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Φ–MOMENT B–VALUED MARTINGALE INEQUALITIES ON LORENTZ SPACES

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Abstract. In this paper, with the help of some new atomic decomposition theorems, several Φ-moment Banach space valued martingale inequalities associated with concave functions in the context of Lorentz spaces are deduced. Our results are closely related with the geometrical properties of the underlying Banach spaces.

1. Introduction

Martingale inequalities are essential in the development of harmonic analysis, probability and other aspects of analysis. Many classical scalar-valued martingale inequalities have been extended to the Banach space valued (**B**-valued) martingale setting. We refer the reader to [3, 4, 10, 19, 21, 28, 29] and the references therein for more information on **B**-valued martingales. The main topic we shall discuss here is the Φ-moment **B**-valued martingale inequalities associated with concave functions in the framework of Lorentz spaces.

In 1970, Burkholder and Gundy [6] firstly discussed the Φ-moment inequalities for martingales. Later, the well known Φ-moment version of the Burkholder-Davis-Gundy inequality was discovered by Burkholder et al. in [5]. To better explain our motivation and results, let us briefly recall the main inequality they achieved there. Suppose that Φ is a strictly convex Orlicz function on $[0,\infty)$ satisfying the Δ_2-condition. Then for any L_Φ-bounded martingale $f=(f_n)_{n\geqslant 0}$,

$$\mathbb{E}(\Phi(S(f)))\lesssim\mathbb{E}(\Phi(M(f)))\lesssim\mathbb{E}(\Phi(S(f))), \tag{1}$$

where $M(f)=\sup_{m\geqslant 0}|f_m|$ denotes the maximal function of martingale f and $S(f)=\left(\sum_{m=0}^{\infty}|df_m|^2\right)^{1/2}$ means the square function of martingale f. Using the techniques of Doob's decomposition and the averaging operators, Kikuchi [15, Theorem 3] obtained an extension of (1) amid rearrangement invariant Banach function spaces. Recently, several Φ-moment inequalities related to concave functions have been studied. This was initiated by Jiao and Yu [14]. Subsequently, Peng and Li [27] generalized their

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results to the framework of Lorentz spaces. One of their results reads as follows: let Φ be a concave Orlicz function, $0<p<2$ and $0<q\leqslant\infty$. Then for every martingale $f=(f_n)_{n\geqslant 0}$,

$$\|\Phi(M(f))\|_{p,q}\lesssim\|\Phi(s(f))\|_{p,q}, \tag{2}$$

where $s(f)=\left(\sum_{m=0}^{\infty}\mathbb{E}_{m-1}(|df_m|)^2\right)^{1/2}$ is the conditional square function of martingale f. Clearly, martingales in the articles mentioned above are scalar-valued martingales. Motivated by these various results, one natural question arises, that is, does the type of (2) also hold for $\mathbf{B}$-valued martingales? In this paper, we give an affirmative answer. We should also mention that martingale inequalities in the $\mathbf{B}$-valued case are closely connected with the geometrical properties of the underlying Banach spaces. For instance, one of our main results which should be compared to (2) states as follows (see Section 2 for any unexplained terminology; the proof can be found in Theorem 4): let $\mathbf{B}$ be a Banach space, $\Phi\in\mathscr{G}$ be a concave function, $1<r\leqslant 2$, $0<p<r$ and $0<q\leqslant\infty$. Then the following statements are equivalent:

(i) $\mathbf{B}$ is isomorphic to a r-uniformly smooth space;

(ii) If the $\mathbf{B}$-valued martingale $f=(f_n)_{n\geqslant 0}$ satisfies $\|\Phi(s^r(f))\|_{p,q}<\infty$, then

$$\|\Phi(M(f))\|_{p,q}\lesssim\|\Phi(s^r(f))\|_{p,q}. \tag{3}$$

We note that if $r=2$ and $\mathbf{B}=\mathbb{R}$, then (3) recovers (2). Moreover, if we take $\Phi(t)=t$ in item (ii), then we recover [18, Theorem 5.4 (ii)] while item (ii) gives [20, Theorem 5 (ii)] when $\Phi(t)=t$ and $p=q$.

Our main approach is based on new atomic decompositions. Recall that atomic decompositions were introduced by Herz [9] and Bernard and Maisonneuve [2] for scalar-valued martingales. After that, this method was generalized by Weisz [30, 31] and developed by many other authors (see e.g. [8, 11, 12, 13, 22, 26]). As for $\mathbf{B}$-valued martingales, Liu and Hou [20] firstly investigated the atomic decomposition of $\mathbf{B}$-valued martingale Hardy spaces. Recently, Liu et al. [18] obtained some martingale inequalities in the setting of $\mathbf{B}$-valued martingale Hardy-Lorentz spaces with the help of atomic decompositions. For more details of atomic decompositions of various $\mathbf{B}$-valued martingale spaces see [16, 17, 19, 21, 24, 25, 32] and the references therein. It should be noticed that the atomic decomposition theorems of this paper improve those in [18, 20].

The paper is structured as follows. In Section 2, some preliminary lemmas and basic knowledge will be introduced. Some atomic decompositions of the $\mathbf{B}$-valued martingale Hardy-Lorentz spaces are established in Section 3. These theorems are closely connected with the geometrical properties of the underlying Banach space $\mathbf{B}$. In the last section, with the help of atomic decompositions, we deduce some Φ-moment $\mathbf{B}$-valued martingale inequalities in frame of Lorentz spaces.

Throughout this paper, the set of nonnegative integers, the set of integers, the real number field and the complex number field are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. The letter C denotes a positive real number, not necessarily the same number from line to line. $f\lesssim g$ means there exists a positive constant C such that $f\leqslant Cg$. If $f\lesssim g\lesssim f$, then we write $f\approx g$.

2. Preliminaries

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space and $(\mathbf{B},\|\cdot\|)$ be a Banach space. For a scalar-valued function $f:\Omega\to\mathbb{R}$ (or $\mathbb{C}$), let

$$\|f\|_p=\left(\int_\Omega |f|^p d\mathbb{P}\right)^{\frac{1}{p}}\ (0<p<\infty) \text{ and } \|f\|_\infty = ess\ \sup|f|.$$

For a $\mathbf{B}$-valued function $f:\Omega\to\mathbf{B}$, let

$$\|f\|_{L_p(\mathbf{B})}=\left(\int_\Omega \|f\|^p d\mathbb{P}\right)^{\frac{1}{p}}\ (0<p<\infty) \text{ and } \|f\|_{L_\infty(\mathbf{B})} = ess\ \sup\|f\|.$$

2.1. Lorentz spaces

In this subsection, we start with the following definition of Lorentz spaces. For basic properties of Lorentz spaces, see [1, 7, 23].

DEFINITION 1. Given a measurable function f on a measure space $(\Omega,\mathscr{F},\mathbb{P})$, $0<p<\infty$ and $0<q\leqslant\infty$, define

$$\|f\|_{p,q}=\begin{cases}\left(p\int_0^\infty \left(t\mathbb{P}(|f|>t)^{\frac{1}{p}}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } 0<q<\infty;\\ \sup_{t>0} t\mathbb{P}(|f|>t)^{\frac{1}{p}}, & \text{if } q=\infty.\end{cases}$$

The set of all f with $\|f\|_{p,q}<\infty$ is denoted by $L_{p,q}$ and is called the Lorentz space with indices p and q.

REMARK 1. ([7]) (i) It is well known that if p, q are bigger than 1, then $L_{p,q}$ is a Banach space. However, for other values of p and q, $L_{p,q}$ is only a quasi-Banach space.

(ii) If $p=q$ for $0<p<\infty$, then $L_{p,q}$ is the usual L_p space. In this case, we denote $\|\cdot\|_{p,q}$ by $\|\cdot\|_p$.

(iii) Observe that for all $0<p,r<\infty$ and $0<q\leqslant\infty$ we have $\||f|^r\|_{p,q}=\|f\|^r_{pr,qr}$.

2.2. Orlicz functions

Recall that a function $\Phi:[0,\infty)\to[0,\infty)$ is called an Orlicz function if it is non-decreasing, $\Phi(t)>0$ for all $t>0$, $\Phi(0)=0$ and $\Phi(t)\to\infty$ as $t\to\infty$. Let $\mathscr{G}$ be the set of all Orlicz functions. We have the following simple but useful lemmas.

LEMMA 1. *Let $\Phi\in\mathscr{G}$ be concave. Then it follows from* [26, 27] *that:*

(i) *If $0<s\leqslant1$ and $t\geqslant0$, then $s\Phi(t)\leqslant\Phi(st)$;*

(ii) *If $s\geqslant1$ and $t\geqslant0$, then $\Phi(st)\leqslant s\Phi(t)$.*

Moreover, one can prove that Φ is subadditive, continuous and bijective from $[0,\infty)$ to $[0,\infty)$.

LEMMA 2. ([27]) *Let $\Phi \in \mathscr{G}$ be concave, $0<p<\infty$, $0<q\leqslant\infty$ and $p,q<r\leqslant\infty$. For $f\in L_r$, if there exists $A\in\mathscr{F}$ with $\mathbb{P}(A)\neq 0$ such that $\{f\neq 0\}\subset A$, then*

$$\|\Phi(|f|)\|_{p,q}\lesssim \mathbb{P}(A)^{\frac{1}{p}}\Phi\Big(\frac{\|f\|_r}{\mathbb{P}(A)^{\frac{1}{r}}}\Big).$$

2.3. B-valued martingales

Let $\{\mathscr{F}_n\}_{n\geqslant 0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F}=\sigma\big(\bigcup_{n\geqslant 0}\mathscr{F}_n\big)$. The expectation operator and the conditional expectation operator related to $\mathscr{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. A sequence $f=(f_n)_{n\geqslant 0}$ in $L_1(\mathbf{B})$ is called a $\mathbf{B}$-valued martingale if f_n is $\mathscr{F}_n$-measurable and satisfies $\mathbb{E}_n(f_{n+1})=f_n$ for each $n\geqslant 0$.

