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MOVING AROUND THE SUMS OF ORTHOGONAL UNIT VECTORS

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Abstract. We discuss the extreme values of the sum of two Birkhoff orthogonal unit vectors in a normed space. In addition, we obtain some relationships between these values with some moduli of convexity and smoothness, as well as with the notions of uniform convexity or uniform non-squareness. Finally, we present some illustrative examples.

1. Introduction

Let $(X, \|.\|)$ be a real Banach space with unit ball B and sphere S. If X is Hilbert we shall denote it by H. Given $u, v \in X$, we say u is Birkhoff orthogonal to v (denoted by $u \perp v$) if $\|u\| \leqslant \|u + \lambda v\|$ for every $\lambda \in \mathbb{R}$. If S is a regular n-gon wiht $n = 4$ or $n = 6$, we can easily find pairs of unit orthogonal vectors such that $u + v$ or $\frac{u+v}{2}$ belong to S. Along this paper we study the relationship between the range of $u + v$ (where u and v are orthogonal unit vectors) and concepts like modulus of convexity, uniform convexity, uniform non-squareness or modulus of smoothness.

X is *strictly convex* (SC for short) if $\|x + y\| < 2$ for all different unit vectors x, y. X is *uniformly convex* (UC for short) if for every $0 < \varepsilon \leqslant 2$ there exists $\delta > 0$ such that $\|x + y\| \leqslant 2(1 - \delta)$ for all $x, y \in B$ with $\|x - y\| \geqslant \varepsilon$. X is *uniformly non-square* (UNS for short) if there exists $\delta > 0$ such that $\|x + y\| \leqslant 2(1 - \delta)$ for all $x, y \in B$ with $\|x - y\| > 2(1 - \delta)$. Uniform convexity implies strict convexity and uniform non-squareness. Uniform convexity and strict convexity are equivalent concepts if X is finite dimensional, but there exist infinite dimensional normed spaces that are SC but not UC. $\mathbb{R}^2$ endowed with the hexagonal norm is UNS but not SC. If X is UC (or also UNS) then X is reflexive. See [14] or [4] for details.

For $\varepsilon \in [0, 2]$, the *modulus of convexity* $\delta(\varepsilon)$, the *coefficient of convexity* ε_0, and the *modulus of smoothness* $\rho(\varepsilon)$ of X are defined in the following way:

$$\begin{aligned}\delta(\varepsilon) &= \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S;\ \|x - y\| = \varepsilon\right\} \\ &= \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leqslant 1;\ \|y\| \leqslant 1;\ \|x - y\| \geqslant \varepsilon\right\},\end{aligned}$$

$$\varepsilon_0 = \sup\{\varepsilon \in [0, 2] : \delta(\varepsilon) = 0\},$$

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$$\rho(\varepsilon) = \sup\left\{1 - \frac{\|x+y\|}{2} : x,y \in S;\ \|x-y\| = \varepsilon\right\}$$
$$= \sup\left\{1 - \frac{\|x+y\|}{2} : \|x\| \geqslant 1;\ \|y\| \geqslant 1;\ \|x-y\| \leqslant \varepsilon\right\}.$$

These are some properties for $\delta(\varepsilon)$, ε_0, and $\rho(\varepsilon)$ (see [14], [2], and [4]):

PROPOSITION 1. *Let X be a normed space. Then:*

(1) *X is UC* $\iff \delta(\varepsilon) > 0 \quad \forall \varepsilon \in (0,2] \iff \varepsilon_0 = 0$.

(2) *X is SC* $\iff \delta(2) = 1$.

(3) *X is not UNS iff one of the following conditions holds:*

(a) $\delta(\varepsilon) = 0$ *for all* $\varepsilon \in [0,2)$

(b) $\varepsilon_0 = 2$

(c) $\rho(\varepsilon) = \frac{\varepsilon}{2}$ *for some* $\varepsilon \in (0,2)$

(d) $\rho(\varepsilon) = \frac{\varepsilon}{2}$ *for all* $\varepsilon \in [0,2]$

(4) $\delta(\varepsilon)$ *and* $\frac{\delta(\varepsilon)}{\varepsilon}$ *are non-decreasing. Besides,* $\delta(\varepsilon)$ *is strictly increasing for* $\varepsilon \in [\varepsilon_0, 2]$. $\delta(\varepsilon)$ *is continuous in* $[0,2)$ *and* $\lim_{\varepsilon\to 2^-} \delta(\varepsilon) = 1 - \frac{\varepsilon_0}{2}$. *In general,* $\delta(\varepsilon)$ *is not convex.*

(5) $\rho(\varepsilon)$ *is strictly increasing, continuous and convex in* $[0,2]$. $\frac{\rho(\varepsilon)}{\varepsilon}$ *is non-decreasing in* $(0,2]$.

(6) $\delta(\varepsilon) \leqslant 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \leqslant \rho(\varepsilon)$ *for every* $\varepsilon \in [0,2]$, *and equalities hold (for all* ε*) if and only if X is Hilbert. Besides,* $\rho(\varepsilon) \leqslant \frac{\varepsilon}{2}$.

Our discussion in the next sections will be deeply concerned with the following not yet well studied conjugate parameters:

$$h_X^- = \inf\{\|x+y\| : x,y \in S;\ x \perp y\}$$

$$h_X^+ = \sup\{\|x+y\| : x,y \in S;\ x \perp y\}.$$

We shall write simply h^- and h^+ when no confusion can arise. Clearly $1 \leqslant h^- \leqslant h^+ \leqslant 2$ by the properties of the norm and the definition of Birkhoff orthogonality, but also (see Proposition 3 in Section 3) the following is true:

$$1 \leqslant h_X^- \leqslant h_H^- = \sqrt{2} = h_H^+ \leqslant h_X^+ \leqslant 2.$$

2. The starting point

Our paper was partly suggested by the following result proved in [13] (p. 590):

PROPOSITION 2. *Let X be a UC-normed space and $z = u+v$, with $\|u\| = \|v\|$ and $u \perp v$. Then there exists $\varepsilon > 0$ such that:*

(1) *If $z \in S$, then $\|u\| = \|v\| \leqslant 1-\varepsilon$.*

(2) *If $u,v \in S$, then $\|z\| \geqslant 1+\varepsilon$.*

Proof. (1) is proved in [13]. For (2), apply (1) to $z' = \frac{z}{\|z\|}$, $u' = \frac{u}{\|z\|}$ and $v' = \frac{v}{\|z\|}$. □

We will show that the above result can be generalized, with rather simple proofs, in several ways.

