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ON THE NUMERICAL RADIUS OF AN OPERATOR MATRIX

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Abstract. This paper investigates possible upper and lower bounds for the numerical radii of 2×2 operator matrices.

The obtained results improve and generalize many results from the literature in accessible forms. Moreover, many numerical examples will be given to support the feasibility of our findings.

1. Introduction

Let $\mathscr{H},\mathscr{K}$ be two Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_{\mathscr{H}},\langle\cdot,\cdot\rangle_{\mathscr{K}}$, respectively, and let $\mathscr{B}(\mathscr{H},\mathscr{K})$ denote the Banach space of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$. When $\mathscr{H}=\mathscr{K}$, we write $\mathscr{B}(\mathscr{H})$ instead of $\mathscr{B}(\mathscr{H},\mathscr{K})$. The zero element in $\mathscr{B}(\mathscr{H},\mathscr{K})$ or $\mathscr{B}(\mathscr{H})$ will be denoted by O, and the identity operator in $\mathscr{B}(\mathscr{H})$ will be denoted by I.

For $i=1,\cdots,n$, let $\mathscr{H}_i$ denote Hilbert spaces, and let $\mathscr{H}=\oplus_{i=1}^{n}\mathscr{H}_i$. An operator $\mathbb{T}\in\mathscr{B}(\mathscr{H})$ can be represented by an $n\times n$ operator matrix $\mathbb{T}=[T_{ij}]$, in which $T_{ij}\in\mathscr{B}(\mathscr{H}_i,\mathscr{H}_j)$.

While studying the problem of positive completions of certain partial operator matrices, Hou and Du proved the following significant result, in which the notations $\|\cdot\|,\omega(\cdot)$ and $r(\cdot)$ refer, respectively, to the usual operator norm, the numerical radius, and the spectral radius.

LEMMA 1. [11] *Let $\mathscr{H}_1,\mathscr{H}_2,\ldots,\mathscr{H}_n$ be Hilbert spaces, and let $\mathbb{T}=[T_{ij}]$ be an $n\times n$ operator matrix with $T_{ij}\in\mathscr{B}(\mathscr{H}_i,\mathscr{H}_j)$. Then*

$$\|\mathbb{T}\|\leqslant\left\|\left[\|T_{ij}\|\right]\right\|,$$

$$\omega(\mathbb{T})\leqslant\omega\left(\left[\|T_{ij}\|\right]\right),$$

$$r(\mathbb{T})\leqslant r\left(\left[\|T_{ij}\|\right]\right).$$

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One significance of the above result is how the calculations of operator matrices are bounded by the same calculations for certain scalar (non-negative) matrices. However, we notice that, in general, $\omega(\mathbb{T}) \nleqslant \omega[\omega(T_{ij})]$ and $r(\mathbb{T}) \nleqslant r(r(T_{ij}))$. We refer the reader to [11, 14] for a list of references in which this and related problems were treated.

In this paper, we are interested in obtaining sharp and accessible bounds for the numerical radius of a 2×2 operator matrix. Among many results, we will be able to show that if $A,B \in \mathscr{B}(\mathscr{H})$, then

$$\max\left\{\omega\left(A\right),\omega\left(D\right),\frac{1}{2}\left\|B+C^*\right\|,\frac{1}{2}\left\|B-C^*\right\|\right\} \leqslant \omega\left(\begin{bmatrix} A & B \\ C & D\end{bmatrix}\right), \tag{1}$$

$$\max\left\{\omega\left(AB\right),\omega\left(BA\right)\right\} \leqslant \frac{1}{4}\left(\left\|A+B^*\right\|^2+\left\|A-B^*\right\|^2\right), \tag{2}$$

and that

$$\frac{1}{2}\left\|T\right\| \leqslant \omega\left(\begin{bmatrix} O & \Re T \\ \mathrm{i}\Im T & O\end{bmatrix}\right) \leqslant \omega\left(T\right),$$

where $\Re T$ and $\Im T$ denote the real and imaginary parts of $T \in \mathscr{B}(\mathscr{H})$. The significance of the latter bound lies in refining the first inequality in

$$\frac{1}{2}\|T\| \leqslant \omega(T) \leqslant \|T\|, \tag{3}$$

known for any Hilbert space operator T. On the other hand, the significance of (2) is the way it generalizes an important result from [6].

It is of particular interest to study the equality cases in (3). In particular, it is well known that if $T^2 = O$, then $\frac{1}{2}\|T\| = \omega(T)$. In the sequel, we will show some equality scenarios. Among those results, the Aluthge transform will have its role. We recall that if $T \in \mathscr{B}(\mathscr{H})$ has the polar decomposition $T = U|T|$, then the weighted Aluthge transform of T is defined by $\widetilde{T}_t = |T|^t U|T|^{1-t}$ for $0 \leqslant t \leqslant 1$. This transform was defined for $t = \frac{1}{2}$ in [3], then was extended in [5] to the weighted form. The Aluthge transform was used by many authors to find sharper bounds for the numerical radius, as one can see in [2, 13, 17, 20]. Among them, Yamazaki [19] showed that $\omega\left(T\right) = \frac{1}{2}\left\|T\right\|$, whenever $\widetilde{T} = O$, where $\widetilde{T} = \widetilde{T}_{\frac{1}{2}}$.

The bound in (1) is a substantial reverse of the corresponding upper bound [9]

$$\omega\left(\begin{bmatrix} A & B \\ C & D\end{bmatrix}\right) \leqslant \max\{\omega(A),\omega(D)\} + \frac{\omega(B+C)+\omega(B-C)}{2}. \tag{4}$$

In particular,

$$\omega\left(\begin{bmatrix} A & B \\ B & D\end{bmatrix}\right) \leqslant \max\{\omega(A),\omega(D)\} + \omega(B).$$

The following lemma, which gives alternative definitions for the numerical radius, will be used frequently in the sequel. This lemma can be found in [12, (2.1)] and [8, pp. 75–76]

LEMMA 2. *Let $T \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega(T) = \sup_{\theta \in \mathbb{R}} \left\| \Re\left(e^{\mathrm{i}\theta} T\right) \right\| = \sup_{\alpha^2+\beta^2=1} \|\alpha \Re T + \beta \Im T\|,$$

where α, β vary over all such real numbers.

