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SUCCESSIVE COEFFICIENTS AND TOEPLITZ DETERMINANT FOR CONCAVE UNIVALENT FUNCTIONS

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Abstract. Let $Co(p)$ be the class of all functions f defined in the unit disc $\mathbb{D}$ having a simple pole at $z = p$ where $0 < p < 1$ and analytic in $\mathbb{D}\setminus\{p\}$ with $f(0) = 0 = f'(0)-1$ such that f maps $\mathbb{D}$ onto a domain whose complement with respect to the extended complex plane is a bounded convex set. These functions are called concave univalent functions. Each $f \in Co(p)$ has the following Taylor expansion:

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n, \quad |z| < p.$$

In this article, we first determine the regions of variability of the difference of successive coefficients $(a_{n+1}-a_n)$ for $n \geqslant 3$. We also find sharp upper bounds of the Toeplitz determinants, the entries of which are the Taylor coefficients of functions in $Co(p)$.

1. Introduction

Throughout this article, we will use the following notations. Let $\mathbb{C}$ be the finite complex plane, $\overline{\mathbb{C}} := \mathbb{C}\cup\{\infty\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathscr{A}$ denote the class of all analytic functions f defined in $\mathbb{D}$ with the normalization $f(0) = 0 = f'(0)-1$ and $\mathscr{S} = \{f \in \mathscr{A} : f \text{ is univalent}\}$. Each $f \in \mathscr{S}$ has the Taylor expansion shown below.

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Geometric function theory generated numerous interesting and exciting results over the previous century. The Bieberbach conjecture, introduced in 1916, was one of the major challenges in this field. According to this conjecture, each $f \in \mathscr{S}$ with the expansion (1.1) must meet the inequality $|a_n| \leqslant n$ for all $n \geqslant 2$. L. de Branges ([5]) proved this conjecture in 1985. In order to settle the Bieberbach conjecture prior to de Branges' effort, various geometric subclasses of $\mathscr{S}$ were established, and the claim was proved

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for these subclasses. The classes of convex functions, starlike functions, and close-to-convex functions were among the particular subclasses of $\mathscr{S}$ for which this conjecture was resolved (c.f. [6]). We remark here that, a function $f \in \mathscr{S}$ is convex if f transfers the unit disc onto a convex set, i.e. $f(\mathbb{D})$ is convex, and a function $f \in \mathscr{S}$ is starlike if $f(\mathbb{D})$ is starlike with respect to the origin. $\mathscr{S}^*$ and $\mathscr{K}$ are commonly used to designate the classes of starlike and convex functions in $\mathscr{S}$. More information on the convex and starlike classes can be found in Duren's book [6]. To prove the Bieberbach conjecture, another approach was to investigate the expression $|a_{n+1} - a_n|$, that is, difference of the successive coefficients. It is known that for $f \in \mathscr{S}$, $|a_{n+1} - a_n|$ is bounded. In fact, in 1963, Hayman (see f.i. [8]) obtained that, for $f \in \mathscr{S}$,

$$||a_{n+1}| - |a_n|| \leqslant A, \tag{1.2}$$

where, $A \geqslant 1$ is an absolute constant. Here, finding the minimal value of the constant A is still open. The best result till date is $A < 3.61$ which was proved by Grinspan (see [7]). We mention here that the following sharp bound is known (see [6, Theorem 3.11]) only for $n = 2$,

$$-1 \leqslant |a_3| - |a_2| \leqslant 1.029\ldots.$$

In [14], the author conjectured that $|a_{n+1} - a_n| \leqslant 1$ for the class $\mathscr{S}^*$ which was solved in the article [9] by Leung in 1978. Leung proved that, if $f \in \mathscr{S}^*$, then for $n \geqslant 1$,

$$-1 \leqslant |a_{n+1}| - |a_n| \leqslant 1,$$

i.e $A = 1$ in the inequality (1.2) and both the inequalities are sharp. From Leung's result, it is clear that the following quantities for the class $\mathscr{S}^*$