Let $\big([0,1),\mathscr{F},\mu\big)$ be a probability space, μ be Lebesgue measure and filtration $\{\mathscr{F}_n\}_{n\geqslant 0}$ be generated by:

$$\mathscr{F}_n=\Big\{\sigma\text{-algebra generated by atoms } \Big[\frac{i}{2^n},\frac{i+1}{2^n}\Big) : i=0,\dots,2^n-1\Big\}.$$

Remind that all martingales with respect to the above filtration $\{\mathscr{F}_n\}_{n\geqslant 0}$ are called dyadic martingales.

Denote by $\mathscr{M}$ the set of all $\mathbf{B}$-valued martingales $f=(f_n)_{n\geqslant 0}$ relative to $\{\mathscr{F}_n\}_{n\geqslant 0}$ such that $f_0=0$. For $f=(f_n)_{n\geqslant 0}\in\mathscr{M}$, we define the martingale difference by $df_n=f_n-f_{n-1}$ ($n\geqslant 0$, with convention $f_{-1}=0$ and $\mathscr{F}_{-1}=\{\emptyset,\Omega\}$). The $\mathbf{B}$-valued martingale $f=(f_n)_{n\geqslant 0}\in\mathscr{M}$ is said to be $L_p(\mathbf{B})$-bounded if $f_n\in L_p(\mathbf{B})$ for all $n\geqslant 0$ and

$$\|f\|_{L_p(\mathbf{B})}:=\sup_{n\geqslant 0}\|f_n\|_{L_p(\mathbf{B})}<\infty.$$

Let $\mathscr{V}$ be the set of all stopping times relative to $\{\mathscr{F}_n\}_{n\geqslant 0}$. For $f\in\mathscr{M}$ and $\upsilon\in\mathscr{V}$, the stopped martingale $f^{\upsilon}=(f_n^{\upsilon})_{n\geqslant 0}$ is defined by

$$f_n^{\upsilon}:=\sum_{m=1}^{n}\chi_{\{m\leqslant\upsilon\}}df_m.$$

The maximal function, the r-variation and the conditional r-variation ($1\leqslant r<\infty$) of a $\mathbf{B}$-valued martingale $f=(f_n)_{n\geqslant 0}$ are respectively defined by

$$M_n(f)(\omega):=\sup_{0\leqslant m\leqslant n}\|f_m(\omega)\|,\quad M(f)(\omega):=\sup_{m\geqslant 0}\|f_m(\omega)\|;$$

$$S_n^r(f)(\omega):=\Big(\sum_{m=0}^{n}\|df_m(\omega)\|^r\Big)^{\frac{1}{r}},\quad S^r(f)(\omega):=\Big(\sum_{m=0}^{\infty}\|df_m(\omega)\|^r\Big)^{\frac{1}{r}};$$

$$s_n^r(f)(\omega):=\Big(\sum_{m=0}^{n}\mathbb{E}_{m-1}\big(\|df_m\|^r\big)(\omega)\Big)^{\frac{1}{r}},\quad s^r(f)(\omega):=\Big(\sum_{m=0}^{\infty}\mathbb{E}_{m-1}\big(\|df_m\|^r\big)(\omega)\Big)^{\frac{1}{r}}.$$

Let Λ be the collection of all sequences $(\lambda_n)_{n\geqslant 0}$ of nondecreasing, nonnegative and adapted functions with respect to $\{\mathscr{F}_n\}_{n\geqslant 0}$. Set $\lambda_\infty:=\lim_{n\to\infty}\lambda_n$. For $f\in\mathscr{M}$, $\Phi\in\mathscr{G}$, $0<p<\infty, 0<q\leqslant\infty$ and $1\leqslant r<\infty$, we define

$$\Lambda[\mathscr{Q}^{S^r}_{p,q,\Phi}(f)](\mathbf{B}):=\Big\{(\lambda_n)_{n\geqslant 0}\in\Lambda : S_n^r(f)\leqslant\lambda_{n-1}\ (n\geqslant 1),\Phi(\lambda_\infty)\in L_{p,q}\Big\}$$

and

$$\Lambda[\mathscr{D}_{p,q,\Phi}(f)](\mathbf{B}) := \Big\{(\lambda_n)_{n\geqslant 0} \in \Lambda : \|f_n\| \leqslant \lambda_{n-1}\ (n \geqslant 1), \Phi(\lambda_\infty) \in L_{p,q}\Big\}.$$

Set

$$\|\Phi(f)\|_{\mathscr{Q}^{S^r}_{p,q}(\mathbf{B})} := \inf\Big\{\|\Phi(\lambda_\infty)\|_{p,q} : (\lambda_n)_{n\geqslant 0} \in \Lambda[\mathscr{Q}^{S^r}_{p,q,\Phi}(f)](\mathbf{B})\Big\}$$

and

$$\|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})} := \inf\Big\{\|\Phi(\lambda_\infty)\|_{p,q} : (\lambda_n)_{n\geqslant 0} \in \Lambda[\mathscr{D}_{p,q,\Phi}(f)](\mathbf{B})\Big\}.$$

One should note that the inequalities of $\mathbf{B}$-valued martingales are closely related with the geometrical properties of Banach spaces. We now consider definitions of p-uniformly smooth, q-uniformly convex and Radon-Nikodým property (in short **RNP**) of Banach spaces.

DEFINITION 2. ([29]) Let $\mathbf{B}$ be a Banach space and $t > 0$. The modulus of uniform smoothness $\rho_{\mathbf{B}}(t)$ is defined as

$$\rho_{\mathbf{B}}(t) := \sup\left\{\frac{\|x+ty\|+\|x-ty\|}{2} - 1 :\ x, y \in \mathbf{B},\ \|x\| = \|y\| = 1\right\}.$$

We shall say that $\mathbf{B}$ is p-uniformly smooth if there is a constant $C > 0$ such that $\rho_{\mathbf{B}}(t) \leqslant Ct^p$ for all $t > 0$.

DEFINITION 3. ([29]) Let $\mathbf{B}$ be a Banach space and $0 < \theta \leqslant 2$. The modulus of uniform convexity $\delta_{\mathbf{B}}(\theta)$ is defined as

$$\delta_{\mathbf{B}}(\theta) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| :\ x, y \in \mathbf{B},\ \|x\| \leqslant 1,\ \|y\| \leqslant 1,\ \|x-y\| \geqslant \theta\right\}.$$

We shall say that $\mathbf{B}$ is q-uniformly convex if there is a constant $C > 0$ such that $\delta_{\mathbf{B}}(\theta) \geqslant C\theta^q$ for all $0 < \theta \leqslant 2$.

DEFINITION 4. ([29]) A Banach space $\mathbf{B}$ is said to have the **RNP** with respect to $(\Omega, \mathscr{F}, \mathbb{P})$ if for each $\mathbb{P}$-continuous vector measure $F : \mathscr{F} \to \mathbf{B}$ of bounded variation there exists $g \in L_1(\mathbf{B})$ such that $F(E) = \int_E g d\mathbb{P}$ for all $E \in \mathscr{F}$. A Banach space $\mathbf{B}$ has the **RNP** if $\mathbf{B}$ has the **RNP** with respect to every finite measure space.

REMARK 2. ([19]) If $\mathbf{B}$ is isomorphic to a p-uniformly smooth (q-uniformly convex) space, then $\mathbf{B}$ has the **RNP**.

We end this subsection by recording the following lemmas which will be frequently used in the sequel.

LEMMA 3. *Let $1 < r \leqslant 2$. Then the following properties of a Banach space* $\mathbf{B}$ *are equivalent:*

(i) $\mathbf{B}$ *is isomorphic to a r-uniformly smooth space;*

(ii) *For any* $\mathbf{B}$*-valued martingale* $f=(f_n)_{n\geqslant 0}$ *with* $\mathbb{E}\Big(\sum_{m=0}^{\infty}\|df_m\|^r\Big)<\infty$, $f=(f_n)_{n\geqslant 0}$ *converges in probability;*

(iii) *There is a positive constant* C *such that all* $\mathbf{B}$*-valued martingales* $f=(f_n)_{n\geqslant 0}$ *in* $L_p(\mathbf{B})$ *(*$1\leqslant p<\infty$*) satisfy*

$$\|M(f)\|_p\leqslant C\|S^r(f)\|_p;$$

(iv) *There is a positive constant* C *such that all* $\mathbf{B}$*-valued martingales* $f=(f_n)_{n\geqslant 0}$ *in* $L_r(\mathbf{B})$ *satisfy*

$$\sup_{n\geqslant 0}\mathbb{E}\big(\|f_n\|^r\big)\leqslant C^r\sum_{n=0}^{\infty}\mathbb{E}\big(\|df_n\|^r\big);$$

(v) *Same as* (iv) *for all* $\mathbf{B}$*-valued dyadic martingales.*

For Lemma 3, the proof of (i) $\Leftrightarrow$ (ii) comes from [19] and the proof of (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) were showed in [29].

LEMMA 4. *Let* $2\leqslant r<\infty$*. Then the following properties of a Banach space* $\mathbf{B}$ *are equivalent:*

(i) $\mathbf{B}$ *is isomorphic to a* r*-uniformly convex space;*

(ii) *There is a positive constant* C *such that all* $\mathbf{B}$*-valued martingales* $f=(f_n)_{n\geqslant 0}$ *in* $L_p(\mathbf{B})$ $(1\leqslant p<\infty)$ satisfy

$$\|S^r(f)\|_p\leqslant C\|M(f)\|_p;$$

(iii) *For every* $\mathbf{B}$*-valued martingale* $f=(f_n)_{n\geqslant 0}$ *with* $\sup_{n\geqslant 0}\|f_n\|_{L_\infty(\mathbf{B})}<\infty$, $S^r(f)<\infty$ *a.e.;*

(iv) *There is a positive constant* C *such that all* $\mathbf{B}$*-valued martingales* $f=(f_n)_{n\geqslant 0}$ *in* $L_r(\mathbf{B})$ *satisfy*

$$\sum_{n=0}^{\infty}\mathbb{E}\big(\|df_n\|^r\big)\leqslant C^r\sup_{n\geqslant 0}\mathbb{E}\big(\|f_n\|^r\big);$$

(v) *Same as (iv) for all* $\mathbf{B}$*-valued dyadic martingales.*

For Lemma 4, the proof of (i) $\Leftrightarrow$ (iii) was showed in [19] and (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) can be found in [29].

