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# JENSEN–TYPE INEQUALITIES IN TERMS OF LIPSCHITZIANITY

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*Abstract.* The main focus of this paper is a study of Jensen-type inequalities for the Lipschitzian functions. We establish the reverse of the Jensen inequality expressed in terms of the corresponding Lipschitz constant. In addition, we also obtain the reverse of the superadditivity relation for a convex function, expressed in the same way. As an application, we obtain reverses of power mean inequalities, the Hölder inequality, and the Hermite-Hadamard inequality, expressed in terms of the Lipschitzianity. In particular, we derive reverses of the arithmetic-geometric mean inequality in both difference and quotient forms.

## 1. Introduction

Let $J \subseteq \mathbb{R}$ be an interval. The function $f : J \to \mathbb{R}$ is said to be convex if the relation

$$f((1-\nu)a+\nu b) \leqslant (1-\nu)f(a)+\nu f(b) \tag{1}$$

holds for all $a,b \in J$ and $0 \leqslant \nu \leqslant 1$. Conversely, $f$ is concave if the sign of (1) is reversed. There is also one more case for which the sign of (1) is reversed. Namely, if $a,b \in J$ and $\nu \notin [0,1]$ are such that $(1-\nu)a+\nu b \in J$, then holds the reverse of (1), provided that $f$ is convex on $J$. This represents the so-called external form of (1).

The above inequality can be rewritten as

$$f(w_1 a + w_2 b) \leqslant w_1 f(a) + w_2 f(b), \tag{2}$$

provided that $w_1 + w_2 = 1$, and $w_1, w_2 \geqslant 0$. A multivariable version of (2) is the celebrated Jensen inequality which asserts that if $f : J \to \mathbb{R}$ is a convex function and $a_1, a_2, \ldots, a_n \in J$, then holds the inequality

$$f\left(\sum_{i=1}^{n} w_i a_i\right) \leqslant \sum_{i=1}^{n} w_i f(a_i)$$

where $w_1, w_2, \ldots, w_n \geqslant 0$ are such that $\sum_{i=1}^{n} w_i = 1$.

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If $f:[0,\infty)\to\mathbb{R}$ is a convex function such that $f(0)\leqslant 0$, then (1) implies the inequality $f(vx)\leqslant vf(x)$, where $x\geqslant 0$ and $0\leqslant v\leqslant 1$. Now, by putting $x=a+b$, $v=\frac{a}{a+b}$, $a,b\geqslant 0$, and then, $v=\frac{a}{a+b}$, in this inequality, we obtain

$$f(a)+f(b)\leqslant f(a+b). \tag{3}$$

This means that $f$ is superadditive on $[0,\infty)$.

Although classical, the Jensen inequality is still of interest to numerous authors. For a comprehensive overview of convexity and the Jensen inequality, including proofs, generalizations, and diverse applications, the reader is referred to monographs [7, 12, 13] and the references cited therein. Furthermore, for the latest research regarding this topic, we recommend referring to [1, 3, 8, 9, 10, 14, 15, 16, 17].

The main goal of this paper is to establish the connection between convex functions and another important class of real functions named after the 19th-century German mathematician Rudolf Lipschitz. A function $f:J\to\mathbb{R}$ is said to satisfy a Lipschitz condition on the interval $J\subseteq\mathbb{R}$ if there exists a constant $L>0$ such that the inequality

$$|f(a)-f(b)|\leqslant L|a-b| \tag{4}$$

holds for all $a,b\in J$. We also say that $f$ is $L$-Lipschitzian (see, e.g. [13]). For example, since $|b^2-a^2|=|b+a||b-a|\leqslant 2\max\{|a|,|b|\}|b-a|$, it follows that $f(x)=x^2$ is 2-Lipschitzian function on the unit interval.

The function belonging to this class is characterized by the fact that its rate of change is bounded by the corresponding Lipschitz constant. In fact, the Lipschitz constant measures how much the function can change as the input values change.