$$\sup_{f \in \mathscr{S}^*} (|a_{n+1}(f)| - |a_n(f)|) \quad \text{and} \quad \sup_{f \in \mathscr{S}^*} (|a_n(f)| - |a_{n+1}(f)|)$$

are the same, that is 1. But, these quantities are not same for the class of convex univalent functions $\mathscr{K}$ which prompted Li and Sugawa (compare [10]) to consider the quantities like

$$D_n^+ := \sup_{f \in \mathscr{K}} (|a_{n+1}(f)| - |a_n(f)|) \quad \text{and} \quad D_n^- := \sup_{f \in \mathscr{K}} (|a_n(f)| - |a_{n+1}(f)|).$$

Indeed, it is easy to see that $D_1^+ = 0$ and $D_1^- = 1$. In the same article, they determined that $D_n^+ = 1/(n+1)$, for $n \geqslant 2$, and $D_n^- = 1/n$, for $n = 1,2,3$. In addition, Li and Sugawa obtained the estimate $(1/n) < D_n^- < 2/(n+1)$ for each $n \geqslant 4$. A recent article of Arora, et. al. (see [1]) and references therein show that there is still considerable amount of interest in obtaining the bounds for successive coefficients for various subclasses of $\mathscr{S}$.

To the best of our knowledge, estimates for the successive coefficients of meromorphic univalent functions has not been found so far. Motivated by this and the recent results obtained by Li and Sugawa in [10], in the first part of the paper, we wish to explore this problem for concave univalent functions which can be thought as meromorphic analogs of convex univalent functions. To this end, let $\mathscr{A}(p)$ be the class that is defined as the collection of functions in $\mathbb{D}$ having a simple pole at $z = p$ where $p \in (0,1)$ and analytic in $\mathbb{D} \setminus \{p\}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$. We define $\Sigma(p) := \{f \in \mathscr{A}(p) : f \text{ is univalent}\}$. In this article, we will consider mainly $Co(p)$, the class of concave univalent functions which can be seen as a meromorphic analogs of the class of convex univalent functions in the analytic set up. Here, note that $Co(p) := \{f \in \Sigma(p) : \widehat{\mathbb{C}} \setminus f(\mathbb{D}) \text{ is a compact convex set}\}$. It is well-known that

$$k_p(z) = \frac{-pz}{(z-p)(1-pz)}, \quad z \in \mathbb{D}, \tag{1.3}$$

belongs to $Co(p)$ and $k_p(\mathbb{D}) = \overline{\mathbb{C}} \setminus [-p/(1-p)^2, -p/(1+p)^2]$. For more information about the class $Co(p)$, we refer to the articles [2, 3, 11] and references therein. Each $f \in Co(p)$ is analytic in the disc $\mathbb{D}_p := \{z : |z| < p\}$, and has the Taylor expansion of the form (1.1) valid in $\mathbb{D}_p$. For $n \geqslant 2$ and $f \in Co(p)$ having the Taylor expansion (1.1), one has (see [2])

$$\left| a_n - \frac{1-p^{2n+2}}{p^{n-1}(1-p^4)} \right| \leqslant \frac{p^2(1-p^{2n-2})}{p^{n-1}(1-p^4)}. \tag{1.4}$$

Equality is attained in the above inequality for the function (1.3). In particular, we get from (1.4) that

$$|a_n| \leqslant \frac{1-p^{2n}}{p^{n-1}(1-p^2)}. \tag{1.5}$$

Though the exact regions of variability of a_n, $n \geqslant 2$ for functions in the class $Co(p)$ have already been found, but, it is always interesting to determine sharp estimates of the difference of the successive coefficients for functions in this class. In [4], the first author of the present article obtained the exact set of variability of the linear combination of the Taylor coefficients $\mu a_2 - a_3$, $\mu \in \mathbb{C}$ for functions in $Co(p)$.