THEOREM 1. *Let X be a normed space. Then for all $a \in \left[\frac{1}{h^+}, \frac{1}{h^-}\right]$ we have*

$$\delta(2a-1) \leqslant \frac{1-a}{2}. \tag{i}$$

Proof. Let $u,v \in S$, $u \perp v$, and $z = u+v$. Set $a = 1/\|z\| \in \left(\frac{1}{h^+}, \frac{1}{h^-}\right)$ and $z' = az \in S$. We have

$$\|z'-u\| = \|a(u+v)-u\| = \|av-(1-a)u\| \geqslant a-(1-a) = 2a-1.$$

Thus $\|\frac{z'+u}{2}\| \leqslant 1-\delta(2a-1)$. On the other hand,

$$\begin{aligned}\left\|\frac{z'+u}{2}\right\| &= \left\|\frac{a(u+v)+u}{2}\right\| \\ &= \left\|\frac{(1+a)u+av}{2}\right\| \\ &= \frac{(1+a)}{2}\left\|u+\frac{a}{(1+a)}v\right\| \geqslant \frac{1+a}{2}.\end{aligned}$$

Therefore $\frac{1+a}{2} \leqslant 1-\delta(2a-1)$, and $\delta(2a-1) \leqslant \frac{1-a}{2}$ for all $a \in \left(\frac{1}{h^+}, \frac{1}{h^-}\right)$. Since $\delta(2a-1)$ and $\frac{1-a}{2}$ are continuous functions, we obtain the thesis. □

REMARK 1. Since $\left[\frac{1}{h^+}, \frac{1}{h^-}\right] \subset \left[\frac{1}{2}, 1\right]$, Theorem 1 can be applied at most for $a \in [\frac{1}{2}, 1]$. Besides, from (6) in Proposition 1 we have

$$\delta(2a-1) \leqslant 1-\sqrt{1-\frac{(2a-1)^2}{4}} \leqslant \frac{1-a}{2}$$

for all $a \in [\frac{1}{2}, \frac{\sqrt{11}+1}{5}]$. Thus Theorem 1 has some interest only for $a \in (\frac{\sqrt{11}+1}{5}, 1]$.

REMARK 2. A reason for (i) in Theorem 1 being not sharp: If X is Hilbert we have always $\frac{1}{a} = \|u+v\| = \sqrt{2} \approx 1.414$ for any orthogonal vectors $u,v \in S$. Formula (i) only gives $\frac{1}{a} \geqslant \frac{5}{\sqrt{11}+1} \approx 1.158$; in proving Theorem 1 we use $\|u(1+a)+av\| \geqslant 1+a$, while in Hilbert spaces we have $\|u(1+a)+av\| = \sqrt{(1+a)^2+a^2} > 1+a$, and $\|u(a-1)+av\| \geqslant 2a-1$, while in Hilbert spaces $\|u(a-1)+av\| = \sqrt{(a-1)^2+a^2} > 2a-1$.

For X finite dimensional, $h^- = 1$ or $h^+ = 2$ implies $\varepsilon_0 \geqslant 1$. This can be justified as follows. If $h^- = 1$ there exist $u,v \in S$ with $u \perp v$ such that $\|u+v\| = 1$, and then the segment joining u and $u+v$ (of length 1) must be on S. Therefore, $\delta(1) = 0$. If $h^+ = 2$, there exist $u,v \in S$, with $u \perp v$ such that $\|\frac{u+v}{2}\| = 1$, and then the whole segment joining u and v (of length at least 1) must be on S. Thus $\delta(1) = 0$. Theorem 2 generalizes all the above results for any normed space.

THEOREM 2. *Let X be a normed space. Then $\varepsilon_0 < 1 \implies h^- > 1$ and $h^+ < 2$.*

Proof. The function $f(x) := \delta(2x-1)$ is non decreasing respect to x, while $g(x) := \frac{1-x}{2}$ is strictly decreasing. Both functions are continuous for $x \in [\frac{1}{2}, \frac{3}{2})$.

For $x = \frac{\varepsilon_0+1}{2} < 1$ we have $f(\frac{\varepsilon_0+1}{2}) = \delta(\varepsilon_0) = 0 < g(\frac{\varepsilon_0+1}{2}) = \frac{1-\varepsilon_0}{4}$.

For $x = 1$ we have $f(1) = \delta(1) > g(1) = 0$.

Thus, there exists a unique $\bar{a} \in (\frac{\varepsilon_0+1}{2}, 1)$ such that $\delta(2\bar{a}-1) = \frac{1-\bar{a}}{2}$, and $\delta(2x-1) \leqslant \frac{1-x}{2}$ only for $x \leqslant \bar{a}$.

Let $z = u+v$ with $u,v \in S$ and $u \perp v$. Denote $a = \frac{1}{\|z\|}$. According to Theorem 1, $\delta(2a-1) \leqslant \frac{1-a}{2}$ and then $a \leqslant \bar{a}$. Therefore $\|z\| = \frac{1}{a} \geqslant \frac{1}{\bar{a}} > 1$, and $h^- \geqslant \frac{1}{\bar{a}} > 1$.