Lipschitzianity is an important concept in various fields of mathematics, including calculus, functional analysis, and optimization theory. In particular, it is often used to prove both the existence and uniqueness of solutions to various boundary value problems. Further, the Lipschitz continuous functions are frequently used to analyze numerical methods in computational mathematics. One of the most significant implications of Lipschitzian mappings is that they provide means to quantify the stability of solutions of differential equations. By bounding the rate of change of a function via the Lipschitz constant, one can show that small changes in the initial conditions result in correspondingly small changes in the solution. This is a powerful tool for analyzing the behavior of complex systems in physics, engineering, and other disciplines.

This paper's main objective is to study Jensen-type inequalities for the Lipschitzian functions. It turns out that the reverses of the Jensen-type inequalities can be expressed in terms of Lipschitzianity. The paper's outline is as follows: after this Introduction, in Section 2, we establish a class of Jensen-type inequalities for Lipschitzian functions. We obtain a reverse of the Jensen inequality if these functions are also convex. In other words, we derive the reverse of the Jensen inequality, expressed via the corresponding Lipschitz constant. In particular, we also obtain the reverse of the superadditivity relation for a convex function. As an application, in Section 3, our main results are applied in deriving reverses of power mean inequalities. In particular, we establish reverses of the arithmetic-geometric mean inequality, in terms of the Lipschitzianity, in both difference and quotient forms. Further, in Section 4, we give a reverse of the Hölder inequality. Finally, Section 5 deals with further extensions in this topic. First, we give an external form of the Jensen inequality for Lipschitzian functions. Then, we establish an even more precise Jensen-type inequality for this class of functions. Finally, as an application, we also derive a reverse of the Hermite-Hadamard inequality in terms of the corresponding Lipschitz constant.

We believe that our results have potential applications in various fields, including optimization, mathematical finance, and machine learning. This paper contributes to the existing literature by providing a deeper understanding of Lipschitzian mappings and their properties. We refer the reader to [4, 11, 18] as a list of some treatments of Lipschitzian and convex functions with their applications.

## 2. Main results

The starting point in our study is the following basic property of Lipschitzian mappings.

THEOREM 1. *Let $J \subseteq \mathbb{R}$ be an interval such that $a, b \in J$, and let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a $L$-Lipschitzian mapping on $J$. Then the inequality*

$$|(1-v) f(a) + v f(b) - f((1-v) a + v b)| \leqslant 2L(1-v) v |b-a| \tag{5}$$

*holds for all $0 \leqslant v \leqslant 1$. In particular, if $0 \in J$ and $f(0) = 0$, then the inequality*

$$|v f(x) - f(vx)| \leqslant 2L(1-v) v |x| \tag{6}$$

*holds for any $x \in J$ and $0 \leqslant v \leqslant 1$.*

*Proof.* Utilizing the triangle inequality, as well as the $L$-Lipschitzian condition (4), we have

$$\begin{aligned}
&|(1-v) f(a) + v f(b) - f((1-v) a + v b)| \\
&= |(1-v)(f(a) - f((1-v) a + v b)) + v (f(b) - f((1-v) a + v b))| \\
&\leqslant (1-v) |f(a) - f((1-v) a + v b)| + v |f(b) - f((1-v) a + v b)| \\
&\leqslant L(1-v) |a - ((1-v) a + v b)| + L v |b - ((1-v) a + v b)| \\
&= 2L(1-v) v |b-a|,
\end{aligned}$$

so (5) holds. Furthermore, by putting $a = 0$, $b = x$ in (5), and taking into account that $f(0) = 0$, we get $|v f(x) - f(vx)| \leqslant 2L(1-v) v |x|$, as claimed. □

REMARK 1. It should be noticed here that if $f : J \to \mathbb{R}$ is a convex function, then the modulus function on the left-hand side of (5) is redundant. This means that (5) represents a reverse of a convexity definition (1), expressed in terms of the Lipschitz constant of the corresponding function. The same conclusion can be drawn for concave functions.

Theorem 1 can be exploited in obtaining superadditivity relations for Lipschitzian mappings. We start our discussion with the following simple result.

