (1, 0)\) and \(y = \left( \frac{1}{\sqrt{8}}, \frac{\sqrt{2} + 1}{\sqrt{8}} \right)\). We have that \(x, y \in S\) and \(x \perp y\); \(\|x + y\| = \frac{1 + \sqrt{8}}{2\sqrt{2} - 2} \approx 1.7071\), which is the value of \(h^+\); and \(\|x - y\| = \frac{7(\sqrt{2} + 1)}{5\sqrt{2} + 8} \approx 1.1213\), which is the value of \(h^-\). Moreover, \(g_\perp = J_\perp = \sqrt{2}\) (see Example 5.3 in [5]), but \(\|\cdot\|_{8n}\) is not Hilbert: \(1.1213 \approx h^- < g_\perp = \sqrt{2} = J_\perp < h^+ \approx 1.7071\). Of course it is not SC, \(\delta(1) > 0\). Moreover, \(\varepsilon_0 = \frac{2}{\sqrt{2} + 1} \approx 0.83\), which is the distance between two consecutive vertices.

EXAMPLE 6. \((\mathbb{R}^2, \|\cdot\|_{10n})\). \(\varepsilon_0 \approx 0.6180 < h^- \approx 1.2361 < h^+ \approx 1.6180\).

Let us consider \(\mathbb{R}^2\) endowed with the norm \(\|\cdot\|_{10n}\) whose unit ball is a regular decagon whose vertices are \((\pm 1, 0), (\pm \cos(\pi/5), \sin(\pi/5)), (\pm \cos(2\pi/5), \sin(2\pi/5)), (\pm \cos(\pi/5), -\sin(\pi/5)), (\pm \cos(2\pi/5), -\sin(2\pi/5))\). Let us consider \(x = (1, 0)\) and \(y = (-\cos(2\pi/5), \sin(2\pi/5))\). We have that \(x \perp y\); \(\|x + y\| = \frac{2\sin(\pi/5)}{\sin(2\pi/5) + \sin(\pi/5)} \approx 1.2361\), which is the value of \(h^-\); and \(\|x - y\| = 2\cos(\pi/5) \approx 1.6180\), which is the value of
Moreover, $\varepsilon_0 = \frac{\sin(\pi/5)}{\sin(2\pi/5)} \approx 0.6180$, which is the distance between two consecutive vertices.

**Example 7.** $(\mathbb{R}^2, \|\cdot\|_{12n})$. $\varepsilon_0 \approx 0.5359 < h^- \approx 1.2321 < h^+ \approx 1.5981$.

Let us consider $\mathbb{R}^2$ endowed with the norm $\|\cdot\|_{12n}$ whose unit ball is a regular $n$-gon with $n = 12$ whose vertices are $(\cos(k\pi/6), \sin(k\pi/6))$, where $k \in \{0, \ldots, 11\}$. Let $x = (1,0)$, $y' = (-\cos(7\pi/12),\sin(7\pi/12))$ and $y = \frac{y'}{\|y'\|}$. We have that $x \perp y$; $\|x + y\| \approx 1.2321$, which is the value of $h^-$; and $\|x - y\| \approx 1.5981$, which is the value of $h^+$. Moreover, $\varepsilon_0 = \frac{2}{\sqrt{3} + 2} \approx 0.5359$, which is the distance between two consecutive vertices.

**Example 8.** $(\mathbb{R}^2, \|\cdot\|_p)$. $0 = \varepsilon_0 < 1 < h^- < h^+ < 2$

Given $p > 1$, let us consider $\mathbb{R}^2$ endowed with the norm $\|(x_1, x_2)\|_p = (|x_1|^p + |x_2|^p)^{1/p}$. The following approximate values of $h^+$ and $h^-$ have been computed for some $p$:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$h^-$</th>
<th>$h^+$</th>
<th>$h^- \cdot h^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.10</td>
<td>1.052273</td>
<td>1.942347</td>
<td>2.04388</td>
</tr>
<tr>
<td>1.20</td>
<td>1.102418</td>
<td>1.872711</td>
<td>2.06451</td>
</tr>
<tr>
<td>1.25</td>
<td>1.126613</td>
<td>1.836794</td>
<td>2.06937</td>
</tr>
<tr>
<td>1.3333333</td>
<td>1.165565</td>
<td>1.777520</td>
<td>2.07182</td>
</tr>
<tr>
<td>1.50</td>
<td>1.238135</td>
<td>1.666284</td>
<td>2.06308</td>
</tr>
<tr>
<td>1.65</td>
<td>1.297327</td>
<td>1.577695</td>
<td>2.04679</td>
</tr>
<tr>
<td>1.75</td>
<td>1.336529</td>
<td>1.525049</td>
<td>2.03388</td>
</tr>
<tr>
<td>2.00</td>
<td>1.414214</td>
<td>1.414214</td>
<td>2.00000</td>
</tr>
<tr>
<td>2.50</td>
<td>1.262378</td>
<td>1.538204</td>
<td>1.94179</td>
</tr>
<tr>
<td>3.00</td>
<td>1.170664</td>
<td>1.625597</td>
<td>1.90303</td>
</tr>
<tr>
<td>4.00</td>
<td>1.076115</td>
<td>1.735040</td>
<td>1.86710</td>
</tr>
<tr>
<td>5.00</td>
<td>1.035052</td>
<td>1.798064</td>
<td>1.86109</td>
</tr>
<tr>
<td>6.00</td>
<td>1.016343</td>
<td>1.838066</td>
<td>1.86811</td>
</tr>
<tr>
<td>10.00</td>
<td>1.000818</td>
<td>1.911642</td>
<td>1.91321</td>
</tr>
<tr>
<td>15.00</td>
<td>1.000022</td>
<td>1.944524</td>
<td>1.94457</td>
</tr>
</tbody>
</table>

It seems that in general $h^- \cdot h^+ \neq 2$, and $h^- \cdot h^+$ is greater than 2 for $1 < p < 2$ and smaller than 2 for $p > 2$. For some conjugate indices $p$ and $q$ (namely, $1/p + 1/q = 1$), for example $p = 1.2$ and $q = 6$, the values of $h^-$ and $h^+$ are different.

6. **Concluding and resuming. Open questions**

For many parameters in Banach spaces, the values at the extremes of their ranges are related to *very bad* spaces, like non-UNS spaces. The situation concerning $h^+$ and $h^+$ seems to be slightly different. Instead, the *central value* ($h^- = \sqrt{2}$ and $h^+ = \sqrt{2}$) characterizes Hilbert spaces. Figure 1 summarizes the general situation concerning extreme values.
In particular, $h^+ = 2$ implies $\delta(1) = 0$, which is equivalent to $h^- = 1$ when $X$ is finite dimensional. We are not able to prove if, in general, $\delta(1) = 0$ implies $h^- = 1$, or at least $h^+ = 2$ implies $h^- = 1$. Also: is Theorem 5 valid in any normed space?

For the sake of completeness, we recall that a recent paper (see [8]) indicates some facts considered here. In particular, Proposition 2.14 there proves that $h^+ < 2$ implies the space is UNS (but not conversely).

It would be interesting to clarify the relationship between $h^-$ and $\epsilon_0$ for infinite dimensional spaces when $h^+ < 2$ (see Corollary 4 and the examples in Section 5).

Finally, it is not clear if the functions $h^-$ and $h^+$ are continuous with respect to the Banach-Mazur distance.

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