Now let $\delta(1) > 0$. Let us consider any pair $u,v \in S$ such that $u \perp v$. Since $\|u-v\| \geqslant 1$, we have $\frac{\|u+v\|}{2} \leqslant 1-\delta(1)$ and $\|u+v\| \leqslant 2-2\delta(1)$. Thus, we conclude $h^+ \leqslant 2-2\delta(1) < 2$. $\square$

COROLLARY 1. *Let X be a normed space. Then $\varepsilon_0 < 1 \implies \delta(2/h^- - 1) \leqslant \frac{1}{2} - \frac{1}{2h^-}$.*

Proof. From the proof of Theorem 2 we know that $\bar{a} \geqslant \frac{1}{h^-}$, where $\bar{a}$ verifies $\delta(2\bar{a}-1) = \frac{1-\bar{a}}{2}$. Since $\delta(2x-1)$ is non decreasing and $\frac{1-x}{2}$ is strictly decreasing we have

$$\delta(2/h^- - 1) \leqslant \delta(2\bar{a}-1) = \frac{1-\bar{a}}{2} \leqslant \frac{1-\frac{1}{h^-}}{2} = \frac{1}{2} - \frac{1}{2h^-}. \quad \square$$

Due to (6) in Proposition 1 (see Remark 1), the value denoted $\bar{a}$ in the above proof belongs to $[\frac{\sqrt{11}+1}{5}, 1)$.

Concerning the reverse implication of Theorem 2, see the discussion in the last section.

3. Some history and known results about h^- and h^+

These two numbers, or at least the first one, have been considered in a few papers during the years starting from [6]. In that paper the inequality $J \leqslant \frac{2}{h^-}$ between h^- and the Jung constant J was indicated. A short history of h^- up to 1989 and some connections with other parameters was traced in [12], a not easily available paper. But a detailed study of h^- and h^+ seems to be still lacking. For a recent attempt for generalizing them, see [10], Section 3. The following result is known.

PROPOSITION 3. *Let X be a normed space. Then*

$$1 \leqslant h_X^- \leqslant h_H^- = \sqrt{2} = h_H^+ \leqslant h_X^+ \leqslant 2.$$

Proof. Let X_2 be a 2-dimensional subspace of X and $S_2 = S \cap X_2$. For $x, y \in S_2$, let $[x, y] = \sigma$ denote the orientation of x, y respect to a fixed orientation σ in S_2. Let us consider this set:

$$S_B = \{x + y: \quad x, y \in S_2;\ x \perp y;\ [x, y] = \sigma\}.$$

Joly ([11]) (see also [1]) proves that S_B is a rectifiable Jordan curve that is symmetric respect to the origin and delimits an area in X_2 equal to two times the area delimited by S_2. Since $\sqrt{2}S$ has the same properties, S_B and $\sqrt{2}S$ must share at least two common points. Therefore, for every normed space X there exists a vector $x + y \in S_B \cap \sqrt{2}S$, and $h^- \leqslant \sqrt{2} \leqslant h^+$.

The rest of the equalities and inequalities are obvious consequences of the properties of the norms and the definition of Birkhoff orthogonality. □

From Proposition 3 we obtain the following (characterizations (10.3) and $(10.3')$ in [3], p. 79).

THEOREM 3. *Let X be a normed space. The following are equivalent:*

(1) *X is Hilbert.*

(2) *$h^- \geqslant \sqrt{2}$ (equivalently, $h^- = \sqrt{2}$).*

(3) *$h^+ \leqslant \sqrt{2}$ (equivalently, $h^+ = \sqrt{2}$).*

(4) *$x, y \in S$, $x \perp y$ imply $\|x - y\| = \sqrt{2}$.*

(5) *$h^- = h^+$.*

Therefore, if X is not Hilbert, there exist $x_1, y_1, x_2, y_2 \in S$ such that $x_1 \perp y_1$, $x_2 \perp y_2$ with $\|x_1 + y_1\| < \sqrt{2}$ and $\|x_2 + y_2\| > \sqrt{2}$.

We recall the following result, given in [6], with an awkward proof (based also on a result concerning h^- and the Jung's constant of X).

PROPOSITION 4. *Let X be a normed space and $t_0 \in \mathbb{R}$ such that $t_0 + 2\delta(t_0) = 1$. Then $h^- \geqslant \frac{1}{t_0}$.*

If $\varepsilon_0 \geqslant 1$ then $t_0 = 1$ and we only obtain $h^- \geqslant 1$. If $\delta(1) > 0$ then $t_0 < 1$ and we obtain $h^- > 1$, which is one of the statements of Theorem 2.

REMARK 3. Note that the proof of Theorem 2 gives the estimate $h^- \geqslant \frac{1}{\bar{a}}$ when $\varepsilon_0 < 1$, where $\bar{a} \in [\frac{\sqrt{11}+1}{5}, 1)$ verifies $\delta(2\bar{a}-1) = \frac{1-\bar{a}}{2}$. Since $2\bar{a}-1 < \bar{a}$ and $\delta(.)$ is non decreasing, we have

$$1 = t_0 + 2\delta(t_0) = \bar{a} + 2\delta(2\bar{a}-1) \leqslant \bar{a} + 2\delta(\bar{a}).$$

Therefore, for $\varepsilon_0 < 1$ we conclude that $t_0 \leqslant \bar{a} \in [\frac{\sqrt{11}+1}{5}, 1)$, and $h^- \geqslant \frac{1}{t_0} \geqslant \frac{1}{\bar{a}}$.

REMARK 4. Also the estimation provided by Proposition 4 is not sharp. If $t_0 + 2\delta(t_0) = 1$ then

$$t_0 + 2\left(1 - \sqrt{1 - \frac{t_0^2}{4}}\right) \geqslant 1 \iff 2t_0^2 + 2t_0 - 3 \geqslant 0 \iff t_0 \geqslant \frac{\sqrt{7}-1}{2}$$

and the best lower bound from the above proposition would be $h^- \geqslant \frac{1}{t_0} \geqslant \frac{2}{\sqrt{7}-1} \approx 1.215$, which would be the estimation for a Hilbert space. Instead $h_H^- = \sqrt{2} \approx 1.414$.

4. Our constants and moduli

In this section we focus on the modulus of smoothness $\rho(\varepsilon)$ and its connections with $\delta(\varepsilon)$, h^-, and h^+.