THEOREM 2. *Let* $f:[0,\infty)\to\mathbb{R}$ *be a* $L$*-Lipschitzian mapping and let* $a,b\geqslant 0$*. If* $f(0)=0$*, then holds the inequality*

$$|f(a+b)-f(a)-f(b)|\leqslant\frac{4Lab}{a+b},$$

*provided that* $a+b\neq 0$*.*

*Proof.* Since $0\leqslant\frac{a}{a+b}\leqslant 1$, by putting $v=\frac{a}{a+b}$ and $x=a+b$ in (6), we obtain the inequality

$$\begin{aligned}\left|\frac{a}{a+b}f(a+b)-f\left(\frac{a}{a+b}(a+b)\right)\right|&=\left|\frac{a}{a+b}f(a+b)-f(a)\right|\\&\leqslant\frac{2Lab}{a+b}.\end{aligned}\tag{7}$$

Similarly, by changing the roles of variables $a$ and $b$, we also have

$$\left|\frac{b}{a+b}f(a+b)-f(b)\right|\leqslant\frac{2Lab}{a+b}.\tag{8}$$

Finally, utilizing the triangle inequality, as well as relations (7) and (8), we have that

$$\begin{aligned}&|f(a+b)-f(a)-f(b)|\\&=\left|\frac{b}{a+b}f(a+b)-f(b)+\frac{a}{a+b}f(a+b)-f(a)\right|\\&\leqslant\left|\frac{b}{a+b}f(a+b)-f(b)\right|+\left|\frac{a}{a+b}f(a+b)-f(a)\right|\\&\leqslant\frac{4Lab}{a+b},\end{aligned}$$

and the proof is completed. □

However, by a more precise analysis, Theorem 2 can be improved in the following way:

THEOREM 3. *Let* $f:[0,\infty)\to\mathbb{R}$ *be a* $L$*-Lipschitzian mapping and let* $a,b\geqslant 0$*. If* $f(0)=0$*, then holds the inequality*

$$\begin{aligned}&\frac{4Lab}{a+b}-|f(a+b)-f(a)-f(b)|\\&\geqslant\left|\left|\frac{a}{a+b}f(a+b)-f(a)\right|-\left|\frac{b}{a+b}f(a+b)-f(b)\right|\right|,\end{aligned}\tag{9}$$

*provided that* $a+b\neq 0$*.*

*Proof.* Due to (7) and (8), we have that

$$\max\left\{\left|\frac{a}{a+b}f(a+b)-f(a)\right|,\left|\frac{b}{a+b}f(a+b)-f(b)\right|\right\}\leqslant\frac{2Lab}{a+b}. \tag{10}$$

On the other hand, if $x,y\in\mathbb{R}$, then

$$\max\{|x|,|y|\}=\frac{|x|+|y|+||x|-|y||}{2}\geqslant\frac{|x+y|+||x|-|y||}{2},$$

i.e., we arrive at the inequality

$$2\max\{|x|,|y|\}\geqslant|x+y|+||x|-|y||.$$

Now, by substituting $x=\frac{a}{a+b}f(a+b)-f(a)$ and $y=\frac{b}{a+b}f(a+b)-f(b)$ in the last inequality, as well as, taking into account (10), we obtain (9), as claimed. □

We have already discussed in Remark 1 that if $f$ is a convex function, then the left-hand sides of inequalities (5) and (6) are non-negative, and we may drop the modulus function. In the sequel, we deal with functions that are also differentiable. Recall that the Lagrange mean value theorem asserts that if $f:[x,y]\to\mathbb{R}$ is a continuous function, that is differentiable on $(x,y)$, then there exists $\xi\in(x,y)$ such that

$$f(y)-f(x)=f'(\xi)(y-x).$$

The following result, referring to both differentiable and convex functions, is a consequence of Theorems 2 and 3.

COROLLARY 1. *Let $f:[0,\infty)\to\mathbb{R}$ be a differentiable convex function, let $a,b>0$, and $L=\sup_{t\in[a,b]}|f'(t)|<\infty$. If $f(0)=0$, then hold the inequalities*

$$0\leqslant f(a+b)-f(a)-f(b)\leqslant\frac{4Lab}{a+b} \tag{11}$$

*and*

$$\begin{aligned}&\frac{4Lab}{a+b}-f(a+b)+f(a)+f(b)\\ &\geqslant\left|\left|\frac{a}{a+b}f(a+b)-f(a)\right|-\left|\frac{b}{a+b}f(a+b)-f(b)\right|\right|,\end{aligned} \tag{12}$$

*provided that $a+b\neq0$.*

*Proof.* Let $x,y\in[a,b]$, $x<y$. Then, by the Lagrange mean value theorem, we have that $|f(y)-f(x)|=|f'(\xi)||y-x|\leqslant L|y-x|$, since $\xi\in(x,y)\subset[a,b]$. This means that $f$ is $L$-Lipschitzian on $[a,b]$. The rest of the proof follows from Theorems 2 and 3 due to the convexity of $f$. □

REMARK 2. It should be noticed here that relations (11) and (12) represent reverses of the superadditivity relation (3).