LEMMA 5. *Let* $\mathbf{B}$ *be a Banach space. Then the following properties of* $\mathbf{B}$ *are equivalent:*

(i) $\mathbf{B}$ *has the* **RNP***;*

(ii) *Fixing* $p\in(1,\infty)$, *every* $\mathbf{B}$*-valued martingale bounded in* $L_p(\mathbf{B})$ *converges almost surely* (*a.s.*) and in $L_p(\mathbf{B})$;

(iii) *If there exists a positive constant* C *such that* $\sup_{n\geqslant 0}\|f_n\|_{L_\infty(\mathbf{B})}\leqslant C$ *for any* $\mathbf{B}$*-valued martingale* $f=(f_n)_{n\geqslant 0}$, *then* f_n *converges a.e.*

For Lemma 5, we refer to [29] for the proof of (i) $\Leftrightarrow$ (ii); the proof of (i) $\Leftrightarrow$ (iii) can be seen in [19].

3. Atomic decompositions

In this section, we shall construct some new atomic decomposition theorems. Firstly, let us review the definitions for atoms.

DEFINITION 5. Let $\Phi \in \mathscr{G}$ be a concave function, $1 \leqslant r < \infty$, $0 < p < \infty$ and $0 < \kappa \leqslant \infty$. A $\mathbf{B}$-valued measurable function a is called a $(\Phi, p, \kappa)^{s^r}$-atom $\big($resp. $(\Phi, p, \kappa)^{S^r}$-atom, $(\Phi, p, \kappa)^{M}$-atom$\big)$, if there exists a stopping time $\upsilon \in \mathscr{V}$ such that

(1) $a_n = \mathbb{E}_n(a) = 0$, if $n \leqslant \upsilon$;

(2) $\|s^r(a)\|_\kappa$ $\big($resp. $\|S^r(a)\|_\kappa, \|M(a)\|_\kappa\big)$ $\leqslant \mathbb{P}(\upsilon < \infty)^{\frac{1}{\kappa}} \Phi^{-1}\Big(\mathbb{P}(\upsilon < \infty)^{-\frac{1}{p}}\Big)$.

For $0 < q \leqslant \infty$, let $\mathscr{A}^{s^r}(\Phi, p, q, \kappa)$ $\Big($resp. $\mathscr{A}^{S^r}(\Phi, p, q, \kappa)$, $\mathscr{A}^{M}(\Phi, p, q, \kappa)\Big)$ be the set of all sequences of triples (μ^k, a^k, υ^k), where

$$\mu^k = \frac{\Phi^{-1}(2^{k+1})}{\Phi^{-1}\Big(\mathbb{P}(\upsilon^k < \infty)^{-\frac{1}{p}}\Big)},$$

a^k is a $(\Phi, p, \kappa)^{s^r}$-atom $\Big($resp. $(\Phi, p, \kappa)^{S^r}$-atom, $(\Phi, p, \kappa)^{M}$-atom$\Big)$ and υ^k is the stopping time corresponding to a^k, satisfying

$$\Big\{\mathbb{P}(\upsilon^k < \infty)^{\frac{1}{p}} \Phi\Big(\mu^k \Phi^{-1}\Big(\mathbb{P}(\upsilon^k < \infty)^{-\frac{1}{p}}\Big)\Big)\Big\}_{k \in \mathbb{Z}} \in l_q.$$

Now we can present the atomic decompositions for $\mathbf{B}$-valued martingale Hardy-Lorenz spaces.

THEOREM 1. *Let $\mathbf{B}$ be a Banach space, $\Phi \in \mathscr{G}$ be a concave function, $1 < r \leqslant 2$, $0 < p \leqslant r$, $0 < q \leqslant \infty$ and $\max\{1, p\} < \kappa \leqslant \infty$. The following assertions are equivalent:*

(i) *$\mathbf{B}$ is isomorphic to a r-uniformly smooth space;*

(ii) *If the $\mathbf{B}$-valued martingale $f = (f_n)_{n \geqslant 0}$ satisfies $\|\Phi(s^r(f))\|_{p,q} < \infty$, then there exists a sequence of triples $(\mu^k, a^k, \upsilon^k) \in \mathscr{A}^{s^r}(\Phi, p, q, \kappa)$ such that for $n \geqslant 0$,*

$$f_n = \sum\nolimits_{k \in \mathbb{Z}} \mu^k \mathbb{E}_n(a^k) \quad a.e., \tag{4}$$

$$\sup\nolimits_{k \in \mathbb{Z}} \Big\|M(a^k)\Big\|_p < \infty \tag{5}$$

and

$$\|\Phi(s^r(f))\|_{p,q} \approx \inf \Big\|\Big\{\mathbb{P}(\upsilon^k < \infty)^{\frac{1}{p}} \Phi\Big(\mu^k \Phi^{-1}\Big(\mathbb{P}(\upsilon^k < \infty)^{-\frac{1}{p}}\Big)\Big)\Big\}_{k \in \mathbb{Z}}\Big\|_{l_q}, \tag{6}$$

where the infimum is taken over all the decompositions of the form (4).

Proof. (i) $\Rightarrow$ (ii). Given an arbitrary $\mathbf{B}$-valued martingale $f=(f_n)_{n\geqslant 0}$ such that $\|\Phi(s^r(f))\|_{p,q}<\infty$. For $k\in\mathbb{Z}$, the stopping time υ^k is defined as:

$$\upsilon^k := \inf\left\{n\in\mathbb{N} : s^r_{n+1}(f) > \Phi^{-1}(2^k)\right\}\ (\inf\emptyset=\infty).$$

Indeed, $\upsilon^k\leqslant\upsilon^{k+1}$. For $k\in\mathbb{Z}$ and $n\in\mathbb{N}$, set

$$\mu^k := \frac{\Phi^{-1}(2^{k+1})}{\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)} \quad\text{and}\quad a_n^k := \frac{f_n^{\upsilon^{k+1}}-f_n^{\upsilon^k}}{\mu^k} \quad (\text{if } \mu^k=0 \text{ then let } a_n^k=0).$$

Note that for any fixed $k\in\mathbb{Z}$,

$$da_n^k = \frac{df_n^{\upsilon^{k+1}}-df_n^{\upsilon^k}}{\mu^k} = \frac{df_n\chi_{\{\upsilon^k<n\leqslant\upsilon^{k+1}\}}}{\mu^k},\ \forall n\in\mathbb{N}.$$

Then

$$\mathbb{E}_{n-1}\left(da_n^k\right) = \frac{\mathbb{E}_{n-1}\left(df_n\right)\chi_{\{\upsilon^k<n\leqslant\upsilon^{k+1}\}}}{\mu^k} = 0.$$

Hence, $(a_n^k)_{n\geqslant 0}$ is a $\mathbf{B}$-valued martingale. From the definition of a_n^k, we have $s^r\left((a_n^k)_{n\geqslant 0}\right) = 0$ on $\{\upsilon^k=\infty\}$. Moreover, $s^r(f^{\upsilon^k}) = s^r_{\upsilon^k}(f)\leqslant\Phi^{-1}(2^k)$. Therefore,

$$\begin{aligned} s^r\left((a_n^k)_{n\geqslant 0}\right) &= \left(\sum_{m=0}^{\infty}\mathbb{E}_{m-1}\left(\|da_m^k\|^r\right)\right)^{\frac{1}{r}}\chi_{\{\upsilon^k<\infty\}} \\ &\leqslant \frac{s^r_{\upsilon^{k+1}}(f)}{\mu^k}\chi_{\{\upsilon^k<\infty\}} \\ &= \Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\chi_{\{\upsilon^k<\infty\}}. \end{aligned} \tag{7}$$

Then by Lemma 3 (iii), we get

$$\begin{aligned} \left\|M\left((a_n^k)_{n\geqslant 0}\right)\right\|_r &\leqslant C\left\|S^r\left((a_n^k)_{n\geqslant 0}\right)\right\|_r = C\left\|s^r\left((a_n^k)_{n\geqslant 0}\right)\right\|_r \\ &\leqslant C\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{r}}\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)<\infty. \end{aligned} \tag{8}$$

Thus, $(a_n^k)_{n\geqslant 0}$ is $L_r(\mathbf{B})$-bounded martingale. Since condition (i) implies $\mathbf{B}$ has the **RNP** (see Remark 2), then a_n^k converges *a.s.* to a limit a^k in $L_r(\mathbf{B})$ by Lemma 5. Therefore, $a_n^k=\mathbb{E}_n(a^k)$ (see [19, p. 27]). Combining this with (7), we obtain

$$\left\|s^r(a^k)\right\|_\kappa \leqslant \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{\kappa}}\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right). \tag{9}$$

For $n\leqslant\upsilon^k$,

$$\mathbb{E}_n(a^k) = a_n^k = \frac{f_n^{\upsilon^{k+1}}-f_n^{\upsilon^k}}{\mu^k} = \frac{f_n-f_n}{\mu^k} = 0. \tag{10}$$