THEOREM A. *Let $p \in (0,1)$ and $f \in Co(p)$ have the expansion (1.1). Set $\alpha := (1+p^2)/p$. Then the domain of variability of $\mu a_2 - a_3$, $\mu \in \mathbb{C}$ is determined by the inequality*

$$\left| \mu a_2 - a_3 - \left(\left(\frac{\alpha^2 - 1}{\alpha} \right) \mu + (2 - \alpha^2) \right) \right|$$
$$\leqslant \begin{cases} \frac{1}{\alpha} \left[\frac{1}{3} + \frac{3|\mu-\alpha|^2}{4} \right] & \text{for} \quad |\mu - \alpha| < 2/3; \\ \left| 1 - \frac{\mu}{\alpha} \right| & \text{for} \quad |\mu - \alpha| \geqslant 2/3. \end{cases}$$

Substituting $\mu = 1$ in the above theorem, we get

$$\left| (a_3 - a_2) - \frac{(1-p)(1+p^7)}{p^2(1-p^4)} \right| \leqslant \frac{(1-p)(1+p^3)}{(1-p^4)}, \tag{1.6}$$

and the equality holds in the above inequality for the function k_p. In Theorem 1 of this article, we obtain the exact region of variability of $(a_{n+1} - a_n)$, $n \geqslant 3$, whenever

$f \in Co(p)$ and $p \in (0,1)$. Our obtained results answer the open problem (see [4, Remark 2]) raised by the first author of the present article. Next, we consider the following two quantities for $n \geqslant 1$:

$$\Omega_n^+ = \sup_{f \in Co(p)} (|a_{n+1}(f)| - |a_n(f)|) \quad \text{and} \quad \Omega_n^- = \sup_{f \in Co(p)} (|a_n(f)| - |a_{n+1}(f)|).$$

Using Theorem 1, we estimate the above quantities in Theorem 2 of this article. As another application of Theorem 1, we estimate the second and the third order Toeplitz determinant whose entries are the Taylor coefficients of $f \in Co(p)$. We wish to elaborate this problem in order to be little more precise. For $n \geqslant 1$, $q \geqslant 1$, the symmetric Toeplitz determinant $T_q(n)$ for analytic functions f of the form (1.1), is defined as

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q-2} & \dots & a_n, \end{vmatrix}$$

where, $a_1 = 1$. In particular,

$$T_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_1 \end{vmatrix}, \quad T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \quad T_2(4) = \begin{vmatrix} a_4 & a_5 \\ a_5 & a_4 \end{vmatrix}, \tag{1.7}$$

and

$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, \quad T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

We refer to the article [20] for various applications of Toeplitz matrices in the field of pure and applied mathematics. In this paper, we wish to investigate the Toeplitz determinant $T_2(n)$, $n \geqslant 1$ and $T_3(1)$ for the functions in the class $Co(p)$ and obtain the sharp upper bounds of above determinants.

2. Main results

In the following theorem, we obtain the regions of variability of $(a_{n+1} - a_n)$, for $n \geqslant 3$ and $p \in (0,1)$ whenever $f \in Co(p)$ with the Taylor expansion (1.1) in $\mathbb{D}_p$.

THEOREM 1. *Let $f \in Co(p)$ be of the form (1.1) in $\mathbb{D}_p$. Then*

$$\left| (a_{n+1} - a_n) - \frac{(1-p)(1+p^{2n+3})}{p^n(1-p^4)} \right| \leqslant \frac{(1-p)(1+p^{2n-1})}{p^{n-2}(1-p^4)},$$

for all $n \geqslant 3$ and $p \in (0,1)$. In particular, we have

$$|a_{n+1} - a_n| \leqslant \frac{1+p^{2n+1}}{p^n(1+p)}.$$

Equality holds in all the above inequalities for the function k_p which is defined in (1.3).