REMARK 3. Let $a,b \in \mathbb{R}$, $0 \leqslant a < b$, and let $f(x) = x^p$, where $p \geqslant 1$. Clearly, $f$ is a differentiable convex function on $[0,\infty)$ and $f(0) = 0$. Moreover, since $f'(t) = pt^{p-1}$ is increasing function, it follows that $L = \sup_{t\in[a,b]} \left|pt^{p-1}\right| = pb^{p-1}$, so in this setting Corollary 1 yields the inequalities

$$0 \leqslant (a+b)^p - a^p - b^p \leqslant \frac{pb^p a}{a+b}$$

and

$$\frac{pb^p a}{a+b} - (a+b)^p + a^p + b^p \geqslant \left| \left|a(a+b)^{p-1} - a^p\right| - \left|b(a+b)^{p-1} - b^p\right| \right|.$$

Our next intention is to extend inequality (5) to a multivariable version. In fact, keeping in mind Remark 1, we obtain a reverse of the Jensen inequality expressed in terms of the Lipschitz constant of the corresponding function.

THEOREM 4. *Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a $L$-Lipschitzian mapping on $J$, and let $a_1, a_2, \ldots, a_n \in J$. If $w_1, w_2, \ldots, w_n$ are non-negative scalars such that $\sum_{i=1}^n w_i = 1$, then holds the inequality*

$$\left| \sum_{i=1}^n w_i f(a_i) - f\left( \sum_{i=1}^n w_i a_i \right) \right| \leqslant L \sum_{i=1}^n w_i \left| a_i - \sum_{j=1}^n w_j a_j \right|. \tag{13}$$

*Proof.* Rewriting the left-hand-side of (13) in a suitable form and then using the triangle inequality, as well as relation (4), we have respectively,

$$\begin{aligned} \left| \sum_{i=1}^n w_i f(a_i) - f\left( \sum_{i=1}^n w_i a_i \right) \right| &= \left| \sum_{i=1}^n w_i \left( f(a_i) - f\left( \sum_{i=1}^n w_i a_i \right) \right) \right| \\ &\leqslant \sum_{i=1}^n w_i \left| f(a_i) - f\left( \sum_{i=1}^n w_i a_i \right) \right| \\ &\leqslant L \sum_{i=1}^n w_i \left| a_i - \sum_{j=1}^n w_j a_j \right|, \end{aligned}$$

and the proof is completed. □

REMARK 4. Since $|\cos y - \cos x| \leqslant |y - x|$, for all $x, y \in \mathbb{R}$, by the Lagrange mean value theorem, it follows that the cosine function is 1-Lipschitzian on $\mathbb{R}$. Hence, in this setting, relation (13) reads

$$\left| \sum_{i=1}^n w_i \cos(a_i) - \cos\left( \sum_{i=1}^n w_i a_i \right) \right| \leqslant \sum_{i=1}^n w_i \left| a_i - \sum_{j=1}^n w_j a_j \right|,$$

where $a_1, a_2, \ldots, a_n$ are arbitrary real numbers.

If $f$ is a convex function, then the modulus function on the left-hand side of (13) is redundant, which represents a reverse of the Jensen inequality. In fact, similarly to Corollary 1, we arrive at the following consequence of Theorem 4.