THEOREM 4. *Let X be a normed space. Then the following inequalities hold:*

(1) $2(1-\rho(h^-)) \leqslant h^+$

(2) $h^+ \leqslant 2(1-\delta(h^-))$

(3) $2(1-\rho(h^+)) \leqslant h^-$

(4) $h^- \leqslant 2(1-\delta(h^+))$ *if $h^+ < 2$.*
For finite dimensional spaces, the inequality holds also if $h^+ = 2$.

(5) *If $h^+ = 2$, then $h^- \leqslant 2 - 2\lim_{\varepsilon\to 2^-} \delta(\varepsilon) = \varepsilon_0$; thus, $1 \leqslant h^- \leqslant \varepsilon_0$ and $\delta(1) = 0$.*

Proof. (1). Given $\sigma > 0$, we can find $x, y \in S$, $x \perp y$, such that $\|x-y\| < h^- + \sigma$. This implies

$$1 - \rho(h^- + \sigma) \leqslant \frac{\|x+y\|}{2},$$

and $2(1-\rho(h^-+\sigma)) \leqslant h^+$. Since $\rho(.)$ is continuous we obtain $2(1-\rho(h^-)) \leqslant h^+$.

(2). Take $x, y \in S$ with $x \perp y$. We have $\|x-y\| \geqslant h^-$, so $\frac{\|x+y\|}{2} \leqslant 1 - \delta(h^-)$. Thus $h^+ \leqslant 2(1-\delta(h^-))$.

(3). Given $\sigma > 0$, we can take $x, y \in S$, $x \perp y$ such that $\|x+y\| < h^- + \sigma$. We have

$$1-\rho(h^+) \leqslant 1-\rho(\|x-y\|) \leqslant \frac{\|x+y\|}{2} < \frac{h^- + \sigma}{2}.$$

Since σ is arbitrary, we obtain $2(1-\rho(h^+)) \leqslant h^-$.

(4). Given $\sigma > 0$, we can take $x, y \in S$ such that $x \perp y$ and $\|x-y\| > h^+ - \sigma$. By the definition of $\delta(.)$, we have

$$\frac{\|x+y\|}{2} \leqslant 1-\delta(h^+ - \sigma)$$

and $h^- \leqslant 2(1-\delta(h^+ - \sigma))$. Since $\delta(.)$ is continuous in $[0,2)$, if $h^+ < 2$ we obtain $h^- \leqslant 2(1-\delta(h^+))$.

If X is finite dimensional and $h^+ = 2$, there exist $x, y \in S$ with $x \perp y$ and $\|x-y\| = 2$. Then $\frac{\|x+y\|}{2} \leqslant 1-\delta(2)$ and $h^- \leqslant 2(1-\delta(h^+))$ also holds for $h^+ = 2$.

(5). If $h^+ = 2$, given $\sigma > 0$ we conclude that $h^- \leqslant 2(1-\delta(h^+ - \sigma))$ similarly to (4). Since $\lim_{\varepsilon \to 2^-} \delta(\varepsilon) = 1 - \frac{\varepsilon_0}{2}$ (see Proposition 1), we obtain

$$h^- \leqslant \lim_{\sigma \to 0} 2(1-\delta(h^+ - \sigma)) = \lim_{\varepsilon \to 2^-} 2(1-\delta(\varepsilon)) = \varepsilon_0,$$

so $1 \leqslant h^- \leqslant \varepsilon_0$ and $\delta(1) = 0$. □

REMARK 5. We recall that the following equalities are true:

$(*)$ $\rho(2(1-\rho(\varepsilon))) = 1 - \frac{\varepsilon}{2}$ (see (4.3) in [4]).

$(**)$ $\delta(2(1-\delta(\varepsilon))) = 1 - \frac{\varepsilon}{2}$ if $\varepsilon \geqslant \varepsilon_0$ (see [9], p. 56).

By means of $(*)$, the inequalities (1) and (3) of Theorem 4 are equivalent. In fact, applying $(*)$ to inequality (1) we obtain

$$\frac{1-h^-}{2} \leqslant \rho(h^+),$$

that is (3); and applying $(*)$ to inequality (3) we deduce

$$\frac{1-h^+}{2} \leqslant \rho(h^-),$$

that is (1).

Besides, (2) and (4) are *almost* equivalent by $(**)$: if $h^- \geqslant \varepsilon_0$ applying $(**)$ to (2) we have

$$\delta(h^+) \leqslant 1 - \frac{h^-}{2},$$

that is (4); if $h^+ \geqslant \varepsilon_0$, similarly (4) implies

$$\delta(h^-) \leqslant 1 - \frac{h^+}{2},$$

that is (2).

We briefly discuss in which range of values the inequalities in Theorem 4 are interesting.

(1) is useful only when $2(1-\rho(h^-)) > \sqrt{2}$, and the greatest value of $2(1-\rho(h^-))$ is achieved for the minimum value of h^-. Since $h^+ \geqslant \sqrt{2}$ and

$$2(1-\rho(h^-)) \leqslant 2(1-\rho(1)) \leqslant 2\left(1-\left(1-\sqrt{1-\frac{1}{4}}\right)\right) = \sqrt{3},$$

inequality (1) is interesting when $2(1-\rho(h^-)) \in (\sqrt{2},\sqrt{3}]$. The best lower bound deduced from (1) would be $h^+ \geqslant \sqrt{3}$. If $h^- = \sqrt{2}$, then X is Hilbert, $h^+ = \sqrt{2}$, and (1) is sharp:

$$2(1-\rho(\sqrt{2})) = 2\left(1-\left(1-\sqrt{1-\frac{1}{2}}\right)\right) = \sqrt{2}.$$

(2) is always non-trivial if $\delta(1) > 0$ because $h^- \geqslant 1$. The range of values of interest for $2(1-\delta(h^-))$ is $[\sqrt{2},2]$. If $h^- = 1$, then

$$2(1-\delta(1)) \geqslant 2\left(1-\left(1-\sqrt{1-\frac{1}{4}}\right)\right) = \sqrt{3},$$

and $2(1-\delta(h^-)) \in [\sqrt{3},2]$. We present examples of spaces (see Example 4 in Section 5) with $h^- = 1$ and h^+ slightly greater than $\sqrt{3}$, and also h^- slightly greater than 1 and h^+ smaller than $\sqrt{3}$. From (2) we deduce that $h^+ = 2$ implies $\delta(h^-) = 0$ and $1 \leqslant h^- \leqslant \varepsilon_0$, which is (5).