Proof. From [19], we have that for any $f \in Co(p)$, there exists a function w-holomorphic in $\mathbb{D}$ such that $|w(z)| \leqslant 1$ and

$$f(z) = \frac{z - \frac{p}{1+p^2}(1+w(z))z^2}{\left(1-\frac{z}{p}\right)(1-pz)}, \quad z \in \mathbb{D}. \tag{2.1}$$

Let each w have the following Taylor expansion in $\mathbb{D}$:

$$w(z) = \sum_{k=0}^{\infty} c_k z^k.$$

Now inserting the above expression for w and the series expansion (1.1) for f in the representation formula (2.1), we get

$$f(z) = \frac{z - \left(\frac{p}{1+p^2}\right)z^2}{\left(1-\frac{z}{p}\right)(1-pz)} - \frac{\left(\frac{p}{1+p^2}\right)z^2 w(z)}{\left(1-\frac{z}{p}\right)(1-pz)} = \sum_{n=1}^{\infty} \frac{1-p^{2n+2}}{p^{n-1}(1-p^4)} z^n + \sum_{n=2}^{\infty} b_n z^n,$$

where

$$b_n = -\sum_{k=0}^{n-2} c_k \frac{p^2(1-p^{2(n-k)-2})}{p^{n-k-1}(1-p^4)}.$$

Next, comparing the coefficients of z^n, $n \geqslant 1$ in the above equality, we get

$$a_n = \frac{1-p^{2n+2}}{p^{n-1}(1-p^4)} + b_n, \quad n \geqslant 2, \tag{2.2}$$

which gives

$$\begin{aligned} a_{n+1} - a_n &= \frac{1-p^{2n+4}}{p^n(1-p^4)} + b_{n+1} - \frac{1-p^{2n+2}}{p^{n-1}(1-p^4)} - b_n \\ &= \frac{(1-p)(1+p^{2n+3})}{p^n(1-p^4)} + b_{n+1} - b_n. \end{aligned}$$

Now, we compute

$$\begin{aligned} b_{n+1} - b_n &= -\sum_{k=0}^{n-1} c_k \frac{p^2(1-p^{2(n-k)})}{p^{n-k}(1-p^4)} + \sum_{k=0}^{n-2} c_k \frac{p^2(1-p^{2(n-k)-2})}{p^{n-k-1}(1-p^4)} \\ &= -\frac{p^2(1-p)}{1-p^4} \left[\sum_{k=0}^{n-1} c_k \frac{1+p^{2(n-k)-1}}{p^{n-k}} \right], \end{aligned}$$

and hence

$$\left| (a_{n+1} - a_n) - \frac{(1-p)(1+p^{2n+3})}{p^n(1-p^4)} \right| = \frac{p^2(1-p)}{1-p^4} \left| \sum_{k=0}^{n-1} c_k \frac{1+p^{2(n-k)-1}}{p^{n-k}} \right|. \tag{2.3}$$

From (2.3) and the statement of the theorem it is clear that we need to prove for $n \geqslant 3$ and $p \in (0,1)$,

$$\left|\sum_{k=0}^{n-1} c_k \frac{1+p^{2(n-k)-1}}{p^{n-k}}\right| \leqslant \frac{1+p^{2n-1}}{p^n},$$

or, equivalently,

$$\left|\sum_{k=0}^{n-1} p^k \frac{1+p^{2(n-k)-1}}{1+p^{2n-1}} c_k\right| \leqslant 1. \tag{2.4}$$

For $p \in (0,1)$, let us consider the function

$$g(z) := \sum_{k=0}^{n-1} \alpha_k z^k, \;\; z \in \mathbb{D},$$

where

$$\alpha_k = \left(\frac{1+p^{2(n-k)-1}}{1+p^{2n-1}}\right) p^k,$$

and $0 \leqslant k \leqslant (n-1)$. Now, for $0 \leqslant m, m+1 \leqslant (n-1)$,

$$\begin{aligned}
&\alpha_{m+1} \leqslant \alpha_m \\
\Leftrightarrow\ & p\left(1+p^{2n-2m-3}\right) \leqslant \left(1+p^{2n-2m-1}\right) \\
\Leftrightarrow\ & (1-p)(1-p^{2n-2m-2}) \geqslant 0,
\end{aligned}$$