Numerous researchers have used the formulas in the above lemma in the literature to improve some well-known bounds.

A related result to Lemma 2 can be stated as follows.

LEMMA 3. [9, (4.6)] *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then*

$$\frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| A + e^{\mathrm{i}\theta} B^* \right\| = \omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right).$$

At this point, we remark that such alternative identities for ω help calculate the numerical radius using computer algebras. For example, the identity in Lemma 3 is used to calculate the numerical radius in Example 1 below.

Other lemmas that we will need are as follows.

LEMMA 4. [18, Theorem 2.14] *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then*

$$\|A+B\| \leqslant \sqrt{\max\{\|A\|, \|B\|\} + \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|} \sqrt{\max\{\|A\|, \|B\|\} + \left\| |A^*|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|}.$$

LEMMA 5. [1, Corollary 2] *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces, and let $\mathbb{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be an operator matrix with $A \in \mathscr{B}(\mathscr{H}_1)$, $B \in \mathscr{B}(\mathscr{H}_2, \mathscr{H}_1)$, $C \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$, and $D \in \mathscr{B}(\mathscr{H}_2)$. Then*

$$\omega(\mathbb{T}) \leqslant \frac{1}{2} \left(\omega(A) + \omega(D) + \sqrt{(\omega(A) - \omega(D))^2 + 4\omega\left(\begin{bmatrix} O & B \\ C & O \end{bmatrix} \right)^2} \right).$$

Moreover, the following lemma will be used to obtain some comparisons with the existing literature.

LEMMA 6. [13] *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then*

$$\max\{\omega(AB), \omega(BA)\} \leqslant \omega^2\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right).$$

Now, we show the following formula for the off-diagonal block operator matrix $\begin{bmatrix} O & A \\ B & O \end{bmatrix}$. This will be used as a key tool in obtaining our results. For the remainder of this paper, α, β refer to real numbers.

PROPOSITION 1. *Let $A,B \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) = \sup_{\alpha^2+\beta^2=1} \left\| \alpha \frac{A+B^*}{2} + \beta \frac{A-B^*}{2\mathrm{i}} \right\|.$$

Proof. We have

$$\begin{aligned} \sup_{\alpha^2+\beta^2=1} \left\| \alpha \frac{A+B^*}{2} + \beta \frac{A-B^*}{2\mathrm{i}} \right\| &= \frac{1}{2} \sup_{\alpha^2+\beta^2=1} \|(\alpha-\mathrm{i}\beta)A + (\alpha+\mathrm{i}\beta)B^*\| \\ &= \frac{1}{2} \sup_{\alpha^2+\beta^2=1} |\alpha-\mathrm{i}\beta| \left\| A + \frac{\alpha+\mathrm{i}\beta}{\alpha-\mathrm{i}\beta} B^* \right\|. \end{aligned}$$

Since $\alpha^2+\beta^2=1$, we may let $\alpha=\cos\theta$, $\beta=\sin\theta$, $\theta\in[0,2\pi]$ to get

$$\begin{aligned} \sup_{\alpha^2+\beta^2=1} \left\| \alpha \frac{A+B^*}{2} + \beta \frac{A-B^*}{2\mathrm{i}} \right\| &= \frac{1}{2}\sup_{\theta} \|A+e^{-2\mathrm{i}\theta}B^*\| \\ &= \omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right), \end{aligned}$$

where we have used Lemma 3 to obtain the last equality. This completes the proof. □

The following corollary is well known. However, we derive it from Proposition 1.

COROLLARY 1. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega\left(\begin{bmatrix} O & A \\ A^* & O \end{bmatrix}\right) = \|A\|,$$

and (see Lemma 2)

$$\omega(A) = \sup_{\alpha^2+\beta^2=1} \|\alpha \Re A + \beta \Im A\|.$$

Proof. The first conclusion follows from Proposition 1, by letting $B=A^*$. On the other hand, letting $B=A$, and noting that $\omega\left(\begin{bmatrix} O & A \\ A & O \end{bmatrix}\right) = \omega(A)$ imply the second assertion. □

2. Upper bounds for the numerical radius of an operator matrix

Now we present several new upper bounds for the numerical radii of different important forms of 2×2 operator matrices, including the following forms.

$$\begin{bmatrix} A & B \\ O & O \end{bmatrix}, \quad \begin{bmatrix} O & A \\ B & O \end{bmatrix}, \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

We begin with the following upper bound for $\omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right)$.

THEOREM 1. *Let $A,B \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right) \leqslant \frac{1}{2}\omega(A) + \frac{1}{2}\left\| |A^*|^2 + |B^*|^2 \right\| + \frac{1}{8},$$

and, in particular,

$$\|B\| \leqslant \left\| |B^*|^2 \right\| + \frac{1}{4} = \|B\|^2 + \frac{1}{4},$$

with equality if and only if $\|B\| = \frac{1}{2}$.

Proof. We have

$$\begin{aligned}
&\left\| \Re\left(e^{\mathrm{i}\theta}\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right)\right\| \\
&= \frac{1}{2}\left\| e^{\mathrm{i}\theta}\begin{bmatrix} A & B \\ O & O \end{bmatrix} + e^{-\mathrm{i}\theta}\begin{bmatrix} A^* & O \\ B^* & O \end{bmatrix}\right\| \\
&= \frac{1}{2}\left\| (\cos\theta + \mathrm{i}\sin\theta)\begin{bmatrix} A & B \\ O & O \end{bmatrix} + (\cos\theta - \mathrm{i}\sin\theta)\begin{bmatrix} A^* & O \\ B^* & O \end{bmatrix}\right\| \\
&= \left\| \cos\theta\begin{bmatrix} \frac{A+A^*}{2} & \frac{B}{2} \\ \frac{B^*}{2} & O \end{bmatrix} - \sin\theta\begin{bmatrix} \frac{A-A^*}{2\mathrm{i}} & \frac{B}{2\mathrm{i}} \\ -\frac{B^*}{2\mathrm{i}} & O \end{bmatrix}\right\|
\end{aligned}$$

i.e.,

$$\left\| \Re\left(e^{\mathrm{i}\theta}\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right)\right\| = \left\| \cos\theta\begin{bmatrix} \Re A & \frac{B}{2} \\ \frac{B^*}{2} & O \end{bmatrix} - \sin\theta\begin{bmatrix} \Im A & \frac{B}{2\mathrm{i}} \\ -\frac{B^*}{2\mathrm{i}} & O \end{bmatrix}\right\|.$$