From (3) we obtain a non-trivial estimate when $2(1-\rho(h^+)) > 1$. In that case

$$\frac{1}{2} > \rho(h^+) \geqslant 1-\sqrt{1-\frac{(h^+)^2}{4}} \iff h^+ < \sqrt{3}.$$

The range of values of interest of $2(1-\rho(h^+))$ in (3) is $[1,\sqrt{2}]$.

(4) is non-trivial if $2(1-\delta(h^+)) \leqslant \sqrt{2}$, and the range of values of interest for $2(1-\delta(h^+))$ in (4) is $[1,\sqrt{2}]$.

Inequality (4) is not always true when $h^+ = 2$: there are spaces with $\delta(2) = 1$, $h^+ = 2$, and obviously $h^- \geqslant 1$ (see Example 1 in Section 5).

We indicate two consequences of Theorem 4. Since $\delta(\sqrt{3}) \leqslant 1-\sqrt{1-\frac{(\sqrt{3})^2}{4}} = \frac{1}{2}$, we have that $\delta(h^+)$ could be greater than $\frac{1}{2}$ for $h^+ > \sqrt{3}$, and the next result is non-trivial when $h^+ > \sqrt{3}$.

COROLLARY 2. *Let X be a normed space. If one of the following conditions hold:*

(1) $h^+ < 2$,

(2) *X is finite dimensional,*

(3) $h^- \geqslant \varepsilon_0$,

then $\delta(h^+) \leqslant 1 - \frac{h^-}{2} \leqslant \frac{1}{2}$.

Proof. If $h^+ < 2$ or X is finite dimensional, then this is (4) of Theorem 4. If $h^- \geqslant \varepsilon_0$, then we can apply $(**)$ as explained in Remark 5. □

Therefore $\delta(h^+) \leqslant \frac{1}{2}$ if any of the conditions of Corollary 2 holds. The inequality $\delta(h^+) \leqslant 1 - \frac{h^-}{2}$ is not always true (see again Example 1 in Section 5).

COROLLARY 3. *Let X be a normed space. We always have:*

(1) $2\left(1-\frac{\sqrt{2}}{2}\right) \leqslant 2(1-\rho(\sqrt{2})) \leqslant 2(1-\rho(h^-)) \leqslant h^+ \leqslant 2(1-\delta(h^-)) \leqslant 2(1-\delta(1))$.

(2) $0 \leqslant 2(1-\rho(h^+)) \leqslant h^- \leqslant \begin{cases} 2(1-\delta(h^+)) & \textit{if } \dim(X) < \infty \textit{ or } h^+ < 2 \\ 2(1-\lim_{\varepsilon\to 2^-}\delta(\varepsilon)) & \textit{otherwise.} \end{cases}$

Proof. Since $\delta(.)$ and $\rho(.)$ are non-decreasing functions, (1) and (2) of Theorem 4 joint with Proposition 1 imply the first chain of inequalities. The second one is consequence of (3), (4), and (5) of Theorem 4. □

REMARK 6. The inequalities of Theorem 4 are mostly sharp in the sense that equality is possible (in Hilbert spaces or when h^- and/or h^+ has one of its extreme values). All the inequalities are not equalities in general.

New result is connected with ε_0.

THEOREM 5. *Let X a finite dimensional normed space and $\alpha = \min\{\varepsilon_0, 1\}$. Then*

$$h^- \leqslant 2 - \alpha.$$

In particular, $\varepsilon_0 \geqslant 1$ implies $h^- = 1$.

Proof. Note that $\alpha \leqslant \varepsilon_0$, so $\delta(\alpha) = 0$.

For every $n \in \mathbb{N}$ we can find $x_n, y_n \in S$ with $\|x_n - y_n\| = \alpha$ and $\frac{\|x_n+y_n\|}{2} > 1 - \frac{1}{n}$. By compactness, we can find $x, y \in S$ such that $\|x-y\| = \alpha$ and $\frac{\|x+y\|}{2} \geqslant 1$, so $\frac{\|x+y\|}{2} = 1$. This implies that S contains x, y, and $\frac{x+y}{2}$. If we set $f(\lambda) = \|x + \lambda(y-x)\|$, we have $f(0) = f(\frac{1}{2}) = f(1)$, and then $1 = \min\{\|x+\lambda(y-x)\| : \lambda \in \mathbb{R}\}$ by convexity. Therefore, $x \perp y - x$.

Set $z = \frac{y-x}{\alpha} \in S$. Then

$$\begin{aligned} \|x+z\| &\leqslant \|x+y-x\| + \|z-(y-x)\| \\ &= 1 + \left\|\frac{y-x}{\alpha} + (x-y)\right\| = 1 + \left(\frac{1-\alpha}{\alpha}\right)\|x-y\| = 2-\alpha. \quad \square \end{aligned}$$

REMARK 7. Clearly the above inequality is not sharp. Indeed, since $h^- \leqslant \sqrt{2}$ for every X (in particular whenever $\varepsilon_0 > 0$), the above estimate is trivial for $\alpha \leqslant 2-\sqrt{2}$.