According to (9) and (10), we conclude that a^k is a $(\Phi,p,\kappa)^{s^r}$-atom. Furthermore,

$$\sum_{k\in\mathbb{Z}}\mu^k\mathbb{E}_n(a^k)=\sum_{m=1}^{n}\Big(\sum_{k\in\mathbb{Z}}df_m\chi_{\{\upsilon^k<m\leqslant\upsilon^{k+1}\}}\Big)=f_n,$$

which implies that f has a decompositon of the form (4). Since $0<p\leqslant r$, by applying (8) and Hölder's inequality, then (5) holds. It is easy to check that

$$\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big)=2^{k+1} \tag{11}$$

and

$$\{\upsilon^k<\infty\}=\Big\{\Phi\big(s^r(f)\big)>2^k\Big\}.$$

Then we get the following inequality for $0<q<\infty$:

$$\begin{aligned}
&\Big\|\Big\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big)\Big\}_{k\in\mathbb{Z}}\Big\|_{l_q}\\
&=\Big\|\Big\{\mathbb{P}\big(\Phi\big(s^r(f)\big)>2^k\big)^{\frac{1}{p}}2^{k+1}\Big\}_{k\in\mathbb{Z}}\Big\|_{l_q}\\
&=\Big(\sum_{k\in\mathbb{Z}}\mathbb{P}\big(\Phi\big(s^r(f)\big)>2^k\big)^{\frac{q}{p}}2^{(k+1)q}\Big)^{\frac{1}{q}}\\
&\lesssim\left(\sum_{k\in\mathbb{Z}}p\int_{2^{k-1}}^{2^k}\mathbb{P}\big(\Phi\big(s^r(f)\big)>t\big)^{\frac{q}{p}}t^{q-1}dt\right)^{\frac{1}{q}}\\
&=\|\Phi\big(s^r(f)\big)\|_{p,q}.
\end{aligned}$$

This also shows that $(\mu^k,a^k,\upsilon^k)\in\mathscr{A}^{s^r}(\Phi,p,q,\kappa)$. Standard modifications can be made for $q=\infty$. Consequently,

$$\Big\|\Big\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big)\Big\}_{k\in\mathbb{Z}}\Big\|_{l_q}\lesssim\|\Phi\big(s^r(f)\big)\|_{p,q}. \tag{12}$$

On the other hand, it follows from the sublinearity of the conditional r-variation s^r and the subadditivity of Φ that

$$\Phi\big(s^r(f)\big)\leqslant\Phi\Big(\sum_{k\in\mathbb{Z}}\mu^ks^r(a^k)\Big)\leqslant\sum_{k\in\mathbb{Z}}\Phi\big(\mu^ks^r(a^k)\big).$$

For an arbitrary integer k_0, we define

$$\sum_{k\in\mathbb{Z}}\Phi\big(\mu^ks^r(a^k)\big)=\sum_{k=-\infty}^{k_0-1}\Phi\big(\mu^ks^r(a^k)\big)+\sum_{k=k_0}^{\infty}\Phi\big(\mu^ks^r(a^k)\big):=T_1+T_2.$$

Then

$$\Phi(s^r(f))\leqslant T_1+T_2$$

and

$$\|\chi_{\{\Phi(s^r(f))>2^{k_0+1}\}}\|_p\lesssim\|\chi_{\{T_1>2^{k_0}\}}\|_p+\|\chi_{\{T_2>2^{k_0}\}}\|_p. \tag{13}$$

Estimation for $\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\|\chi_{\{T_1>2^{k_0}\}}\|_p^q$.

Let $0<\theta<\min\{p,q,1\}$. Choose λ such that $1<\lambda<\min\{\frac{1}{\theta},\frac{\kappa}{p}\}$. By Chebyshev's inequality, Remark 1 (iii) and Lemma 2, we obtain

$$\begin{aligned}
\|\chi_{\{T_1>2^{k_0}\}}\|_p &\leqslant \frac{1}{2^{k_0\lambda}}\left\|\left[\sum_{k=-\infty}^{k_0-1}\Phi\left(\mu^k s^r(a^k)\right)\right]^{\lambda}\right\|_p \\
&= \frac{1}{2^{k_0\lambda}}\left\|\left[\sum_{k=-\infty}^{k_0-1}\Phi\left(\mu^k s^r(a^k)\right)\right]^{\lambda\theta}\right\|_{\frac{p}{\theta}}^{\frac{1}{\theta}} \\
&\leqslant \frac{1}{2^{k_0\lambda}}\left\|\sum_{k=-\infty}^{k_0-1}\Phi\left(\mu^k s^r(a^k)\right)^{\lambda\theta}\right\|_{\frac{p}{\theta}}^{\frac{1}{\theta}} \\
&\leqslant \frac{1}{2^{k_0\lambda}}\left\{\sum_{k=-\infty}^{k_0-1}\left\|\Phi\left(\mu^k s^r(a^k)\right)^{\lambda\theta}\right\|_{\frac{p}{\theta}}\right\}^{\frac{1}{\theta}} \\
&= \frac{1}{2^{k_0\lambda}}\left\{\sum_{k=-\infty}^{k_0-1}\left\|\Phi(\mu^k s^r(a^k))\right\|_{\lambda p}^{\lambda\theta}\right\}^{\frac{1}{\theta}} \\
&\lesssim \frac{1}{2^{k_0\lambda}}\left\{\sum_{k=-\infty}^{k_0-1}\mathbb{P}(\upsilon^k<\infty)^{\frac{\theta}{p}}\Phi\left(\frac{\mu^k\|s^r(a^k)\|_\kappa}{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{\kappa}}}\right)^{\lambda\theta}\right\}^{\frac{1}{\theta}}.
\end{aligned}$$

By the definition of $(\Phi,p,\kappa)^{s^r}$-atom, it is easy to see that

$$\Phi\left(\frac{\mu^k\|s^r(a^k)\|_\kappa}{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{\kappa}}}\right)\leqslant 2^{k+1}. \tag{14}$$

Next we divide the proof into two cases according to the value of q.

Case 1*:* $0<q<\infty$. Set $1<\eta<\lambda$, then one can further deduce that

$$\begin{aligned}
&\|\chi_{\{T_1>2^{k_0}\}}\|_p \qquad (15)\\
&\lesssim \frac{1}{2^{k_0\lambda}}\left\{\sum_{k=-\infty}^{k_0-1}\mathbb{P}(\upsilon^k<\infty)^{\frac{\theta}{p}}2^{k\lambda\theta}\right\}^{\frac{1}{\theta}} \\
&= \frac{1}{2^{k_0\lambda}}\left\{\sum_{k=-\infty}^{k_0-1}\mathbb{P}(\upsilon^k<\infty)^{\frac{\theta}{p}}2^{k\eta\theta}2^{k(\lambda-\eta)\theta}\right\}^{\frac{1}{\theta}} \\
&\leqslant \frac{1}{2^{k_0\lambda}}\left\{\sum_{k=-\infty}^{k_0-1}\mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}}2^{k\eta q}\right\}^{\frac{1}{q}}\left\{\sum_{k=-\infty}^{k_0-1}2^{k(\lambda-\eta)\theta\frac{q}{q-\theta}}\right\}^{\frac{q-\theta}{q\theta}} \\
&= \frac{2^{\eta-\lambda}}{2^{k_0\eta}\left(1-2^{\frac{q\theta(\eta-\lambda)}{q-\theta}}\right)^{\frac{q-\theta}{q\theta}}}\left\{\sum_{k=-\infty}^{k_0-1}\mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}}2^{k\eta q}\right\}^{\frac{1}{q}},
\end{aligned}$$

where the first "$\leqslant$" is due to Hölder's inequality and $\frac{\theta}{q}+\frac{q-\theta}{q}=1$. By using (15), the Abel transformation and (11), we get

$$\begin{aligned}
&\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\|\chi_{\{T_1>2^{k_0}\}}\|_p^q \qquad (16)\\
&\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q(1-\eta)} \sum_{k=-\infty}^{k_0-1} \mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}} 2^{k\eta q}\\
&= \sum_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}} 2^{k\eta q} \sum_{k_0=k+1}^{\infty} 2^{k_0 q(1-\eta)}\\
&= \frac{1}{2^{q\eta}-2^q} \sum_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}} 2^{(k+1)q}\\
&= \frac{1}{2^{q\eta}-2^q} \sum_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}} \Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big)^q.
\end{aligned}$$

Estimation for $\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\|\chi_{\{T_2>2^{k_0}\}}\|_p^q$.

For the above symbol θ, let $0<\beta<1$. It follows from Chebyshev's inequality, Remark 1 (iii) and Lemma 2 that

$$\begin{aligned}
\|\chi_{\{T_2>2^{k_0}\}}\|_p &\leqslant \frac{1}{2^{k_0\beta}} \left\| \Big[\sum_{k=k_0}^{\infty} \Phi\Big(\mu^k s^r(a^k)\Big)\Big]^{\beta} \right\|_p\\
&= \frac{1}{2^{k_0\beta}} \left\| \Big[\sum_{k=k_0}^{\infty} \Phi\Big(\mu^k s^r(a^k)\Big)\Big]^{\beta\theta} \right\|_{\frac{p}{\theta}}^{\frac{1}{\theta}}\\
&\leqslant \frac{1}{2^{k_0\beta}} \left\| \sum_{k=k_0}^{\infty} \Phi\Big(\mu^k s^r(a^k)\Big)^{\beta\theta} \right\|_{\frac{p}{\theta}}^{\frac{1}{\theta}}\\
&\leqslant \frac{1}{2^{k_0\beta}} \left\{ \sum_{k=k_0}^{\infty} \left\| \Phi\Big(\mu^k s^r(a^k)\Big)^{\beta\theta} \right\|_{\frac{p}{\theta}} \right\}^{\frac{1}{\theta}}\\
&= \frac{1}{2^{k_0\beta}} \left\{ \sum_{k=k_0}^{\infty} \left\| \Phi\Big(\mu^k s^r(a^k)\Big) \right\|_{\beta p}^{\beta\theta} \right\}^{\frac{1}{\theta}}\\
&\lesssim \frac{1}{2^{k_0\beta}} \left\{ \sum_{k=k_0}^{\infty} \mathbb{P}(\upsilon^k<\infty)^{\frac{\theta}{p}} \Phi\left(\frac{\mu^k\|s^r(a^k)\|_\kappa}{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{\kappa}}}\right)^{\beta\theta} \right\}^{\frac{1}{\theta}}.
\end{aligned}$$