On the other hand, by letting $\cos\theta = \alpha$ and $-\sin\theta = \beta$, we have

$$\begin{aligned}
\left\| \Re\left(e^{\mathrm{i}\theta}\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right)\right\| &= \left\| \begin{bmatrix} \alpha\Re A + \beta\Im A & \frac{1}{2}(\alpha - \mathrm{i}\beta)B \\ \frac{1}{2}(\alpha + \mathrm{i}\beta)B^* & O \end{bmatrix}\right\| \\
&= r\left(\begin{bmatrix} \alpha\Re A + \beta\Im A & \frac{1}{2}(\alpha - \mathrm{i}\beta)B \\ \frac{1}{2}(\alpha + \mathrm{i}\beta)B^* & O \end{bmatrix}\right) \\
&= r\left(\begin{bmatrix} A^* & \frac{1}{2}(\alpha - \mathrm{i}\beta)I \\ B^* & O \end{bmatrix}\begin{bmatrix} \frac{1}{2}(\alpha + \mathrm{i}\beta)I & O \\ A & B \end{bmatrix}\right) \\
&= r\left(\begin{bmatrix} \frac{1}{2}(\alpha + \mathrm{i}\beta)I & O \\ A & B \end{bmatrix}\begin{bmatrix} A^* & \frac{1}{2}(\alpha - \mathrm{i}\beta)I \\ B^* & O \end{bmatrix}\right) \\
&= r\left(\begin{bmatrix} \frac{1}{2}(\alpha + \mathrm{i}\beta)A^* & \frac{\alpha^2+\beta^2}{4}I \\ |A^*|^2 + |B^*|^2 & \frac{1}{2}(\alpha - \mathrm{i}\beta)A \end{bmatrix}\right).
\end{aligned}$$

Therefore, by Lemma 5, we obtain

$$\begin{aligned}
\left\| \Re\left(e^{\mathrm{i}\theta}\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right)\right\| &\leqslant \omega\left(\begin{bmatrix} \frac{\alpha+\mathrm{i}\beta}{2}A^* & \frac{\alpha^2+\beta^2}{4}I \\ |A^*|^2 + |B^*|^2 & \frac{\alpha-\mathrm{i}\beta}{2}A \end{bmatrix}\right) \\
&\leqslant \frac{\sqrt{\alpha^2+\beta^2}}{2}\omega(A) + \omega\left(\begin{bmatrix} O & \frac{\alpha^2+\beta^2}{4}I \\ |A^*|^2 + |B^*|^2 & O \end{bmatrix}\right).
\end{aligned}$$

Now, by Lemma 2, we get

$$\begin{aligned}\omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right) &\leqslant \frac{1}{2}\omega(A) + \omega\left(\begin{bmatrix} O & \frac{1}{4}I \\ |A^*|^2 + |B^*|^2 & O \end{bmatrix}\right) \\ &= \frac{1}{2}\omega(A) + \frac{1}{2}\left\| |A^*|^2 + |B^*|^2 + \frac{1}{4}I \right\| \\ &= \frac{1}{2}\omega(A) + \frac{1}{2}\left\| |A^*|^2 + |B^*|^2 \right\| + \frac{1}{8},\end{aligned}$$

where the last equality follows noting that $\|T+I\| = \|T\|+1$, when $T \geqslant O$, as desired.

The second inequality follows from the following fact:

$$\frac{1}{2}\|B\| = \omega\left(\begin{bmatrix} O & B \\ O & O \end{bmatrix}\right) \leqslant \frac{1}{2}\left\| |B^*|^2 + \frac{1}{4}I \right\|.$$

This completes the proof. □

REMARK 1. It has been shown in [16] that

$$\omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|A\| + \|AA^* + BB^*\|\right),$$

and that

$$\omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right) \leqslant \frac{1}{2}\omega(A) + \frac{1}{4}\|I + AA^* + BB^*\|. \tag{5}$$

We give an example to show that our bound in Theorem 1 can be sharper than the above bound. Let

$$A = \begin{bmatrix} 0.06 & 0.02 \\ 0.05 & 0.01 \end{bmatrix}, \quad B = \begin{bmatrix} 0.08 & 0.02 \\ 0.05 & 0.14 \end{bmatrix}.$$

Then

$$\frac{1}{2}\left\| |A^*|^2 + |B^*|^2 + \frac{1}{4}I \right\| \approx 0.139964$$

and

$$\frac{1}{4}\|I + AA^* + BB^*\| \approx 0.257482.$$

This shows that the bound $\frac{1}{2}\omega(A) + \frac{1}{2}\left\| |A^*|^2 + |B^*|^2 + \frac{1}{4}I \right\|$ found in Theorem 1 is better than the bound $\frac{1}{2}\omega(A) + \frac{1}{4}\|I + AA^* + BB^*\|$ from [16], in this example.

In general, it can be seen that Theorem 1 provides sharper bounds than (5), when $AA^* + BB^* \leqslant I$.

In the recent work [6], it is shown that

$$\omega(P_{\mathscr{T}} P_{\mathscr{S}}) \leqslant \frac{1}{4}\left(\|P_{\mathscr{T}} + P_{\mathscr{S}}\|^2 + \|P_{\mathscr{T}} - P_{\mathscr{S}}\|^2\right) \tag{6}$$

for any pair of orthogonal projections $P_{\mathscr{T}}$ and $P_{\mathscr{S}}$. We generalize and refine this result for any pair of operators in the following corollary.