which is true for all $p \in (0,1)$. This shows that the sequence $\{\alpha_k\}$ is a decreasing sequence. Next, we want to show that the sequence $\{\alpha_k\}$ is a convex sequence. Here, we clarify that a real sequence $\{\alpha_k\}$ is called a convex sequence if the inequality $\alpha_{m-1} + \alpha_{m+1} \geqslant 2\alpha_m$ holds for all m. For this sequence $\{\alpha_k\}$, we calculate

$$\begin{aligned}
&\alpha_{m-1} + \alpha_{m+1} \geqslant 2\alpha_m \\
\Leftrightarrow\ & 1+p^{2n-2m+1} + p^2\left(1+p^{2n-2m-3}\right) \geqslant 2p\left(1+p^{2n-2m-1}\right) \\
\Leftrightarrow\ & (1-p)^2\left(1+p^{2n-2m-1}\right) \geqslant 0,
\end{aligned}$$

which is valid for all $p \in (0,1)$. Therefore, $\{\alpha_k\}$ is a convex sequence. Now, by a result of Rogosinski (c.f. [16]), we get

$$\operatorname{Re}(g(z)) > \frac{1}{2}, \;\; z \in \mathbb{D}.$$

From [18, Theorem 1.3], we know that functions with this property are bound preserving i.e. one will have

$$|(g * w)(1)| \leqslant 1,$$

where $*$ denotes the Hadamard convolution. This will immediately prove the inequality (2.4). Thus, the first part of the theorem is established. Next, using the triangle inequality, we obtain

$$\begin{aligned}|a_{n+1}-a_n| &\leqslant \left|(a_{n+1}-a_n)-\frac{(1-p)(1+p^{2n+3})}{p^n(1-p^4)}\right|+\frac{(1-p)(1+p^{2n+3})}{p^n(1-p^4)},\\ &\leqslant \frac{(1-p)(1+p^{2n-1})}{p^{n-2}(1-p^4)}+\frac{(1-p)(1+p^{2n+3})}{p^n(1-p^4)},\\ &=\frac{1+p^{2n+1}}{p^n(1+p)},\end{aligned}$$

for $n \geqslant 3$ and $p \in (0,1)$. Also, it is a simple exercise to check that all the inequalities stated in the theorem are best possible for the function k_p. This completes proof of the theorem. □

We now prove our next result.

THEOREM 2. *Let $f \in Co(p)$ have the expansion of the form (1.1) in $\mathbb{D}_p$. Then*
(a)

$$\Omega_1^+ = \frac{1-p+p^2}{p} \quad \textit{and} \quad \Omega_1^- = \frac{-(1-p)(1-p^3)}{p(1+p^2)}.$$

(b)

$$\Omega_n^+ = \frac{1+p^{2n+1}}{p^n(1+p)} \ \textit{for} \ n \geqslant 2, \ \ p \in (0,1).$$

(c) for $n \geqslant 2$,

$$\begin{aligned}&\frac{p(1-p^2)(1+p^{2n})-(1+p^2)(1-p^{2n+2})}{p^n(1-p^4)}\\ &< \Omega_n^- < \frac{p(1+p^2)(1-p^{2n})-(1-p^2)(1+p^{2n+2})}{p^n(1-p^4)}.\end{aligned}$$

Proof. (a) From (1.4), we have

$$\frac{(1-p^2)(1+p^{2n})}{p^{n-1}(1-p^4)} \leqslant |a_n| \leqslant \frac{(1+p^2)(1-p^{2n})}{p^{n-1}(1-p^4)}. \tag{2.5}$$