Choose ξ such that $\beta<\xi<1$. Taking the same argument as in (14), we have

$$\begin{aligned}
\|\chi_{\{T_2>2^{k_0}\}}\|_p &\lesssim \frac{1}{2^{k_0\beta}} \Big\{\sum_{k=k_0}^{\infty} \mathbb{P}(\upsilon^k<\infty)^{\frac{\theta}{p}} 2^{k\beta\theta}\Big\}^{\frac{1}{\theta}} \qquad (17)\\
&= \frac{1}{2^{k_0\beta}} \Big\{\sum_{k=k_0}^{\infty} \mathbb{P}(\upsilon^k<\infty)^{\frac{\theta}{p}} 2^{k\xi\theta} 2^{k(\beta-\xi)\theta}\Big\}^{\frac{1}{\theta}}\\
&\leqslant \frac{1}{2^{k_0\beta}} \Big\{\sum_{k=k_0}^{\infty} \mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}} 2^{k\xi q}\Big\}^{\frac{1}{q}} \Big\{\sum_{k=k_0}^{\infty} 2^{k(\beta-\xi)\theta\frac{q}{q-\theta}}\Big\}^{\frac{q-\theta}{q\theta}}\\
&= \frac{1}{2^{k_0\xi}\Big(1-2^{\frac{q\theta(\beta-\xi)}{q-\theta}}\Big)^{\frac{q-\theta}{q\theta}}} \Big\{\sum_{k=k_0}^{\infty} \mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}} 2^{k\xi q}\Big\}^{\frac{1}{q}},
\end{aligned}$$

which implies

$$
\begin{aligned}
&\sum_{k_0\in\mathbb{Z}} 2^{k_0 q}\|\chi_{\{T_2>2^{k_0}\}}\|_p^q \\
&\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q(1-\xi)} \sum_{k=k_0}^{\infty} \mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}} 2^{k\xi q} \\
&= \sum_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}} 2^{k\xi q} \sum_{k_0=-\infty}^{k} 2^{k_0 q(1-\xi)} \\
&= \frac{1}{2^q-2^{q\xi}} \sum_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}} 2^{(k+1)q} \\
&= \frac{1}{2^q-2^{q\xi}} \sum_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}} \Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big)^q.
\end{aligned} \tag{18}
$$

As a consequence of (13), (16) and (18), we get

$$
\begin{aligned}
&\|\Phi(s^r(f))\|_{p,q}^q \\
&\approx \sum_{k_0\in\mathbb{Z}} 2^{(k_0+1)q} \Big\|\chi_{\{\Phi(s^r(f))>2^{k_0+1}\}}\Big\|_p^q \\
&\lesssim \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \Big\|\chi_{\{T_1>2^{k_0}\}}\Big\|_p^q + \sum_{k_0\in\mathbb{Z}} 2^{k_0 q} \Big\|\chi_{\{T_2>2^{k_0}\}}\Big\|_p^q \\
&\lesssim \Big\|\Big\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}} \Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big)\Big\}_{k\in\mathbb{Z}}\Big\|_{l_q}^q.
\end{aligned} \tag{19}
$$

Case 2*:* $q=\infty$. Firstly, by (15), we find that

$$
\begin{aligned}
\|\chi_{\{T_1>2^{k_0}\}}\|_p &\lesssim \frac{1}{2^{k_0\lambda}} \Big\{\sum_{k=-\infty}^{k_0-1} \mathbb{P}(\upsilon^k<\infty)^{\frac{\theta}{p}} 2^{k\lambda\theta}\Big\}^{\frac{1}{\theta}} \\
&\leqslant \frac{1}{2^{k_0\lambda}} \sup_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}} 2^k \Big\{\sum_{k=-\infty}^{k_0-1} 2^{k\theta(\lambda-1)}\Big\}^{\frac{1}{\theta}} \\
&\lesssim \frac{1}{2^{k_0}} \sup_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}} \Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big).
\end{aligned}
$$

Thus we get that

$$
\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_1>2^{k_0}\}}\|_p \lesssim \sup_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}} \Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big).
$$

Secondly, by using (17), we have

$$
\begin{aligned}
\|\chi_{\{T_2>2^{k_0}\}}\|_p &\lesssim \frac{1}{2^{k_0\beta}} \Big\{\sum_{k=k_0}^{\infty} \mathbb{P}(\upsilon^k<\infty)^{\frac{\theta}{p}} 2^{k\beta\theta}\Big\}^{\frac{1}{\theta}} \\
&\leqslant \frac{1}{2^{k_0\beta}} \sup_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}} 2^k \Big\{\sum_{k=k_0}^{\infty} 2^{k\theta(\beta-1)}\Big\}^{\frac{1}{\theta}} \\
&\lesssim \frac{1}{2^{k_0}} \sup_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}} \Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big),
\end{aligned}
$$

which means

$$
\sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_2>2^{k_0}\}}\|_p \lesssim \sup_{k\in\mathbb{Z}} \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}} \Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big).
$$

Consequently, we have

$$\begin{aligned}\|\Phi(s^r(f))\|_{p,\infty} &\approx \sup_{k_0\in\mathbb{Z}} 2^{k_0+1}\|\chi_{\{\Phi(s^r(f))>2^{k_0+1}\}}\|_p \\ &\lesssim \sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_1>2^{k_0}\}}\|_p + \sup_{k_0\in\mathbb{Z}} 2^{k_0}\|\chi_{\{T_2>2^{k_0}\}}\|_p \\ &\lesssim \left\|\left\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_\infty}.\end{aligned} \tag{20}$$

Combining with (19) and (20), we obtain that

$$\|\Phi(s^r(f))\|_{p,q} \lesssim \left\|\left\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_q}. \tag{21}$$

Taking the infimum over all decompositions of the form (4), we get (6).

(ii) $\Rightarrow$ (i). Let $f=(f_n)_{n\geqslant 0}$ be a $\mathbf{B}$-valued martingale and satisfy

$$\mathbb{E}\left(\sum_{m=0}^{\infty}\|df_m\|^r\right)<\infty.$$

For $1<r\leqslant 2$, we have

$$\|s^r(f)\|_1 \leqslant \|s^r(f)\|_r = \left(\mathbb{E}\left(\sum_{m=0}^{\infty}\|df_m\|^r\right)\right)^{\frac{1}{r}}<\infty.$$

Choose $\Phi(t)=t$, then $\|\Phi(s^r(f))\|_1=\|s^r(f)\|_1<\infty$ and $\Phi^{-1}(t)=t$. We know $f=(f_n)_{n\geqslant 0}$ has a decomposition as (4). Therefore,

$$\left\|\left\{\mathbb{P}(\upsilon^k<\infty)\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-1}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_1} = \left\|\{\mu^k\}_{k\in\mathbb{Z}}\right\|_{l_1}<\infty$$

and

$$\sup_{k\in\mathbb{Z}}\|M(a^k)\|_1<\infty.$$

Furthermore, for any $\varepsilon>0$, there exists a $k_0\in\mathbb{Z}$ such that

$$\sum_{|k|>k_0}\mu^k<\varepsilon.$$

Notice that $a_n^k=\mathbb{E}_n(a^k)$ converges to the function a^k as $n\to\infty$ in $L_1(\mathbf{B})$ for each $k\in\mathbb{Z}$ (see [19, p. 27]). Thus, there exists $M_k\in\mathbb{N}$ such that

$$\mathbb{E}\left(\left\|a_m^k-a_n^k\right\|\right)<\varepsilon$$

when $m,n>M_k$. Set $\mathbf{N}=\max_{|k|\leqslant k_0}\{M_k\}$. Then for $m,n>\mathbf{N}$, we can state that

$$\begin{aligned}\|f_m-f_n\|_{L_1(\mathbf{B})} &= \mathbb{E}\left(\left\|\sum_{k\in\mathbb{Z}}\mu^k a_m^k-\sum_{k\in\mathbb{Z}}\mu^k a_n^k\right\|\right) \\ &\leqslant \sum_{k\in\mathbb{Z}}\mu^k\mathbb{E}(\|a_m^k-a_n^k\|)\end{aligned}$$

$$= \sum_{|k|>k_0} \mu^k \mathbb{E}(\|a_m^k - a_n^k\|) + \sum_{|k|\leqslant k_0} \mu^k \mathbb{E}(\|a_m^k - a_n^k\|)$$
$$\leqslant 2\sup_{k\in\mathbb{Z}} \|M(a^k)\|_1 \sum_{|k|>k_0} \mu^k + \varepsilon \sum_{|k|\leqslant k_0} \mu^k \lesssim \varepsilon.$$

This implies that $(f_n)_{n\geqslant 0}$ is a Cauchy sequence in $L_1(\mathbf{B})$ and thus converges in $L_1(\mathbf{B})$. Hence, $(f_n)_{n\geqslant 0}$ converges in probability (see [19, p. 14]). Therefore, by Lemma 3, $\mathbf{B}$ is isomorphic to a r-uniformly smooth space. The proof is completed. □

It is observed that the proof of (21) in Theorem 1 is mainly related to the subadditivity of Φ and the sublinearity of s^r. Namely, for $\|\Phi(S^r(f))\|_{p,q}$ $\big($resp. $\|\Phi(M(f))\|_{p,q}$, $\|\Phi(s^r(f))\|_{p,q}\big)$ we have:

COROLLARY 1. *Let $\mathbf{B}$ be a Banach space and $\Phi \in \mathscr{G}$ be a concave function. If $1 \leqslant r < \infty$, $0 < p < \infty$, $0 < q \leqslant \infty$, $\max\{1,p\} < \kappa \leqslant \infty$ and the $\mathbf{B}$-valued martingale $f = (f_n)_{n\geqslant 0}$ has a decomposition of type (4) with $(\mu^k, a^k, \upsilon^k) \in \mathscr{A}^{S^r}(\Phi,p,q,\kappa)$ $\big($resp. $(\mu^k,a^k,\upsilon^k) \in \mathscr{A}^{M}(\Phi,p,q,\kappa), (\mu^k,a^k,\upsilon^k) \in \mathscr{A}^{s^r}(\Phi,p,q,\kappa)\big)$, then*

$$\|\Phi(S^r(f))\|_{p,q} \lesssim \inf \left\| \left\{ \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}} \Phi\Big(\mu^k \Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big) \right\}_{k\in\mathbb{Z}} \right\|_{l_q}$$

$$\Bigg(\text{resp. } \|\Phi(M(f))\|_{p,q} \lesssim \inf \left\| \left\{ \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}} \Phi\Big(\mu^k \Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big) \right\}_{k\in\mathbb{Z}} \right\|_{l_q},$$

$$\|\Phi(s^r(f))\|_{p,q} \lesssim \inf \left\| \left\{ \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}} \Phi\Big(\mu^k \Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big) \right\}_{k\in\mathbb{Z}} \right\|_{l_q} \Bigg),$$

where the infimum is taken over all the decompositions of the form (4).