COROLLARY 2. *Let* $A,B \in \mathscr{B}(\mathscr{H})$. *Then*

$$\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \leqslant \frac{1}{2}\sqrt{\|A+B^*\|^2+\|A-B^*\|^2}.$$

In particular,

$$\max\{\omega(AB),\omega(BA)\} \leqslant \frac{1}{4}\left(\|A+B^*\|^2+\|A-B^*\|^2\right).$$

Proof. The first assertion follows from Proposition 1, by noting that

$$\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \leqslant |\alpha|\frac{\|A+B^*\|}{2}+|\beta|\frac{\|A-B^*\|}{2}; \;\; \alpha^2+\beta^2=1,$$

then finding the maximum value on the right. The second assertion follows from the first, noting Lemma 6. □

In the following, we obtain a refinement of the first inequality of (3) in terms of the real and imaginary parts of T.

PROPOSITION 2. *Let* $T = \Re T + \mathrm{i}\Im T$ *be the Cartesian decomposition of* $T \in \mathscr{B}(\mathscr{H})$. *Then*

$$\frac{1}{2}\|T\| \leqslant \omega\left(\begin{bmatrix} O & \Re T \\ \mathrm{i}\Im T & O \end{bmatrix}\right) \leqslant \min\left\{\omega(T), \frac{1}{\sqrt{2}}\|T\|\right\}.$$

Proof. Let $A = \Re T$ and $B = \mathrm{i}\Im T$ in Proposition 1, and set $\alpha = 0,1$. This yields

$$\frac{1}{2}\max\{\|\Re T+\mathrm{i}\Im T\|,\|\Re T-\mathrm{i}\Im T\|\} \leqslant \omega\left(\begin{bmatrix} O & \Re T \\ \mathrm{i}\Im T & O \end{bmatrix}\right). \tag{7}$$

Now,

$$\begin{aligned}
\frac{1}{2}\|T\| &= \frac{1}{2}\max\{\|T\|,\|T^*\|\} \\
&= \frac{1}{2}\max\{\|\Re T+\mathrm{i}\Im T\|,\|\Re T-\mathrm{i}\Im T\|\} \\
&\leqslant \omega\left(\begin{bmatrix} O & \Re T \\ \mathrm{i}\Im T & O \end{bmatrix}\right) \quad \text{(by (7))} \\
&\leqslant \frac{1}{2}\sqrt{\|\Re T-\mathrm{i}\Im T\|^2+\|\Re T+\mathrm{i}\Im T\|^2} \quad \text{(by Corollary 2)} \\
&= \frac{1}{2}\sqrt{\|T^*\|^2+\|T\|^2} \\
&= \frac{\sqrt{2}}{2}\|T\|.
\end{aligned}$$

Thus, we have shown that

$$\frac{1}{2}\|T\| \leqslant \omega\left(\begin{bmatrix} O & \Re T \\ \mathrm{i}\Im T & O \end{bmatrix}\right) \leqslant \frac{\sqrt{2}}{2}\|T\|. \tag{8}$$

On the other hand, since (see [9, Theorem 2.3])

$$\omega\left(\begin{bmatrix} O & X \\ Y & O \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|X\| + \|Y\|\right),$$

we can write

$$\begin{aligned}\omega\left(\begin{bmatrix} O & \Re T \\ \mathrm{i}\Im T & O \end{bmatrix}\right) &\leqslant \frac{1}{2}\left(\|\Re T\| + \|\Im T\|\right) \\ &= \frac{1}{2}\left(\omega\left(\Re T\right) + \omega\left(\Im T\right)\right) \\ &\leqslant \omega\left(T\right).\end{aligned}$$

Therefore,

$$\frac{1}{2}\|T\| \leqslant \omega\left(\begin{bmatrix} O & \Re T \\ \mathrm{i}\Im T & O \end{bmatrix}\right) \leqslant \omega\left(T\right).$$

This, together with (8), implies the desired result. □

At this point, we recall that if $T \in \mathscr{B}(\mathscr{H})$ is such that $\Re T, \Im T \geqslant O$ (that is, T is accretive-dissipative), then [15]

$$\frac{1}{\sqrt{2}}\|T\| \leqslant \omega(T).$$

In this case, when T is accretive-dissipative, $\min\{\omega(T), \frac{1}{\sqrt{2}}\|T\|\} = \frac{1}{\sqrt{2}}\|T\|$.

REMARK 2. It has been shown in [10, (2.12)] that

$$\frac{1}{2}\omega\left(T\right) \leqslant \omega\left(\begin{bmatrix} O & \Re T \\ \mathrm{i}\Im T & O \end{bmatrix}\right).$$

Of course, Proposition 2 improves the above estimate.

REMARK 3. $\omega\left(\begin{bmatrix} O & \Re T \\ \mathrm{i}\Im T & O \end{bmatrix}\right) = \frac{1}{2}\|T\|$, whenever $T^2 = O$ (equivalently, when $\widetilde{T} = O$).

We have seen how Proposition 2 provides a refinement of the first inequality in (3). Now, in line with the approach of this paper, we present a refinement of the second inequality in (3). First, a lemma.