REMARK 8. Theorem 5 and inequality (4) of Theorem 4 are formally independent. Let us compare both for finite dimensional spaces. If $\varepsilon_0 = 0$, then $2\delta(h^+)$ can be larger than $\alpha = 0$. If $\varepsilon_0 = 1.1$, then $\alpha = 1$ can be larger than $2\delta(h^+)$: in fact, $\lim_{\varepsilon\to 2^-}\delta(\varepsilon) = 1 - \frac{\varepsilon_0}{2}$ always holds (see Proposition 1), thus, if $h^+ < 2$ we obtain

$$2\delta(h^+) \leqslant 2\lim_{\varepsilon\to 2^-}\delta(\varepsilon) = 2\left(1-\frac{1.1}{2}\right) = 0.9 < \alpha.$$

COROLLARY 4. *Let X be a normed space and $\varepsilon_0 \geqslant 1$. If X is finite dimensional or $h^+ = 2$, then $h^- \leqslant \varepsilon_0$.*

Proof. This is consequence of Theorem 5 and (5) of Theorem 4. □

REMARK 9. We do not know if the thesis of Theorem 5 is true for infinite dimensional spaces. We also proved some other results under the assumption $\dim(X) < \infty$.

In finite dimensional spaces the uniform convexity ($\varepsilon_0 = 0$) is equivalent to strict convexity ($\delta(2) = 1$, see Proposition 1). Nevertheless, in this context passing to infinite dimension is *dangerous*. In fact, there exist spaces that are SC but are not UNS ($\varepsilon_0 = 2$). We present Example 1 in Section 5 (see also [14], p. 11).

5. Some examples

Finally we present some examples to show the potential of the results in the above sections.

EXAMPLE 1. *$l^p(\mathbb{R}^2)$: $1 = h^- < \varepsilon_0 = h^+ = 2$.*

Given $p > 1$, let $l^p(\mathbb{R}^2) = \mathbb{R}^2_2 \oplus \mathbb{R}^2_3 \oplus \mathbb{R}^2_4 \ldots \oplus \mathbb{R}^2_n \ldots$ be the space of sequences $\mathbf{X} = (\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n, \ldots)$, where $\mathbb{R}^2_n$ denotes the space $\mathbb{R}^2$ with the norm $\|\mathbf{x}_n\| = \|(x_{n_1}, x_{n_2})\| = (|x_{n_1}|^n + |x_{n_2}|^n)^{1/n}$, and

$$\|(\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n, \ldots)\| = \left(\sum_{n=2}^{\infty}\|\mathbf{x}_n\|^p\right)^{1/p}.$$

S contains curves nearer and nearer to segments of length 1, so $\delta(1) = 0$. Let $\mathbf{X}_n$, $\mathbf{Y}_n$, $\mathbf{U}_n$, $\mathbf{V}_n$, for $n \in \mathbb{N}$ be as follows:

$\mathbf{X}_n = \|(\mathbf{x}_2, \mathbf{x}_3, \ldots)\| \in l^p(\mathbb{R}^2)$ such that $\mathbf{x}_i = (0,0)$ for $i \neq n$, and $\mathbf{x}_n = \frac{(1,1)}{\|(1,1)\|_n}$;

$\mathbf{Y}_n = \|(\mathbf{y}_2, \mathbf{y}_3, \ldots)\| \in l^p(\mathbb{R}^2)$ such that $\mathbf{y}_i = (0,0)$ for $i \neq n$, and $\mathbf{y}_n = \frac{(-1,1)}{\|(-1,1)\|_n}$;

$\mathbf{U}_n = \|(\mathbf{u}_2, \mathbf{u}_3, \ldots)\| \in l^p(\mathbb{R}^2)$ such that $\mathbf{u}_i = (0,0)$ for $i \neq n$, and $\mathbf{u}_n = \frac{(1,0)}{\|(1,0)\|_n}$;

$\mathbf{V}_n = \|(\mathbf{v}_2, \mathbf{v}_3, \ldots)\| \in l^p(\mathbb{R}^2)$ such that $\mathbf{v}_i = (0,0)$ for $i \neq n$, and $\mathbf{v}_n = \frac{(0,1)}{\|(0,1)\|_n}$.

We have that $\mathbf{X}_n, \mathbf{Y}_n, \mathbf{U}_n, \mathbf{V}_n \in S$; $\mathbf{U}_n \perp \mathbf{V}_n$, and $\|\mathbf{U}_n + \mathbf{V}_n\|$ tends to 1 when n tends to infinity, so $h^- = 1$. $\mathbf{X}_n \perp \mathbf{Y}_n$, and $\|\mathbf{X}_n \pm \mathbf{Y}_n\|$ tends to 2 when n tends to infinity, therefore $h^+ = 2$, and given $\varepsilon > 0$ we can find $\mathbf{X}, \mathbf{Y} \in S$, $\mathbf{X} \perp \mathbf{Y}$, with $\|\mathbf{X} \pm \mathbf{Y}\| > 2 - \varepsilon$. Consequently, $l^p(\mathbb{R}^2)$ is not UNS ($\varepsilon_0 = 2$), it is SC ($\delta(2) = 1$) but it is not UC (see [7]; [14], p. 10).

EXAMPLE 2. $(\mathbb{R}^2, \|.\|_\infty)$ *or* $(\mathbb{R}^2, \|.\|_1)$*:* $1 = h^- < \varepsilon_0 = h^+ = 2$.

Let us consider $\mathbb{R}^2$ endowed with the norm $\|x\|_\infty = \|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$ and $\|x\|_1 = \|(x_1, x_2)\|_1 = |x_1| + |x_2|$ for any $x = (x_1, x_2) \in \mathbb{R}^2$.

Let us consider $x = (1,0)$, $y = (0,1)$, $u = (1,1)$, $v = (1,-1) \in \mathbb{R}^2$. We have that $\|x\|_\infty = \|y\|_\infty = \|u\|_\infty = \|v\|_\infty = 1$; $x \perp y$, and $\|x+y\|_\infty = 1$, so $h^- = 1$. $u \perp v$, and $\|u+v\|_\infty = 2$, so $h^+ = 2$.

We can find easily a similar example that justifies $h^- = 1$ and $h^+ = 2$ for $(\mathbb{R}^2, \|.\|_1)$.