COROLLARY 2. *Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping, let $a_1, a_2, \dots, a_n \in J$ and $L = \sup_{t \in J} |f'(t)| < \infty$. If $w_1, w_2, \dots, w_n$ are non-negative real numbers such that $\sum_{i=1}^n w_i = 1$, then holds inequality (13). In addition, if $f$ is a convex function, then holds the inequality*

$$0 \leqslant \sum_{i=1}^n w_i f(a_i) - f\left(\sum_{i=1}^n w_i a_i\right) \leqslant L \sum_{i=1}^n w_i \left| a_i - \sum_{j=1}^n w_j a_j \right|. \tag{14}$$

Inequality (14) represents a reverse of the Jensen inequality expressed in terms of the Lipschitzianity of a convex function $f$. We aim now to exploit this relation in establishing reverses of power mean inequalities, which is the focus of the next section.

## 3. Reverses of power mean inequalities

Here, we aim to derive reverse inequalities for power means based on our Theorem 4 and Corollary 2. Recall that a power mean is defined by

$$M_r(\mathbf{x}, \mathbf{w}) = \begin{cases} \left(\sum_{i=1}^n w_i {x_i}^r\right)^{\frac{1}{r}}, & r \neq 0, \\ \prod_{i=1}^n {x_i}^{w_i}, & r = 0, \end{cases}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$ stands for a strictly positive $n$-tuple and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ is a non-negative $n$-tuple such that $\sum_{i=1}^n w_i = 1$. The case of $w_1 = w_2 = \cdots = w_n = \frac{1}{n}$ yields the corresponding non-weighted mean

$$m_r(\mathbf{x}) = \begin{cases} \left(\frac{1}{n} \sum_{i=1}^n {x_i}^r\right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}, & r = 0. \end{cases}$$

The set of all non-negative $n$-tuples $\mathbf{w} = (w_1, w_2, \dots, w_n)$ such that $\sum_{i=1}^n w_i = 1$ will be denoted by $\mathscr{W}_n^+$, for brevity.

Recall that the values $r = -1, 0, 1$, provide the harmonic, geometric, and arithmetic means, respectively. The basic power mean inequality, describing monotonic behavior of means, asserts that if $r < s$, then

$$M_r(\mathbf{x}, \mathbf{w}) \leqslant M_s(\mathbf{x}, \mathbf{w}). \tag{15}$$

This inequality is still of interest to numerous mathematicians. In particular, it has been proved in [6] that

$$\begin{aligned} n \min_{1 \leqslant i \leqslant n} \{w_i\} \left[m_s^s(\mathbf{x}) - m_r^s(\mathbf{x})\right] &\leqslant M_s^s(\mathbf{x}, \mathbf{w}) - M_r^s(\mathbf{x}, \mathbf{w}) \\ &\leqslant n \max_{1 \leqslant i \leqslant n} \{w_i\} \left[m_s^s(\mathbf{x}) - m_r^s(\mathbf{x})\right]. \end{aligned} \tag{16}$$

The left inequality in (16) represents a refinement of (15), while the right inequality provides the corresponding reverse expressed in terms of the corresponding non-weighted means. For a comprehensive study of power means, including refinements and generalizations, the reader is referred to monographs [7, 12], as well as to papers [5, 6] and the references cited therein.

Now, based on our Theorem 4 and Corollary 2, we give a different class of reverses for the basic power mean inequality (15). In order to do this, we have to choose the appropriate parameters in inequality (13).

We first deal with the case when both parameters $r$ and $s$ in (15) are not equal to zero.

COROLLARY 3. *Let $s,r \neq 0$, $\mathbf{x} \in [a,b]^n$, $0 < a < b$, and $\mathbf{w} \in \mathscr{W}_n^+$. Then holds the inequality*

$$\left|M_s^s(\mathbf{x},\mathbf{w}) - M_r^s(\mathbf{x},\mathbf{w})\right| \leqslant \left|\frac{s}{r}\right| \max\{a^{s-r}, b^{s-r}\} \sum_{i=1}^n w_i \left|x_i^r - M_r^r(\mathbf{x},\mathbf{w})\right|, \tag{17}$$

*where*

$$\left|M_s^s(\mathbf{x},\mathbf{w}) - M_r^s(\mathbf{x},\mathbf{w})\right| = \begin{cases} M_s^s(\mathbf{x},\mathbf{w}) - M_r^s(\mathbf{x},\mathbf{w}), & \frac{s}{r} < 0 \text{ or } \frac{s}{r} > 1, \\ M_r^s(\mathbf{x},\mathbf{w}) - M_s^s(\mathbf{x},\mathbf{w}), & 0 < \frac{s}{r} \leqslant 1. \end{cases}$$