LEMMA 7. *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \leqslant \frac{1}{2}\sqrt{\max\left\{\|A\|, \|B\|\right\} + \sqrt{r\left(|A|\,|B^*|\right)}}\sqrt{\max\left\{\|A\|, \|B\|\right\} + \sqrt{r\left(|A^*|\,|B|\right)}}.$$

In particular,

$$\omega\left(AB\right) \leqslant \frac{1}{4}\left(\max\left\{\|A\|, \|B\|\right\} + \sqrt{r\left(|A|\,|B^*|\right)}\right)\left(\max\left\{\|A\|, \|B\|\right\} + \sqrt{r\left(|A^*|\,|B|\right)}\right). \tag{9}$$

Proof. From Proposition 1 and Lemma 4, we have

$$
\begin{aligned}
&\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \\
&= \sup_{\alpha^2+\beta^2=1} \left\| \alpha \frac{A+B^*}{2} + \beta \frac{A-B^*}{2\mathrm{i}} \right\| \\
&= \frac{1}{2} \sup_{\alpha^2+\beta^2=1} \left\| A + \frac{\alpha+\mathrm{i}\beta}{\alpha-\mathrm{i}\beta} B^* \right\| \\
&\leqslant \frac{1}{2}\sqrt{\max\{\|A\|,\|B\|\} + \left\| |A|^{\frac{1}{2}} |B^*|^{\frac{1}{2}} \right\|}\sqrt{\max\{\|A\|,\|B\|\} + \left\| |A^*|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|}.
\end{aligned}
$$

This proves the first assertion, since

$$
\left\| |S|^{\frac{1}{2}} |T|^{\frac{1}{2}} \right\| = \left\| |S|^{\frac{1}{2}} |T|^{\frac{1}{2}} |T|^{\frac{1}{2}} |S|^{\frac{1}{2}} \right\|^{\frac{1}{2}} = \sqrt{r(|S|\,|T|)}
$$

for any $S, T \in \mathscr{B}(\mathscr{H})$.

The second conclusion follows from Lemma 6. □

REMARK 4. We note that, using the arithmetic-geometric mean inequality,

$$
\begin{aligned}
&\frac{1}{2}\sqrt{\max\{\|A\|,\|B\|\} + \sqrt{r(|A|\,|B^*|)}}\sqrt{\max\{\|A\|,\|B\|\} + \sqrt{r(|A^*|\,|B|)}} \\
&\leqslant \frac{1}{2}\,\frac{\max\{\|A\|,\|B\|\} + \sqrt{r(|A|\,|B^*|)} + \max\{\|A\|,\|B\|\} + \sqrt{r(|A^*|\,|B|)}}{2} \\
&= \frac{1}{2}\left(\max\{\|A\|,\|B\|\} + \frac{\sqrt{r(|A|\,|B^*|)} + \sqrt{r(|A^*|\,|B|)}}{2}\right) \\
&\leqslant \frac{1}{2}\left(\max\{\|A\|,\|B\|\} + \max\left\{\sqrt{r(|A|\,|B^*|)}, \sqrt{r(|A^*|\,|B|)}\right\}\right). \qquad (10)
\end{aligned}
$$

This shows that Lemma 7 improves Theorem 2.5 in [4].

We have the following refinement of the right inequality in (3).

COROLLARY 3. *Let $T \in \mathscr{B}(\mathscr{H})$. Then*

$$
\omega(T) \leqslant \frac{1}{2}\left(\|T\| + \sqrt{\|T\|\, r\left(|T^*|^{\frac{1}{2}} |T|^{\frac{1}{2}}\right)}\right).
$$

Meanwhile, $\omega(T) = \frac{1}{2}\|T\|$, whenever $|T^*|^{\frac{1}{2}} |T|^{\frac{1}{2}} = O$.

Proof. Let $T = U|T|$ be the polar decomposition of T, and let $A = U|T|^{\frac{1}{2}}$ and $B = |T|^{\frac{1}{2}}$. Noting that

$$
AB = T,\ \ \|A\| \leqslant \|B\| = \|T\|^{\frac{1}{2}},\ \ |A|^{\frac{1}{2}} = |B^*|^{\frac{1}{2}} = |B|^{\frac{1}{2}} = |T|^{\frac{1}{4}},\ \ |A^*|^{\frac{1}{2}} = |T^*|^{\frac{1}{4}},
$$

direct substitution in (9) implies the desired result.

The second assertion follows from the obtained inequality and the first inequality in (3). $\square$

Related to Corollary 2 and Lemma 7, we have the following mixed bound. Notice also how this extends (6).

THEOREM 2. *Let $A,B \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \leqslant \frac{1}{2}\sqrt{\max\{\|A+B^*\|, \|A-B^*\|\} + \frac{1}{2}\sqrt{r(|A+B^*|\,|B-A^*|)}} \times \sqrt{\max\{\|A+B^*\|, \|A-B^*\|\} + \frac{1}{2}\sqrt{r(|A^*+B|\,|A-B^*|)}}.$$

Proof. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha^2+\beta^2 = 1$. So, $|\alpha|, |\beta| \leqslant 1$. We have by Lemma 4 that

$$\begin{aligned}
&\left\|\alpha\frac{A+B^*}{2} + \beta\frac{A-B^*}{2\mathrm{i}}\right\| \\
&\leqslant \sqrt{\max\left\{\frac{|\alpha|}{2}\|A+B^*\|, \frac{|\beta|}{2}\|A-B^*\|\right\} + \frac{\sqrt{|\alpha|\,|\beta|}}{2}\left\||A+B^*|^{\frac{1}{2}}|B-A^*|^{\frac{1}{2}}\right\|} \\
&\quad\times\sqrt{\max\left\{\frac{|\alpha|}{2}\|A+B^*\|, \frac{|\beta|}{2}\|A-B^*\|\right\} + \frac{\sqrt{|\alpha|\,|\beta|}}{2}\left\||A^*+B|^{\frac{1}{2}}|A-B^*|^{\frac{1}{2}}\right\|} \\
&\leqslant \sqrt{\max\left\{\frac{|\alpha|}{2}\|A+B^*\|, \frac{|\beta|}{2}\|A-B^*\|\right\} + \frac{|\alpha|^2+|\beta|^2}{4}\left\||A+B^*|^{\frac{1}{2}}|B-A^*|^{\frac{1}{2}}\right\|} \\
&\quad\times\sqrt{\max\left\{\frac{|\alpha|}{2}\|A+B^*\|, \frac{|\beta|}{2}\|A-B^*\|\right\} + \frac{|\alpha|^2+|\beta|^2}{4}\left\||A^*+B|^{\frac{1}{2}}|A-B^*|^{\frac{1}{2}}\right\|} \\
&\leqslant \frac{1}{2}\sqrt{\max\{\|A+B^*\|, \|A-B^*\|\} + \frac{1}{2}\left\||A+B^*|^{\frac{1}{2}}|B-A^*|^{\frac{1}{2}}\right\|} \\
&\quad\times\sqrt{\max\{\|A+B^*\|, \|A-B^*\|\} + \frac{1}{2}\left\||A^*+B|^{\frac{1}{2}}|A-B^*|^{\frac{1}{2}}\right\|}.
\end{aligned}$$