Next we will establish the atomic decompositions for $\Phi(f)$ in $\mathscr{Q}^{S^r}_{p,q}(\mathbf{B})$ and $\mathscr{D}_{p,q}(\mathbf{B})$.

THEOREM 2. *Let $\mathbf{B}$ be a Banach space, $\Phi \in \mathscr{G}$ be a concave function, $1 < r \leqslant 2$, $0 < p \leqslant r$ and $0 < q \leqslant \infty$. The following assertions are equivalent:*

(i) *$\mathbf{B}$ is isomorphic to a r-uniformly smooth space;*

(ii) *If the $\mathbf{B}$-valued martingale $f = (f_n)_{n\geqslant 0}$ satisfies $\|\Phi(f)\|_{\mathscr{Q}^{S^r}_{p,q}(\mathbf{B})} < \infty$, then there exists a sequence of triples $(\mu^k,a^k,\upsilon^k) \in \mathscr{A}^{S^r}(\Phi,p,q,\infty)$ such that for $n \geqslant 0$, (4), (5) hold and*

$$\|\Phi(f)\|_{\mathscr{Q}^{S^r}_{p,q}(\mathbf{B})} \approx \inf \left\| \left\{ \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}} \Phi\Big(\mu^k \Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big) \right\}_{k\in\mathbb{Z}} \right\|_{l_q}, \qquad (22)$$

where the infimum is taken over all the decompositions of the form (4).

Proof. (i) ⇒ (ii). The proof is similar to the one of Theorem 1, so we omit some details. Let $f = (f_n)_{n\geqslant 0}$ be a $\mathbf{B}$-valued martingale with $\|\Phi(f)\|_{\mathscr{Q}^{S^r}_{p,q}(\mathbf{B})} < \infty$. Fix $(\lambda_n)_{n\geqslant 0} \in \Lambda[\mathscr{Q}^{S^r}_{p,q,\Phi}(f)](\mathbf{B})$. For every $k \in \mathbb{Z}$, we define stopping time

$$\upsilon^k := \inf\{n \in \mathbb{N} : \lambda_n > \Phi^{-1}(2^k)\} \ (\inf\emptyset = \infty).$$

Let μ^k and a_n^k be defined as in the proof of Theorem 1. Then $(a_n^k)_{n\geqslant 0}$ is a $\mathbf{B}$-valued martingale with $S_{\upsilon^k}^r(f) \leqslant \lambda_{\upsilon^k-1} \leqslant \Phi^{-1}(2^k)$. Analogously to the proof of (7), we get

$$S^r\Big((a_n^k)_{n\geqslant 0}\Big) \leqslant \chi_{\{\upsilon^k<\infty\}}\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big).$$

Hence, by Lemma 3 (iii), we obtain

$$\Big\|M\Big((a_n^k)_{n\geqslant 0}\Big)\Big\|_r \leqslant C\Big\|S^r\Big((a_n^k)_{n\geqslant 0}\Big)\Big\|_r \leqslant C\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{r}}\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big) < \infty.$$

A similar verification of Theorem 1 shows that there exists a function a^k in $L_r(\mathbf{B})$ such that $a_n^k = \mathbb{E}_n(a^k)$ $(n\in\mathbb{N})$. Furthermore, a^k is a $(\Phi,p,\infty)^{S^r}$-atom with $\sup_{k\in\mathbb{Z}}\|M(a^k)\|_p < \infty$ and (4) holds. For the case of $0<q<\infty$, by using the facts that $\{\upsilon^k<\infty\} = \{\Phi(\lambda_\infty)>2^k\}$ and (11), we have

$$\begin{aligned}
&\sum\nolimits_{k\in\mathbb{Z}}\mathbb{P}(\upsilon^k<\infty)^{\frac{q}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big)^q\\
&=\sum\nolimits_{k\in\mathbb{Z}}\mathbb{P}\big(\Phi(\lambda_\infty)>2^k\big)^{\frac{q}{p}}2^{(k+1)q}\\
&\lesssim\sum\nolimits_{k\in\mathbb{Z}}p\int_{2^{k-1}}^{2^k}\mathbb{P}\big(\Phi(\lambda_\infty)>t\big)^{\frac{q}{p}}t^{q-1}dt\\
&=\|\Phi(\lambda_\infty)\|_{p,q}^q.
\end{aligned}$$

The case $q=\infty$ is obvious. Taking the infimum over all $(\lambda_n)_{n\geqslant 0}\in\Lambda[\mathscr{Q}_{p,q,\Phi}^{S^r}(f)](\mathbf{B})$, we obtain that

$$\Big\|\Big\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\Big(\mu^k\Phi^{-1}\Big(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\Big)\Big)\Big\}_{k\in\mathbb{Z}}\Big\|_{l_q} \lesssim \|\Phi(f)\|_{\mathscr{Q}_{p,q}^{S^r}(\mathbf{B})}.$$

On the other hand, it follows from the definition of $(\Phi,p,\infty)^{S^r}$-atom that

$$\{S^r(a^k)>0\}\subset\{\upsilon^k<\infty\}.$$

Additionally, for all $n\in\mathbb{N}$, let

$$\rho_n := \sum\nolimits_{k\in\mathbb{Z}}\mu^k\|S^r(a^k)\|_\infty\chi_{\{\upsilon^k\leqslant n\}}.$$

Then $(\rho_n)_{n\geqslant 0}\in\Lambda$ and $S_{n+1}^r(f)\leqslant\rho_n$. Fix an integer k_0 and set

$$\rho_\infty^{(1)} := \sum\nolimits_{k=-\infty}^{k_0-1}\Phi\big(\mu^k\|S^r(a^k)\|_\infty\chi_{\{\upsilon^k<\infty\}}\big),\ \rho_\infty^{(2)} := \sum\nolimits_{k=k_0}^{\infty}\Phi\big(\mu^k\|S^r(a^k)\|_\infty\chi_{\{\upsilon^k<\infty\}}\big).$$

Then

$$\Phi(\rho_\infty)\leqslant\rho_\infty^{(1)}+\rho_\infty^{(2)}$$

and

$$\Big\|\chi_{\{\Phi(\rho_\infty)>2^{k_0+1}\}}\Big\|_p \lesssim \Big\|\chi_{\{\rho_\infty^{(1)}>2^{k_0}\}}\Big\|_p + \Big\|\chi_{\{\rho_\infty^{(2)}>2^{k_0}\}}\Big\|_p.$$

Replacing T_1 and T_2 by $\rho_\infty^{(1)}$ and $\rho_\infty^{(2)}$ in Theorem 1, respectively. Then we have

$$\|\Phi(f)\|_{\mathscr{Q}^{Sr}_{p,q}(\mathbf{B})} \approx \inf\left\|\left\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_q},$$

where the infimum is taken over all decompositions of f of the form (4).

(ii) $\Rightarrow$ (i). Suppose that $f=(f_n)_{n\geqslant 0}$ is a $\mathbf{B}$-valued martingale with $S^r(f)\in L_\infty$. Let $\Phi(t)=t$ and $\lambda_n=\|S^r_{n+1}(f)\|_\infty$ for $n\in\mathbb{N}$. Clearly, $(\lambda_n)_{n\geqslant 0}\in\Lambda$ and $S^r_{n+1}(f)\leqslant\lambda_n$. Consequently,

$$\|\Phi(f)\|_{\mathscr{Q}^{Sr}_1(\mathbf{B})}\leqslant\|\Phi(\lambda_\infty)\|_1=\|S^r(f)\|_\infty<\infty.$$

Therefore, $(f_n)_{n\geqslant 0}$ has a decomposition as (4). The rest of the proof is similar to the one in Theorem 1. $\square$

THEOREM 3. *Let $\mathbf{B}$ be a Banach space, $\Phi\in\mathscr{G}$ be a concave function, $0<p<\infty$ and $0<q\leqslant\infty$. The following assertions are equivalent:*

(i) *$\mathbf{B}$ has the* **RNP***;*

(ii) *If the $\mathbf{B}$-valued martingale $f=(f_n)_{n\geqslant 0}$ satisfies $\|\Phi(f)\|_{\mathscr{D}_{p,q}}(\mathbf{B})<\infty$, then there exists a sequence of triples $(\mu^k,a^k,\upsilon^k)\in\mathscr{A}^M(\Phi,p,q,\infty)$ such that for $n\geqslant 0$, (4), (5) hold and*

$$\|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})} \approx \inf\left\|\left\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_q}, \tag{23}$$

where the infimum is taken over all the decompositions of the form (4).