Both $(\mathbb{R}^2, \|.\|_\infty)$ and $(\mathbb{R}^2, \|.\|_1)$ are not SC ($\delta(2) < 1$), not UNS ($\varepsilon_0 = 2$), and not UC.

The above examples can be generalized as follows. In [5] these parameters were defined and studied:

$$g_\perp = \inf\{\max\{\|x-y\|, \|x+y\|\} : \quad x, y \in S,\ x \perp y\}$$

and

$$J_\perp = \sup\{\min\{\|x-y\|, \|x+y\|\} : \quad x, y \in S,\ x \perp y\}.$$

In particular, (6.8) in [5] proves the following.

PROPOSITION 5. *X is not UNS* $\iff J_\perp = 2 \iff g_\perp = 1$.

It is clear that $h^- \leqslant g_\perp \leqslant J_\perp \leqslant h^+$ always, thus we obtain the following.

THEOREM 6. *If X is not UNS, then $h^- = 1$ and $h^+ = 2$.*

We can have $h^- < g_\perp$ and $h^+ > J_\perp$ as the following example shows. Another example where these strict inequalities hold will be Example 5.

EXAMPLE 3. $(\mathbb{R}^2, \|.\|_{6n})$. $h^- = \varepsilon_0 = 1 < h^+ = 2$.

Let us consider $\mathbb{R}^2$ endowed with the norm

$$\|x\|_{6n} = \|(x_1, x_2)\|_{6n} = \begin{cases} \max\{|x_1|, |x_2|\}, & x_1x_2 \geqslant 0 \\ |x_1| + |x_2|, & x_1x_2 < 0 \end{cases},$$

so the norm is determined by a regular hexagon whose vertices are $(\pm 1, 0)$, $(1,1)$, $(0, \pm 1)$, $(-1,-1)$.

Let $x = (1,1)$ and $y = (0,1)$ We have that $\|x\|_{6n} = \|y\|_{6n} = 1$; $x \perp y$, $\|x+y\|_{6n} = 2$, and $\|x-y\|_{6n} = 1$, so $h^- = 1$ and $h^+ = 2$. Besides, $\varepsilon_0 = 1$ and $g_\perp = J_\perp = 3/2$ (see Example 5.1 in [5]).

$(\mathbb{R}^2, \|.\|_{6n})$ is not UC but UNS.

EXAMPLE 4. $(\mathbb{R}^2, \|.\|_\circ)$. $h^- = \varepsilon_0 = 1 < h^+ \approx 1.7602$.

Let us consider $\mathbb{R}^2$ endowed with the norm $\|.\|_\circ$ whose unit sphere is the set

$$S = \{(x,\pm 1) : |x| \leqslant 1\} \cup \left\{ \left(x, \pm\sqrt{1-(x-1)^2}\right) : 1 \leqslant x \leqslant 2 \right\} \\ \cup \left\{ \left(x, \pm\sqrt{1-(x+1)^2}\right) : -2 \leqslant x \leqslant -1 \right\},$$

and the vectors:

$x = \left(\frac{\sqrt{33}+7}{8}, \frac{\sqrt{\sqrt{33}+15}}{\sqrt{32}}\right) \approx (1.5931, 0.8051),$

$y = \left(-\frac{\sqrt{33}+15}{16}, \frac{\sqrt{9-\sqrt{33}}(\sqrt{33}-1)\sqrt{\sqrt{33}+7}}{\sqrt{32}}\right) \approx (-1.2965, 0.9550),$

$u = (-1,1)$, and $v = (2,0)$.

We have that $x,y,u,v \in S$; $x \perp y$ and $\|x+y\| \approx 1.7602$, which is the value of h^+; $u \perp v$ and $\|u+v\| = 1$, which is the value of h^-. In this example we also have $h^- \cdot h^+ < 2$.

$(\mathbb{R}^2, \|.\|_\circ)$ is not SC and it is UNS ($\varepsilon_0 = 1$).

We obtain $h^- \approx 1.01$ and $h^+ \approx 1.72$ if the unit ball of the space is slightly modified to the following:

$$S = \{(x,\pm 1) : |x| \leqslant 0.8\} \cup \left\{ \left(x, \pm\sqrt{1-(x-0.8)^2}\right) : 0.8 \leqslant x \leqslant 1.8 \right\} \\ \cup \left\{ \left(x, \pm\sqrt{1-(x+0.8)^2}\right) : -1.8 \leqslant x \leqslant -0.8 \right\}.$$

EXAMPLE 5. $(\mathbb{R}^2, \|.\|_{8n})$. $\varepsilon_0 \approx 0.83 < h^- \approx 1.1213 < h^+ \approx 1.7071$.

Let us consider $\mathbb{R}^2$ endowed with the norm $\|.\|_{8n}$ whose unit ball is a regular octagon with vertices $(\pm 1, 0)$, $(\pm\sqrt{2}, \sqrt{2})$, $(0, \pm 1)$, $(\pm\sqrt{2}, -\sqrt{2})$. Let us consider $x = (1,0)$ and $y = (\frac{1}{\sqrt{8}}, \frac{\sqrt{2}+1}{\sqrt{8}})$. We have that $x, y \in S$ and $x \perp y$; $\|x+y\| = \frac{1+\sqrt{8}}{2\sqrt{2}-2} \approx 1.7071$, which is the value of h^+; and $\|x-y\| = \frac{7(\sqrt{2}+1)}{5\sqrt{2}+8} \approx 1.1213$, which is the value of h^-. Moreover, $g_\perp = J_\perp = \sqrt{2}$ (see Example 5.3 in [5]), but $\|.\|_{8n}$ is not Hilbert: $1.1213 \approx h^- < g_\perp = \sqrt{2} = J_\perp < h^+ \approx 1.7071$. Of course it is not SC, $\delta(1) > 0$. Moreover, $\varepsilon_0 = \frac{2}{\sqrt{2}+1} \approx 0.83$, which is the distance between two consecutive vertices.

EXAMPLE 6. $(\mathbb{R}^2, \|.\|_{10n})$. $\varepsilon_0 \approx 0.6180 < h^- \approx 1.2361 < h^+ \approx 1.6180$.