That is,

$$\left\|\alpha\frac{A+B^*}{2} + \beta\frac{A-B^*}{2\mathrm{i}}\right\| \leqslant \frac{1}{2}\sqrt{\max\{\|A+B^*\|, \|A-B^*\|\} + \frac{1}{2}\left\||A+B^*|^{\frac{1}{2}}|B-A^*|^{\frac{1}{2}}\right\|} \times\sqrt{\max\{\|A+B^*\|, \|A-B^*\|\} + \frac{1}{2}\left\||A^*+B|^{\frac{1}{2}}|A-B^*|^{\frac{1}{2}}\right\|}.$$

By taking supremum over $\alpha,\beta\in\mathbb{R}$ with $\alpha^2+\beta^2=1$, we infer that

$$\omega\left(\begin{bmatrix} O & A\\ B & O\end{bmatrix}\right)\leqslant\frac{1}{2}\sqrt{\max\{\|A+B^*\|,\|A-B^*\|\}+\frac{1}{2}\left\||A+B^*|^{\frac{1}{2}}|B-A^*|^{\frac{1}{2}}\right\|}$$
$$\times\sqrt{\max\{\|A+B^*\|,\|A-B^*\|\}+\frac{1}{2}\left\||A^*+B|^{\frac{1}{2}}|A-B^*|^{\frac{1}{2}}\right\|}.$$

Arguing like Lemma 7, this is equivalent to the desired conclusion. □

EXAMPLE 1. We have found different bounds for $\omega\begin{pmatrix} O & A\\ B & O\end{pmatrix}$. In this example, we give numerical examples to show that none of the obtained bounds is uniformly better than the other. This will be summarized in a table as follows. For given A,B, let

$$a=\frac{1}{2}\sqrt{\|A+B^*\|^2+\|A-B^*\|^2},$$

$$b=\frac{1}{2}\sqrt{\max\{\|A\|,\|B\|\}+\left\||A|^{\frac{1}{2}}|B^*|^{\frac{1}{2}}\right\|}\sqrt{\max\{\|A\|,\|B\|\}+\left\||A^*|^{\frac{1}{2}}|B|^{\frac{1}{2}}\right\|},$$

$$c=\frac{1}{2}\sqrt{\max\{\|A+B^*\|,\|A-B^*\|\}+\frac{1}{2}\left\||A+B^*|^{\frac{1}{2}}|B-A^*|^{\frac{1}{2}}\right\|}$$
$$\times\sqrt{\max\{\|A+B^*\|,\|A-B^*\|\}+\frac{1}{2}\left\||A^*+B|^{\frac{1}{2}}|A-B^*|^{\frac{1}{2}}\right\|}.$$

These are the three upper bounds found in Corollary 2, Lemma 7, and Theorem 2, respectively. Further, let $d=\omega\begin{pmatrix} O & A\\ B & O\end{pmatrix}$.

No.	A,B	d	a	b	c
2	$A=\begin{bmatrix}4&0\\1&1\end{bmatrix}$ $B=\begin{bmatrix}5&3\\2&2\end{bmatrix}$	5.1627	5.3762	5.69321	6.49065
1	$A=\begin{bmatrix}5&3\\0&4\end{bmatrix}$ $B=\begin{bmatrix}4&5\\4&0\end{bmatrix}$	6.53824	7.31387	6.76621	7.53419
3	$A=\begin{bmatrix}9&-9\\0&-10\end{bmatrix}$ $B=\begin{bmatrix}-10&1\\2&-7\end{bmatrix}$	10.982	14.324	13.2392	12.9548

These approximate numbers were obtained by Mathematica. These three examples show that none of the three bounds obtained is uniformly better than the other two!

EXAMPLE 2. It has been shown in [12, Therem 2.3] that

$$\omega\begin{pmatrix} O & A \\ B & O \end{pmatrix} \leqslant \frac{1}{2}\left(\|A\|+\|B\|\right).$$

Here we give an example to show that neither this bound nor our bound in Corollary 2 is uniformly better than the other.

(i) Let

$$A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$B=\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\frac{1}{2}\left(\|A\|+\|B\|\right) \approx 1.20711$$

and

$$\frac{1}{2}\sqrt{\|A+B^*\|^2+\|A-B^*\|^2} \approx 1.14412.$$

(ii) Let

$$A=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$B=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\frac{1}{2}\left(\|A\|+\|B\|\right) \approx 1.30902$$

and

$$\frac{1}{2}\sqrt{\|A+B^*\|^2+\|A-B^*\|^2} \approx 1.37491.$$

3. Lower bounds for the numerical radius of an operator matrix

In the previous section, several upper bounds for the numerical radius of certain operator matrices were shown and applied. In this section, we present some lower bounds with applications.

It follows from [9, Theorem 3.7] that

$$\max\left\{\omega(A), \frac{1}{2}\omega(B)\right\} \leqslant \omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right).$$

The following result improves this lower bound.