*Proof.* We consider (13) with $f(t) = t^{\frac{s}{r}}$, $t > 0$, and with $\mathbf{x}^r = (x_1^r, x_2^r, \ldots, x_n^r)$ instead of $n$-tuple $(a_1, a_2, \ldots, a_n)$. Then $\sum_{i=1}^n w_i f(a_i) = \sum_{i=1}^n w_i x_i^s = M_s^s(\mathbf{x},\mathbf{w})$, $f\left(\sum_{i=1}^n w_i a_i\right) = \left(\sum_{i=1}^n w_i x_i^r\right)^{\frac{s}{r}} = M_r^s(\mathbf{x},\mathbf{w})$, and $\sum_{j=1}^n w_j a_j = \sum_{j=1}^n w_j x_j^r = M_r^r(\mathbf{x},\mathbf{w})$. Moreover, since $\mathbf{x} \in [a,b]^n$, it follows that $\mathbf{x}^r \in [\mu_a, \mu_b]$, where $\mu_a = \min\{a^r, b^r\}$ and $\mu_b = \max\{a^r, b^r\}$. Consequently, it follows that

$$L = \sup_{t \in [\mu_a, \mu_b]} \left|f'(t)\right| = \left|\frac{s}{r}\right| \sup_{t \in [\mu_a, \mu_b]} t^{\frac{s}{r}-1} = \left|\frac{s}{r}\right| \max\{a^{s-r}, b^{s-r}\},$$

since $t^{\frac{s}{r}-1}$ is monotonic on the interval $[\mu_a, \mu_b]$. This yields (17). Clearly, $f$ is convex if $\frac{s}{r} < 0$ or $\frac{s}{r} > 1$, while it is concave for $0 < \frac{s}{r} \leqslant 1$. The proof is now completed. □

We proceed with the consequences of (15) when one of the parameters $r$ and $s$ equals zero. If $s = 0$, we obtain the following result:

COROLLARY 4. *Let $r \neq 0$, $\mathbf{x} \in [a,b]^n$, $0 < a < b$, and $\mathbf{w} \in \mathscr{W}_n^+$. Then holds the inequality*

$$\left|\log \frac{M_r(\mathbf{x},\mathbf{w})}{M_0(\mathbf{x},\mathbf{w})}\right| \leqslant \frac{\max\{a^{-r}, b^{-r}\}}{|r|} \sum_{i=1}^n w_i \left|x_i^r - M_r^r(\mathbf{x},\mathbf{w})\right|, \tag{18}$$

*where*

$$\left|\log \frac{M_r(\mathbf{x},\mathbf{w})}{M_0(\mathbf{x},\mathbf{w})}\right| = \begin{cases} \log \frac{M_r(\mathbf{x},\mathbf{w})}{M_0(\mathbf{x},\mathbf{w})}, & r > 0, \\ \log \frac{M_0(\mathbf{x},\mathbf{w})}{M_r(\mathbf{x},\mathbf{w})}, & r < 0. \end{cases}$$

*Proof.* Similar to the previous corollary, (18) is a consequence of inequality (13) when $f(t) = \frac{1}{r}\log t$, $t > 0$, and when the $n$-tuple $(a_1, a_2, \ldots, a_n)$ is replaced by $\mathbf{x}^r = (x_1^r, x_2^r, \ldots, x_n^r)$. Clearly, $f$ is convex for $r < 0$ and concave for $r > 0$. In this setting, we have that $\sum_{i=1}^{n} w_i f(a_i) = \sum_{i=1}^{n} w_i \log x_i = M_0(\mathbf{x}, \mathbf{w})$ and $f\big(\sum_{i=1}^{n} w_i a_i\big) = \log\big(\sum_{i=1}^{n} w_i x_i^r\big)^{\frac{1}{r}} = \log M_r(\mathbf{x}, \mathbf{w})$. In addition, since $\mathbf{x}^r \in [\mu_a, \mu_b]$, where $\mu_a = \min\{a^r, b^r\}$ and $\mu_b = \max\{a^r, b^r\}$, it follows that

$$L = \sup_{t \in [\mu_a, \mu_b]} \left|f'(t)\right| = \frac{1}{|r|} \sup_{t \in [\mu_a, \mu_b]} \frac{1}{t} = \frac{1}{|r|} \max\{a^{-r}, b^{-r}\},$$

due to monotonicity of function $1/t$ on the interval $[\mu_a, \mu_b]$. This completes the proof. □