The case $n=2$ of the above inequality provides

$$\frac{1+p^4}{p(1+p^2)} \leqslant |a_2| \leqslant \frac{(1+p^2)}{p},$$

and consequently, we get

$$\Omega_1^+ = \sup_{f\in Co(p)} (|a_2|-1) = \frac{1-p+p^2}{p},$$

where the supremum is attained by the function k_p and

$$\Omega_1^- = \sup_{f\in Co(p)} (1-|a_2|) = -\frac{(1-p)(1-p^3)}{p(1+p^2)},$$

where the supremum is attained by the following function:

$$f_1(z) := \frac{z - \frac{2p}{1+p^2}z^2}{\left(1-\frac{z}{p}\right)(1-pz)}, \quad z\in\mathbb{D}.$$

(b) Next, from the equation (1.6), we get

$$||a_3|-|a_2|| \leqslant |a_3-a_2| \leqslant \frac{1+p^5}{p^2(1+p)},$$

which in turn implies $\Omega_2^+ = (1+p^5)/(p^2(1+p))$. Again using the inequality $||a_{n+1}|-|a_n|| \leqslant |a_{n+1}-a_n|$, from the second part of Theorem 1, we get for all $n\geqslant 3$,

$$||a_{n+1}|-|a_n|| \leqslant \frac{1+p^{2n+1}}{p^n(1+p)}.$$

Sharpness of this inequality is evident as for the function k_p, we calculate

$$||a_{n+1}|-|a_n|| = |a_{n+1}-a_n| = \frac{1+p^{2n+1}}{p^n(1+p)},$$

for all $n\geqslant 1$. The above discussion immediately yields us for $f\in Co(p)$ and $n\geqslant 3$,

$$\Omega_n^+ = \frac{1+p^{2n+1}}{p^n(1+p)}, \quad p\in(0,1).$$

The function k_p is extremal for Ω_n^+, which proves the second part of the theorem.

(c) Using the maximum and the minimum values of $|a_n|$ and $|a_{n+1}|$ from (2.5), we get

$$\frac{(1-p^2)(1+p^{2n})}{p^{n-1}(1-p^4)} - \frac{(1+p^2)(1-p^{2n+2})}{p^n(1-p^4)}$$
$$< \Omega_n^- < \frac{(1+p^2)(1-p^{2n})}{p^{n-1}(1-p^4)} - \frac{(1-p^2)(1+p^{2n+2})}{p^n(1-p^4)}$$

for $n\geqslant 2$. Upon simplification of the above quantities, the proof of the theorem is complete. $\square$

REMARK 1. We observed that, the sharp upper bounds of $||a_{n+1}|-|a_n||$ and $|a_{n+1}-a_n|$ for the class $\mathscr{K}$ differ (see [10]) whereas we get the same upper bounds for these two quantities whenever $f\in Co(p)$ (the meromorphic analogs of the class $\mathscr{K}$).

REMARK 2. From Theorem 2, we have $\Omega_1^+ = (1-p+p^2)/p$ and $\Omega_1^- = -(1-p)(1-p^3)/p(1+p^2)$ which imply that, in general, $\Omega_n^- \neq \Omega_n^+$ for $f \in Co(p)$ and $n \geqslant 1$. Furthermore, we do not know the exact values of Ω_n^- when $n \geqslant 2$.

REMARK 3. It is worth noting that, unlike in class $\mathscr{K}$, we got Ω_1^- as a negative real number. We also understand that, for larger n, Ω_n^- will be a negative quantity for the class $Co(p)$. This is because, the variability zones for a_n are particular discs lying in the right half plane, and hence, each $|a_n|$ has a non trivial lower bound.