COROLLARY 4. *Let* $A,B \in \mathscr{B}(\mathscr{H})$. *Then*

$$\max\left\{\frac{1}{2}\|B\|,\omega(A)\right\} \leqslant \omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right).$$

Proof. It follows from the proof of Theorem 1 that

$$\left\|\begin{bmatrix} \alpha\Re A+\beta\Im A & \frac{1}{2}(\alpha-\mathrm{i}\beta)B \\ \frac{1}{2}(\alpha+\mathrm{i}\beta)B^* & O \end{bmatrix}\right\| \leqslant \omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right).$$

One can easily see that

$$\frac{\sqrt{\alpha^2+\beta^2}}{2}\|B\| \leqslant \left\|\begin{bmatrix} \alpha\Re A+\beta\Im A & \frac{1}{2}(\alpha-\mathrm{i}\beta)B \\ \frac{1}{2}(\alpha+\mathrm{i}\beta)B^* & O \end{bmatrix}\right\|$$

and

$$\|\alpha\Re A+\beta\Im A\| \leqslant \left\|\begin{bmatrix} \alpha\Re A+\beta\Im A & \frac{1}{2}(\alpha-\mathrm{i}\beta)B \\ \frac{1}{2}(\alpha+\mathrm{i}\beta)B^* & O \end{bmatrix}\right\|.$$

From (3), by letting $\alpha = 1$ and $\beta = 0$, we have

$$\frac{1}{2}\|B\| \leqslant \omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right).$$

From (3) and by taking supremum over $\alpha,\beta \in \mathbb{R}$ with $\alpha^2+\beta^2 = 1$, then noting Lemma 1, we infer that

$$\omega(A) \leqslant \omega\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\right).$$

Combining the last two inequalities imply the desired result. □

Now we are ready to find the following lower bound for the numerical radius of the operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. We notice that this gives a reversed version of (4).

COROLLARY 5. *Let* $A,B \in \mathscr{B}(\mathscr{H})$. *Then*

$$\max\left\{\omega(A),\omega(D),\frac{1}{2}\|B+C^*\|,\frac{1}{2}\|B-C^*\|\right\} \leqslant \omega\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right),$$

with equality when $B = C = O$.

Proof. It follows from Proposition 1 by letting $\alpha = 0,1$, that

$$\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \geqslant \frac{1}{2}\max\{\|A+B^*\|,\|A-B^*\|\}. \tag{11}$$

From this, we have

$$\begin{aligned}\omega\left(\begin{bmatrix} A & B \\ C & D\end{bmatrix}\right) &\geqslant \max\left\{\omega\left(\begin{bmatrix} A & O \\ O & D\end{bmatrix}\right), \omega\left(\begin{bmatrix} O & B \\ C & O\end{bmatrix}\right)\right\} \\ &= \max\left\{\max\left\{\omega(A), \omega(D)\right\}, \omega\left(\begin{bmatrix} O & B \\ C & O\end{bmatrix}\right)\right\} \\ &\geqslant \max\left\{\max\left\{\omega(A), \omega(D)\right\}, \frac{1}{2}\max\left\{\|B+C^*\|, \|B-C^*\|\right\}\right\} \\ &= \max\left\{\omega(A), \omega(D), \frac{1}{2}\|B+C^*\|, \frac{1}{2}\|B-C^*\|\right\},\end{aligned}$$

as desired. □

We remark here that (11) also follows from Lemma 3.

REMARK 5. In [12, Theorem 2.3], it is shown that

$$\omega\left(\begin{bmatrix} O & A \\ B & O\end{bmatrix}\right) \geqslant \frac{1}{2}\|A+B^*\|.$$

Consequently, (11) provides a refinement of this bound.

The first part of the following result has been recently shown in [13, Corollary 2.2]. However, the significance below is gained by the two identities one can find for the stated supremums, and the subsequent equality conditions.

In the proof of the next result, we repeatedly use the property that if $T = U|T|$ is the polar decomposition of T, then $U|T|^q U^* = |T^*|^q, q > 0$, see [7, p. 58] for example.

THEOREM 3. *Let $T \in \mathscr{B}(\mathscr{H})$ and let $0 \leqslant t \leqslant 1$. If $T = U|T|$ is the polar decomposition of T, then*

(i)

$$\omega(T) \leqslant \omega^2\left(\begin{bmatrix} O & U|T|^{1-t} \\ |T|^t & O\end{bmatrix}\right).$$

(ii)

$$\begin{aligned}\omega^2\left(\begin{bmatrix} O & U|T|^{1-t} \\ |T|^t & O\end{bmatrix}\right) &= \sup_{\alpha^2+\beta^2=1}\left\|\frac{1}{4}\left(|T|^{2t} + |T^*|^{2(1-t)}\right) + \frac{\alpha^2-\beta^2}{2}\Re T + \alpha\beta\Im T\right\| \\ &= \sup_{\alpha^2+\beta^2=1}\left\|\frac{1}{4}\left(|T|^{2t} + |T|^{2(1-t)}\right) + \frac{\alpha^2-\beta^2}{2}\Re\widetilde{T}_t + \alpha\beta\Im\widetilde{T}_t\right\|.\end{aligned}$$

Proof. It follows, from Lemma 6 and Proposition 1 that

$$\max\{\omega(AB), \omega(BA)\} \leqslant \sup_{\alpha^2+\beta^2=1}\left\|\alpha\frac{A+B^*}{2} + \beta\frac{A-B^*}{2\mathrm{i}}\right\|^2.$$

For $T \in \mathscr{B}(\mathscr{H})$, let $T = U|T|$ be the polar decomposition of T, and let $A = U|T|^{1-t}$, $B = |T|^t$, $0 \leqslant t \leqslant 1$. Then the first assertion follows from Lemma 6. Next, Proposition 1 implies

$$\begin{aligned}
&\omega^2\left(\begin{bmatrix} O & U|T|^{1-t} \\ |T|^t & O \end{bmatrix}\right)\\
&= \sup_{\alpha^2+\beta^2=1} \left\| \alpha \frac{U|T|^{1-t}+|T|^t}{2} + \beta \frac{U|T|^{1-t}-|T|^t}{2\mathrm{i}} \right\|^2\\
&= \sup_{\alpha^2+\beta^2=1} \left\| \left(\alpha \frac{U|T|^{1-t}+|T|^t}{2} + \beta \frac{U|T|^{1-t}-|T|^t}{2\mathrm{i}}\right)\left(\alpha \frac{U|T|^{1-t}+|T|^t}{2} + \beta \frac{U|T|^{1-t}-|T|^t}{2\mathrm{i}}\right)^* \right\|\\
&= \sup_{\alpha^2+\beta^2=1} \left\| \left(\alpha \frac{U|T|^{1-t}+|T|^t}{2} + \beta \frac{U|T|^{1-t}-|T|^t}{2\mathrm{i}}\right)\left(\alpha \frac{|T|^{1-t}U^*+|T|^t}{2} + \beta \frac{|T|^t-|T|^{1-t}U^*}{2\mathrm{i}}\right) \right\|\\
&= \sup_{\alpha^2+\beta^2=1} \left\| \frac{\alpha^2}{4}\left(|T|^{2t}+|T^*|^{2(1-t)}+2\Re T\right) + \frac{\beta^2}{4}\left(|T|^{2t}+|T^*|^{2(1-t)}-2\Re T\right) + \alpha\beta\Im T \right\|.
\end{aligned}$$