Let us consider $\mathbb{R}^2$ endowed with the norm $\|.\|_{10n}$ whose unit ball is a regular decagon whose vertices are $(\pm 1, 0)$, $(\pm\cos(\pi/5), \sin(\pi/5))$, $(\pm\cos(2\pi/5), \sin(2\pi/5))$, $(\pm\cos(\pi/5), -\sin(\pi/5))$, $(\pm\cos(2\pi/5), -\sin(2\pi/5))$. Let us consider $x = (1,0)$ and $y = (-\cos(2\pi/5), \sin(2\pi/5))$. We have that $x \perp y$; $\|x+y\| = \frac{2\sin(2\pi/5)}{\sin(2\pi/5)+\sin(\pi/5)} \approx 1.2361$, which is the value of h^-; and $\|x-y\| = 2\cos(\pi/5) \approx 1.6180$, which is the value of

h^+. Moreover, $\varepsilon_0 = \frac{\sin(\pi/5)}{\sin(2\pi/5)} \approx 0.6180$, which is the distance between two consecutive vertices.

EXAMPLE 7. $(\mathbb{R}^2, \|.\|_{12n})$. $\varepsilon_0 \approx 0.5359 < h^- \approx 1.2321 < h^+ \approx 1.5981$.

Let us consider $\mathbb{R}^2$ endowed with the norm $\|.\|_{12n}$ whose unit ball is a regular n-gon with $n = 12$ whose vertices are $(\cos(k\pi/6), \sin(k\pi/6))$, where $k \in \{0, \dots, 11\}$. Let $x = (1,0)$, $y' = (-\cos(7\pi/12), \sin(7\pi/12))$ and $y = \frac{y'}{\|y\|}$. We have that $x \perp y$; $\|x+y\| \approx 1.2321$, which is the value of h^-; and $\|x-y\| \approx 1.5981$, which is the value of h^+. Moreover, $\varepsilon_0 = \frac{2}{\sqrt{3}+2} \approx 0.5359$, which is the distance between two consecutive vertices.

EXAMPLE 8. $(\mathbb{R}^2, \|.\|_p)$. $0 = \varepsilon_0 < 1 < h^- < h^+ < 2$

Given $p > 1$, let us consider $\mathbb{R}^2$ endowed with the norm $\|(x_1,x_2)\|_p = (|x_1|^p + |x_2|^p)^{1/p}$. The following approximate values of h^+ and h^- have been computed for some p:

p	h^-	h^+	$h^- \cdot h^+$
1.10	1.052273	1.942347	2.04388
1.20	1.102418	1.872711	2.06451
1.25	1.126613	1.836794	2.06937
1.3333333	1.165565	1.777520	2.07182
1.50	1.238135	1.666284	2.06308
1.65	1.297327	1.577695	2.04679
1.75	1.333652	1.525049	2.03388
2.00	1.414214	1.414214	2.00000
2.50	1.262378	1.538204	1.94179
3.00	1.170664	1.625597	1.90303
4.00	1.076115	1.735040	1.86710
5.00	1.035052	1.798064	1.86109
6.00	1.016343	1.838066	1.86811
10.00	1.000818	1.911642	1.91321
15.00	1.000022	1.944524	1.94457

It seems that in general $h^- \cdot h^+ \neq 2$, and $h^- \cdot h^+$ is greater than 2 for $1 < p < 2$ and smaller than 2 for $p > 2$. For some conjugate indices p and q (namely, $1/p + 1/q = 1$), for example $p = 1.2$ and $q = 6$, the values of h^- and h^+ are different.

6. Concluding and resuming. Open questions

For many parameters in Banach spaces, the values at the extremes of their ranges are related to *very bad* spaces, like non-UNS spaces. The situation concerning h^- and h^+ seems to be slightly different. Instead, the *central value* ($h^- = \sqrt{2}$ and $h^+ = \sqrt{2}$) characterizes Hilbert spaces. Figure 1 summarizes the general situation concerning extreme values.

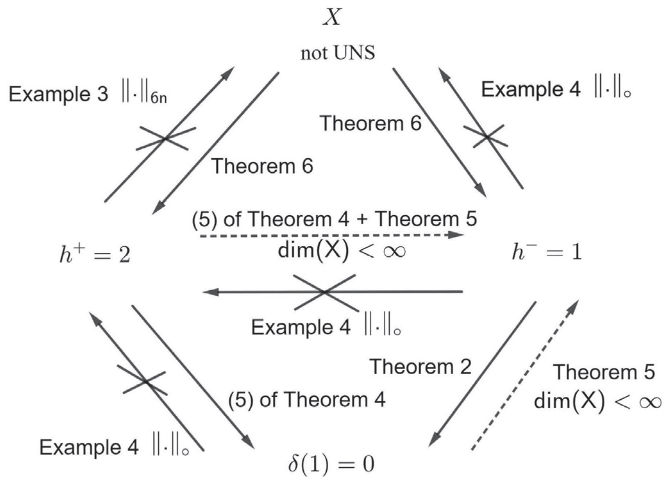

Figure 1: *A diagram of the relationship between the extreme values of h^- and h^+ and some results and examples.*

In particular, $h^+ = 2$ implies $\delta(1) = 0$, which is equivalent to $h^- = 1$ when X is finite dimensional. We are not able to prove if, in general, $\delta(1) = 0$ implies $h^- = 1$, or at least $h^+ = 2$ implies $h^- = 1$. Also: is Theorem 5 valid in any normed space?

For the sake of completeness, we recall that a recent paper (see [8]) indicates some facts considered here. In particular, Proposition 2.14 there proves that $h^+ < 2$ implies that the space is UNS (but not conversely).

It would be interesting to clarify the relationship between h^- and ε_0 for infinite dimensional spaces when $h^+ < 2$ (see Corollary 4 and the examples in Section 5).

Finally, it is not clear if the functions h^- and h^+ are continuous with respect to the Banach-Mazur distance.

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