Proof. (i) $\Rightarrow$ (ii). The proof follows the ideas in Theorem 2, so we only outline the major steps. Suppose that $f=(f_n)_{n\geqslant 0}$ is a $\mathbf{B}$-valued martingale with $\|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})}<\infty$. For every $k\in\mathbb{Z}$, define stopping time υ^k as follows

$$\upsilon^k:=\inf\left\{n\in\mathbb{N}:\lambda_n>\Phi^{-1}(2^k)\right\}\ (\inf\emptyset=\infty),$$

where $(\lambda_n)_{n\geqslant 0}\in\Lambda[\mathscr{D}_{p,q,\Phi}(f)](\mathbf{B})$. Define μ^k and a^k_n as Theorem 1. Thus it is sufficient to prove that

$$\begin{aligned}
\|a^k_n\| &= \frac{\|f_n^{\upsilon^{k+1}}-f_n^{\upsilon^k}\|}{\mu^k}\\
&\leqslant \frac{\|f_n^{\upsilon^{k+1}}\|+\|f_n^{\upsilon^k}\|}{\mu^k}\chi_{\{\upsilon^k<\infty\}}\\
&\leqslant \frac{\lambda_{\upsilon^{k+1}-1}+\lambda_{\upsilon^k-1}}{\mu^k}\chi_{\{\upsilon^k<\infty\}}\\
&\leqslant \frac{\Phi^{-1}(2^{k+1})+\Phi^{-1}(2^k)}{\mu^k}\chi_{\{\upsilon^k<\infty\}}\\
&\leqslant 2\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\chi_{\{\upsilon^k<\infty\}}.
\end{aligned}$$

Therefore,

$$\left\|M\left((a_n^k)_{n\geqslant 0}\right)\right\|_\infty \lesssim \Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right).$$

This shows that there exists a $\mathbf{B}$-valued integrable function a^k such that $a_n^k=\mathbb{E}_n(a^k)$ $(n\in\mathbb{N})$. Then a^k is a $(\Phi,p,\infty)^M$-atom and (4) holds. Referring to the proof of Theorem 2, we can easily get (5) and

$$\left\|\left\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_q} \lesssim \|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})}.$$

On the other hand, set

$$\rho_n := \sum\nolimits_{k\in\mathbb{Z}}\mu^k\|M(a^k)\|_\infty\chi_{\{\upsilon^k\leqslant n\}}.$$

Then $(\rho_n)_{n\geqslant 0}\in\Lambda$ and $\|f_{n+1}\|\leqslant\rho_n$. For a fixed integer k_0, let

$$\rho_\infty^{(1)} := \sum\nolimits_{k=-\infty}^{k_0-1}\Phi\big(\mu^k\|M(a^k)\|_\infty\chi_{\{\upsilon^k<\infty\}}\big),$$

$$\rho_\infty^{(2)} := \sum\nolimits_{k=k_0}^{\infty}\Phi\big(\mu^k\|M(a^k)\|_\infty\chi_{\{\upsilon^k<\infty\}}\big).$$

Consequently,

$$\Phi(\rho_\infty)\leqslant\rho_\infty^{(1)}+\rho_\infty^{(2)}$$

and

$$\left\|\chi_{\left\{\Phi(\rho_\infty)>2^{k_0+1}\right\}}\right\|_p \lesssim \left\|\chi_{\left\{\rho_\infty^{(1)}>2^{k_0}\right\}}\right\|_p+\left\|\chi_{\left\{\rho_\infty^{(2)}>2^{k_0}\right\}}\right\|_p.$$

If we replace T_1 and T_2 by $\rho_\infty^{(1)}$ and $\rho_\infty^{(2)}$ in Theorem 1, respectively, then we obtain

$$\|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})}\approx\inf\left\|\left\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_q},$$

where the infimum is taken over all decompositions of f of the form (4).

(ii) $\Rightarrow$ (i). Choose $\mathbf{B}$-valued martingale $f=(f_n)_{n\geqslant 0}$ such that $\sup_{n\geqslant 0}\|f_n\|_{L_\infty(\mathbf{B})}<\infty$. Let $\Phi(t)=t$ and $\lambda_n=\|M_{n+1}(f)\|_\infty$ for all $n\in\mathbb{N}$. Obviously, $(\lambda_n)_{n\geqslant 0}\in\Lambda$ and $\|f_{n+1}\|\leqslant\lambda_n$. Hence,

$$\|\Phi(f)\|_{\mathscr{D}_1(\mathbf{B})}\leqslant\|\Phi(\lambda_\infty)\|_1\leqslant\sup\nolimits_{n\geqslant 0}\|f_n\|_{L_\infty(\mathbf{B})}<\infty.$$

It is similar to that of (ii) $\Rightarrow$ (i) in Theorem 1, we can prove that $(f_n)_{n\geqslant 0}$ converges in $L_1(\mathbf{B})$. More precisely, $(f_n)_{n\geqslant 0}$ converges *a.e.* According to Lemma 5, we know that $\mathbf{B}$ has the **RNP**. □

REMARK 3. If $\Phi(t)=t$ and $\kappa=\infty$, then Theorem 1 reduces to the corresponding result in [18]; If $\Phi(t)=t$, then Theorems 2 and 3 recover the corresponding results in [18].

REMARK 4. If we consider the special case $r=2$, $\Phi(t)=t$ and $\mathbf{B}=\mathbb{R}$ in Theorems 1, 2 and 3, we get the atomic decomposition of Hardy-Lorentz martingale spaces $H_{p,q}^s$, $\mathscr{Q}_{p,q}$ and $\mathscr{D}_{p,q}$, respectively.

4. Φ-moment B-valued martingale inequalities

In what follows, with the help of atomic decomposition theorems achieved above, we deduce some fundamental Φ-moment $\mathbf{B}$-valued martingale inequalities on Lorentz spaces. Our conclusions strongly depend on the smoothness or convexity of Banach spaces.

THEOREM 4. *Let* $\mathbf{B}$ *be a Banach space,* $\Phi\in\mathscr{G}$ *be a concave function,* $1<r\leqslant 2$, $0<p<r$ *and* $0<q\leqslant\infty$. *The following assertions are equivalent:*

(i) $\mathbf{B}$ *is isomorphic to a* r*-uniformly smooth space;*

(ii) *If the* $\mathbf{B}$*-valued martingale* $f=(f_n)_{n\geqslant 0}$ *satisfies* $\|\Phi(s^r(f))\|_{p,q}<\infty$, *then*

$$\|\Phi(M(f))\|_{p,q}\lesssim\|\Phi(s^r(f))\|_{p,q};\tag{24}$$

(iii) *If the* $\mathbf{B}$*-valued martingale* $f=(f_n)_{n\geqslant 0}$ *satisfies* $\|\Phi(f)\|_{\mathscr{Q}^{s^r}_{p,q}(\mathbf{B})}<\infty$, *then*

$$\|\Phi(M(f))\|_{p,q}\lesssim\|\Phi(f)\|_{\mathscr{Q}^{s^r}_{p,q}(\mathbf{B})}.\tag{25}$$

Proof. (i) $\Rightarrow$ (ii). Suppose that $f=(f_n)_{n\geqslant 0}$ is a $\mathbf{B}$-valued martingale satisfies $\|\Phi(s^r(f))\|_{p,q}<\infty$. By Theorem 1, there exists a sequence of triples $(\mu^k,a^k,\upsilon^k)\in\mathscr{A}^{s^r}(\Phi,p,q,r)$ such that

$$\frac{1}{C}f_n=\sum\nolimits_{k\in\mathbb{Z}}\mu^k\mathbb{E}_n\left(\frac{1}{C}a^k\right)\quad a.e.$$

and

$$\|\Phi(s^r(f))\|_{p,q}\approx\left\|\left\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_q}.$$

Recall that, by Lemma 3 (iii), for any $\mathbf{B}$-valued martingale g,

$$\|M(g)\|_r\leqslant C\|S^r(g)\|_r=C\|s^r(g)\|_r,$$

where $C>1$. Obviously, $a^k=(a^k_n)_{n\geqslant 0}$ is a $\mathbf{B}$-valued martingale. Hence,

$$\left\|M\left(\frac{1}{C}a^k\right)\right\|_r\leqslant\|s^r(a^k)\|_r\leqslant\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{r}}\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right).$$

Therefore, it is clear that $\left(\mu^k,\frac{1}{C}a^k,\upsilon^k\right)\in\mathscr{A}^M(\Phi,p,q,r)$. Additionally, by Corollary 1, we find that

$$\left\|\Phi\left(M\left(\frac{1}{C}f\right)\right)\right\|_{p,q}\lesssim\left\|\left\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_q}.$$

Further on, we conclude by Lemma 1 (i) that

$$\begin{aligned}\frac{1}{C}\|\Phi(M(f))\|_{p,q} &\leqslant \left\|\Phi\left(M\left(\frac{1}{C}f\right)\right)\right\|_{p,q}\\ &\lesssim \left\|\left\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_q}\\ &\approx \|\Phi(s^r(f))\|_{p,q}.\end{aligned}$$

Thus, $\|\Phi(M(f))\|_{p,q}\lesssim\|\Phi(s^r(f))\|_{p,q}$.