This proves that

$$\omega^2\left(\begin{bmatrix} O & U|T|^{1-t} \\ |T|^t & O \end{bmatrix}\right) = \sup_{\alpha^2+\beta^2=1} \left\| \frac{1}{4}\left(|T|^{2t}+|T^*|^{2(1-t)}\right) + \frac{\alpha^2-\beta^2}{2}\Re T + \alpha\beta\Im T \right\|.$$

On the other hand,

$$\begin{aligned}
&\omega^2\left(\begin{bmatrix} O & U|T|^{1-t} \\ |T|^t & O \end{bmatrix}\right)\\
&= \sup_{\alpha^2+\beta^2=1} \left\| \alpha \frac{U|T|^{1-t}+|T|^t}{2} + \beta \frac{U|T|^{1-t}-|T|^t}{2\mathrm{i}} \right\|^2\\
&= \sup_{\alpha^2+\beta^2=1} \left\| \left(\alpha \frac{U|T|^{1-t}+|T|^t}{2} + \beta \frac{U|T|^{1-t}-|T|^t}{2\mathrm{i}}\right)^*\left(\alpha \frac{U|T|^{1-t}+|T|^t}{2} + \beta \frac{U|T|^{1-t}-|T|^t}{2\mathrm{i}}\right) \right\|\\
&= \sup_{\alpha^2+\beta^2=1} \left\| \left(\alpha \frac{|T|^{1-t}U^*+|T|^t}{2} + \beta \frac{|T|^t-|T|^{1-t}U^*}{2\mathrm{i}}\right)\left(\alpha \frac{U|T|^{1-t}+|T|^t}{2} + \beta \frac{U|T|^{1-t}-|T|^t}{2\mathrm{i}}\right) \right\|\\
&= \sup_{\alpha^2+\beta^2=1} \left\| \frac{\alpha^2}{4}\left(|T|^{2t}+|T|^{2(1-t)}+2\Re\widetilde{T}_t\right) + \frac{\beta^2}{4}\left(|T|^{2t}+|T|^{2(1-t)}-2\Re\widetilde{T}_t\right) + \alpha\beta\Im\widetilde{T}_t \right\|.
\end{aligned}$$

This completes the proof. □

For the rest of this section, we discuss some equality conditions. Here we notice that if $T \in \mathscr{B}(\mathscr{H})$, then $T^2 = O$ if and only if $\widetilde{T} = O$.

COROLLARY 6. *Let $T \in \mathscr{B}(\mathscr{H})$. If $T^2 = O$ (equivalently, if $\widetilde{T} = O$), then*

$$\begin{aligned}\|T\| &= \sup_{\alpha^2+\beta^2=1} \left\| \frac{|T|+|T^*|}{2} + \left(\alpha^2-\beta^2\right)\Re T + 2\alpha\beta\Im T \right\| \\ &= \sup_{\alpha^2+\beta^2=1} \left\| |T| + \left(\alpha^2-\beta^2\right)\Re\widetilde{T} + 2\alpha\beta\Im\widetilde{T} \right\| \\ &= \sup_{\alpha^2+\beta^2=1} \left\| \frac{|T|+|T^*|}{2} + \left(\alpha^2-\beta^2\right)\Re T + 2\alpha\beta\Im T \right\|.\end{aligned}$$

Proof. We know that (see [13, Corollary 2.2])

$$\omega^2\left(\begin{bmatrix} O & U|T|^{\frac{1}{2}} \\ |T|^{\frac{1}{2}} & O \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|T\| + \omega\left(\widetilde{T}\right)\right).$$

So, by Theorem 3, we get

$$\begin{aligned}\omega(T) &\leqslant \frac{1}{2}\sup_{\alpha^2+\beta^2=1} \left\| |T| + \left(\alpha^2-\beta^2\right)\Re\widetilde{T} + 2\alpha\beta\Im\widetilde{T} \right\| \\ &= \frac{1}{2}\sup_{\alpha^2+\beta^2=1} \left\| \frac{|T|+|T^*|}{2} + \left(\alpha^2-\beta^2\right)\Re T + 2\alpha\beta\Im T \right\| \\ &\leqslant \frac{1}{2}\left(\|T\| + \omega\left(\widetilde{T}\right)\right).\end{aligned}$$

The result follows by assuming $\widetilde{T} = O$. □

COROLLARY 7. *Let $T \in \mathscr{B}(\mathscr{H})$. If $T^2 = O$ (equivalently, if $\widetilde{T} = O$), then*

$$\max\left\{\left\|\frac{|T|+|T^*|}{2} + \Im T\right\|, \left\|\frac{|T|+|T^*|}{2} + \Re T\right\|\right\} \leqslant \|T\|,$$

and

$$\max\left\{\left\||T| + \Re\widetilde{T}\right\|, \left\||T| + \Im\widetilde{T}\right\|, \left\|\frac{|T|+|T^*|}{2} + \Re T\right\|, \left\|\frac{|T|+|T^*|}{2} + \Im T\right\|\right\} \leqslant \|T\|.$$

Proof. We prove the first inequality. If we set $\alpha = 1$ and $\beta = 0$, we get

$$\left\|\frac{|T|+|T^*|}{2} + \Re T\right\| \leqslant \|T\|.$$

If we set $\alpha = \beta = \frac{1}{\sqrt{2}}$, we obtain

$$\left\|\frac{|T|+|T^*|}{2} + \Im T\right\| \leqslant \|T\|.$$

By combining the above two inequalities, we get the desired inequality. □

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