3. Toeplitz determinant for functions in the class $Co(p)$

The equation (2.3) and the inequality $|c_0| \leqslant 1$ together immediately yield the exact region of variability of $(a_2 - a_1)$ as

$$\left|(a_2-a_1) - \frac{(1-p)(1+p^5)}{p(1-p^4)}\right| \leqslant \frac{(1-p)(1+p)}{p^{-1}(1-p^4)}. \tag{3.1}$$

Now from equations (3.1) and (1.6), we have

$$|a_2-a_1| \leqslant \frac{1+p^3}{p(1+p)} \text{ and } |a_3-a_2| \leqslant \frac{1+p^5}{p^2(1+p)},$$

for $p \in (0,1)$. Therefore, the above inequalities together with the bounds of $|a_2|$ and $|a_3|$ from (1.5) provides

$$|T_2(1)| \leqslant \frac{1-p^6}{p^2(1-p^2)} \text{ and } |T_2(2)| \leqslant \frac{1-p^{10}}{p^4(1-p^2)}.$$

Also, these bounds are best possible for the function k_p. The next theorem deals with the estimates of $|T_q(n)|$ for $q=2, n \geqslant 3$, where $T_q(n)$ is defined in (1.7).

THEOREM 3. *Let $f \in Co(p)$ be of the form (1.1) in $\mathbb{D}_p$. Then for all $n \geqslant 3$ and $p \in (0,1)$, we have*

$$|T_2(n)| \leqslant \frac{1-p^{4n+2}}{p^{2n}(1-p^2)}.$$

The above inequalities are sharp for the function k_p.

Proof. From the definition of the Toeplitz determinant, we have $T_2(n) = a_n^2 - a_{n+1}^2$. Therefore, by using Theorem 1 and bounds of $|a_n|$, we get

$$\begin{aligned} |T_2(n)| &\leqslant (|a_n| + |a_{n+1}|)(|a_{n+1} - a_n|) \\ &\leqslant \left(\frac{1-p^{2n}}{p^{n-1}(1-p^2)} + \frac{1-p^{2(n+1)}}{p^n(1-p^2)}\right)\left(\frac{1+p^{2n+1}}{p^n(1+p)}\right) \\ &= \frac{1-p^{4n+2}}{p^{2n}(1-p^2)}. \end{aligned}$$

Again for the function k_p, it can be computed that $|T_2(n)| = \frac{1-p^{4n+2}}{p^{2n}(1-p^2)}$, which completes proof of the theorem. □

We prove a sharp upper bound for $|T_3(1)|$ whenever $f \in Co(p)$ in the following theorem.

THEOREM 4. *Let $f \in Co(p)$ be of the form (1.1) in $\mathbb{D}_p$. Then for $p \in (0,1)$, we have*

$$|T_3(1)| \leqslant \frac{(1+p^2)^2(1+p^4)}{p^4}.$$

Equality holds in the above inequality for the function k_p.

Proof. A simple computation yields that

$$T_3(1) = 1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2 = 1 + a_2^2(a_3 - 2) - a_3(a_3 - a_2^2).$$

Now, we know that for functions $f \in Co(p)$, $|a_3 - a_2^2| \leqslant 1$ (see [12]). Also, the bounds for $|a_2|$ and $|a_3|$ are known. By using the triangle inequality we get

$$|T_3(1)| \leqslant 1 + |a_2|^2|(a_3 - 2)| + |a_3||(a_3 - a_2^2)|. \tag{3.2}$$

Therefore, to get an upper bound for $|T_3(1)|$, we need to find the maximum value of $|a_3 - 2|$ whenever $f \in Co(p)$. From (2.2), we calculate

$$a_3 - 2 = \frac{1+p^4}{p^2} - c_0 - \frac{p}{1+p^2}c_1 - 2.$$

Hence

$$|a_3 - 2| \leqslant \frac{(1-p^2)^2}{p^2} + |c_0| + \frac{p}{1+p^2}(1 - |c_0|^2).$$

The right hand side of the above inequality has the maximum when $|c_0| = 1$. Therefore,

$$|a_3 - 2| \leqslant \left(\frac{(1-p^2)^2}{p^2} + 1\right). \tag{3.3}$$

Next, plugging in the above bound, and the bounds of $|a_2|$ and $|a_3|$ in (3.2), we get

$$|T_3(1)| \leqslant \frac{(1+p^2)^2(1+p^4)}{p^4}.$$

Again for the function k_p, it is a simple exercise to check that equality holds in the above inequality. This ends the proof of the theorem. □

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