(ii) $\Rightarrow$ (i). Assume that $f=(f_n)_{n\geqslant 0}$ is an arbitrary $\mathbf{B}$-valued martingale with

$$\mathbb{E}\left(\sum_{m=0}^{\infty}\|df_m\|^r\right)=\|S^r(f)\|_r^r<\infty.$$

Let $\Phi(t)=t$. For $n\in\mathbb{N}$, define $\mathbf{B}$-valued martingale $g^n=(g^n_m)_{m\geqslant 0}$ by $g^n_m:=f_{m+n}-f_n$. Actually, $[s^r(g^n)]^r=[s^r(f)]^r-[s^r_{n-1}(f)]^r\to 0$ as $n\to\infty$ and $s^r(g^n)\leqslant s^r(f)$. By the Lebesgue dominated convergence theorem, we have $\|s^r(g^n)\|_r\to 0$ as $n\to\infty$. Applying (24) to g^n, we have

$$\|f_{m+n}-f_n\|_{L_1(\mathbf{B})}\leqslant\|\Phi(M(g^n))\|_1\lesssim\|\Phi(s^r(g^n))\|_1\to 0,\ (n\to\infty).$$

Now we claim that $(f_n)_{n\geqslant 0}$ is a Cauchy sequence in $L_1(\mathbf{B})$. Then $(f_n)_{n\geqslant 0}$ converges in probability (see [19, p. 14]). Using Lemma 3, we obtain that $\mathbf{B}$ is isomorphic to a r-uniformly smooth space.

(i) $\Rightarrow$ (iii). Let $f=(f_n)_{n\geqslant 0}$ be a $\mathbf{B}$-valued martingale and satisfy $\|\Phi(f)\|_{\mathscr{Q}^{S^r}_{p,q}(\mathbf{B})}<\infty$. According to Theorem 2, there exists a sequence of triples $(\mu^k,a^k,\upsilon^k)\in\mathscr{A}^{S^r}(\Phi,p,q,\infty)$ such that

$$\frac{1}{C}f_n=\sum_{k\in\mathbb{Z}}\mu^k\mathbb{E}_n\left(\frac{1}{C}a^k\right)\quad a.e.$$

and

$$\|\Phi(f)\|_{\mathscr{Q}^{S^r}_{p,q}(\mathbf{B})}\approx\left\|\left\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_q}.$$

Moreover, it follows from the definition of $(\Phi,p,\infty)^{S^r}$-atom that $\{S^r(a^k)>0\}\subset\{\upsilon^k<\infty\}$. Therefore, we obtain

$$\|S^r(a^k)\|_r\leqslant\|S^r(a^k)\|_\infty\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{r}}.$$

Using Lemma 4 (ii), we see that

$$\|M(a^k)\|_r\leqslant C\|S^r(a^k)\|_r\ (C>1).$$

Since a^k is a $(\Phi,p,\infty)^{S^r}$-atom, then we can conclude that

$$\left\|M\left(\frac{1}{C}a^k\right)\right\|_r\leqslant\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{r}}\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right).$$

It is easy to check that $\left(\mu^k, \frac{1}{C}a^k, \upsilon^k\right) \in \mathscr{A}^M(\Phi, p, q, r)$. It follows from Corollary 1 and Lemma 1 (i) that

$$\|\Phi(M(f))\|_{p,q} \lesssim \left\| \left\{ \mathbb{P}(\upsilon^k < \infty)^{\frac{1}{p}} \Phi\left(\mu^k \Phi^{-1}\left(\mathbb{P}(\upsilon^k < \infty)^{-\frac{1}{p}} \right) \right) \right\}_{k\in\mathbb{Z}} \right\|_{l_q}.$$

Hence, $\|\Phi(M(f))\|_{p,q} \lesssim \|\Phi(f)\|_{\mathscr{Q}^{S^r}_{p,q}(\mathbf{B})}$.

(iii) $\Rightarrow$ (i). Suppose that $f = (f_n)_{n\geqslant 0}$ is a $\mathbf{B}$-valued dyadic martingale with $S^r(f) \in L_\infty$ and $\Phi(t) = t$. For every $n \in \mathbb{N}$, set $\lambda_n = Cs^r_{n+1}(f)$. Obviously, $(\lambda_n)_{n\geqslant 0} \in \Lambda$. Clearly, since f is a $\mathbf{B}$-valued dyadic martingale, we have $S^r_n(f) \leqslant Cs^r_n(f)$ for each $n \in \mathbb{N}$. Additionally, $S^r_{n+1}(f) \leqslant \lambda_n$. As a consequence, we may infer that

$$\|\Phi(f)\|_{\mathscr{Q}^{S^r}_1(\mathbf{B})} \leqslant \|\Phi(\lambda_\infty)\|_1 = C\|s^r(f)\|_1 < \infty. \tag{26}$$

Let us denote $g^n = (g^n_m)_{m\geqslant 0}$ as (ii) $\Rightarrow$ (i). According to (25) and (26), we get

$$\|f_{m+n} - f_n\|_{L_1(\mathbf{B})} \leqslant \|\Phi(M(g^n))\|_1 \lesssim \|\Phi(g^n)\|_{\mathscr{Q}^{S^r}_1(\mathbf{B})} \lesssim \|s^r(g^n)\|_1 \to 0,\ (n \to \infty).$$

Similarly to (ii) $\Rightarrow$ (i), we obtain $\mathbf{B}$ is isomorphic to a r-uniformly smooth space. □

THEOREM 5. *Let $\mathbf{B}$ be a Banach space, $\Phi \in \mathscr{G}$ be a concave function, $2 \leqslant r < \infty$, $0 < p < r$ and $0 < q \leqslant \infty$. The following assertions are equivalent:*

(i) *$\mathbf{B}$ is isomorphic to a r-uniformly convex space;*

(ii) *If the $\mathbf{B}$-valued martingale $f = (f_n)_{n\geqslant 0}$ satisfies $\|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})} < \infty$, then*

$$\|\Phi(S^r(f))\|_{p,q} \lesssim \|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})}; \tag{27}$$

(iii) *If the $\mathbf{B}$-valued martingale $f = (f_n)_{n\geqslant 0}$ satisfies $\|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})} < \infty$, then*

$$\|\Phi(s^r(f))\|_{p,q} \lesssim \|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})}. \tag{28}$$

Proof. (i) $\Rightarrow$ (ii). Let $f = (f_n)_{n\geqslant 0}$ be a $\mathbf{B}$-valued martingale with $\|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})} < \infty$. Since $\mathbf{B}$ is isomorphic to a r-uniformly convex space implies that $\mathbf{B}$ has the **RNP** (see Remark 2). Then by Theorem 3, there exists a sequence of triples $(\mu^k, a^k, \upsilon^k) \in \mathscr{A}^M(\Phi, p, q, \infty)$ such that

$$\frac{1}{C} f_n = \sum\nolimits_{k\in\mathbb{Z}} \mu^k \mathbb{E}_n\left(\frac{1}{C}a^k\right) \ a.e.$$

and

$$\left\| \left\{ \mathbb{P}(\upsilon^k < \infty)^{\frac{1}{p}} \Phi\left(\mu^k \Phi^{-1}\left(\mathbb{P}(\upsilon^k < \infty)^{-\frac{1}{p}} \right) \right) \right\}_{k\in\mathbb{Z}} \right\|_{l_q} \approx \|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})}.$$

According to Lemma 4 (ii), we know that for any $\mathbf{B}$-valued martingale g,

$$\|s^r(g)\|_r = \|S^r(g)\|_r \leqslant C\|M(g)\|_r,\ (C > 1).$$

Apparently, $a^k = (a_n^k)_{n\geqslant 0}$ is a $\mathbf{B}$-valued martingale. Therefore, a proof similar to (i) $\Rightarrow$ (iii) in Theorem 4, we can get

$$\left\|S^r\left(\frac{1}{C}a^k\right)\right\|_r \leqslant \mathbb{P}(\upsilon^k<\infty)^{\frac{1}{r}}\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right).$$

This shows that $\frac{1}{C}a^k$ is a $(\Phi,p,r)^{S^r}$-atom and $\left(\mu^k,\frac{1}{C}a^k,\upsilon^k\right)\in\mathscr{A}^{S^r}(\Phi,p,q,r)$. Consequently, by Corollary 1 and Lemma 1 (i), we obtain

$$\|\Phi(S^r(f))\|_{p,q} \lesssim \left\|\left\{\mathbb{P}(\upsilon^k<\infty)^{\frac{1}{p}}\Phi\left(\mu^k\Phi^{-1}\left(\mathbb{P}(\upsilon^k<\infty)^{-\frac{1}{p}}\right)\right)\right\}_{k\in\mathbb{Z}}\right\|_{l_q}.$$

Therefore,

$$\|\Phi(S^r(f))\|_{p,q} \lesssim \|\Phi(f)\|_{\mathscr{D}_{p,q}(\mathbf{B})}.$$

(i) $\Rightarrow$ (iii). We can prove (i) $\Rightarrow$ (iii) similarly as above.

(ii), (iii) $\Rightarrow$ (i). Assume that $f=(f_n)_{n\geqslant 0}$ is an arbitrary $\mathbf{B}$-valued dyadic martingale satisfying $\sup_{n\geqslant 0}\|f_n\|_{L_\infty(\mathbf{B})}<\infty$. Set $\Phi(t)=t$ and $\lambda_n=\|M_{n+1}(f)\|_\infty$. It is clear that $(\lambda_n)_{n\geqslant 0}\in\Lambda$ and $\|f_{n+1}\|\leqslant\lambda_n$. Thus, we have

$$\|\Phi(f)\|_{\mathscr{D}_1(\mathbf{B})} \leqslant \|\Phi(\lambda_\infty)\|_1 \leqslant \sup_{n\geqslant 0}\|f_n\|_{L_\infty(\mathbf{B})}<\infty.$$

Then, $S^r(f)<\infty$ *a.e.* by (27) and $s^r(f)<\infty$ *a.e.* by (28). In the latter case, because f is a $\mathbf{B}$-valued dyadic martingale, we have $S^r(f)\leqslant Cs^r(f)<\infty$. Applying Lemma 4, we obtain that $\mathbf{B}$ is isomorphic to a r-uniformly convex space. □

REMARK 5. Let $\Phi(t)=t$ and $0<p=q\leqslant 1$ in Theorems 4 and 5. Then we obtain Theorems 5 and 6 in [20], respectively.

REMARK 6. Let $\Phi(t)=t$ in Theorems 4 and 5. Then we get Theorems 5.4 and 5.6 in [18], respectively.

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