REMARK 5. In particular, if $r = 1$, then inequality (18) reduces to

$$\frac{M_1(\mathbf{x}, \mathbf{w})}{M_0(\mathbf{x}, \mathbf{w})} \leqslant \prod_{i=1}^{n} \exp\left(\frac{w_i |x_i - M_1(\mathbf{x}, \mathbf{w})|}{a}\right),$$

providing the reverse of the arithmetic-geometric mean inequality in the so-called quotient form. On the other hand, if $r = -1$, we obtain the corresponding reverse of the geometric-harmonic inequality:

$$\frac{M_0(\mathbf{x}, \mathbf{w})}{M_{-1}(\mathbf{x}, \mathbf{w})} \leqslant \prod_{i=1}^{n} \exp\left(b w_i \left|x_i^{-1} - M_1^{-1}(\mathbf{x}, \mathbf{w})\right|\right).$$

The above remark provided a reverse of the arithmetic-geometric mean inequality in a quotient form. In the sequel, we will also derive a reverse of the arithmetic-geometric mean inequality in a difference form. The corresponding relation will be covered by the case when parameter $r$ is equal to zero.

COROLLARY 5. *Let $s \in \mathbb{R}$, $\mathbf{x} \in [a, b]^n$, $0 < a < b$, and $\mathbf{w} \in \mathscr{W}_n^+$. Then holds the inequality*

$$0 \leqslant M_s^s(\mathbf{x}, \mathbf{w}) - M_0^s(\mathbf{x}, \mathbf{w}) \leqslant |s| \max\{a^s, b^s\} \sum_{i=1}^{n} w_i \left|\log \frac{x_i}{M_0(\mathbf{x}, \mathbf{w})}\right|. \tag{19}$$

*Proof.* Here, the starting point is also relation (13) equipped with $f(t) = e^{st}$ and with $n$-tuple $\log \mathbf{x} = (\log x_1, \log x_2, \ldots, \log x_n)$ instead of $(a_1, a_2, \ldots, a_n)$. Then, it follows that $\sum_{i=1}^{n} w_i f(a_i) = \sum_{i=1}^{n} w_i e^{s \log x_i} = M_s^s(\mathbf{x}, \mathbf{w})$, $f\big(\sum_{i=1}^{n} w_i a_i\big) = e^{s \sum_{i=1}^{n} w_i \log x_i} = M_0^s(\mathbf{x}, \mathbf{w})$, and $\sum_{j=1}^{n} w_j a_j = \sum_{j=1}^{n} w_j \log x_j = \log M_0(\mathbf{x}, \mathbf{w})$. Furthermore, since $\log \mathbf{x} \in [\log a, \log b]$, it follows that

$$L = \sup_{t \in [\log a, \log b]} \left|f'(t)\right| = |s| \sup_{t \in [\log a, \log b]} e^{st} = |s| \max\{a^s, b^s\}.$$

Finally, $f(t) = e^{st}$ is convex for every $s \in \mathbb{R}$, so (20) holds. □

REMARK 6. If $s=1$, then inequality (19) reduces to

$$0 \leqslant M_1(\mathbf{x},\mathbf{w}) - M_0(\mathbf{x},\mathbf{w}) \leqslant b\sum_{i=1}^{n} w_i \left|\log\frac{x_i}{M_0(\mathbf{x},\mathbf{w})}\right|, \tag{20}$$

which represents the reverse of the arithmetic-geometric mean inequality in a difference form. Similarly, if $s=-1$, we obtain the reverse of the geometric-harmonic mean inequality in a difference form:

$$0 \leqslant M_{-1}^{-1}(\mathbf{x},\mathbf{w}) - M_0^{-1}(\mathbf{x},\mathbf{w}) \leqslant \frac{1}{a}\sum_{i=1}^{n} w_i \left|\log\frac{x_i}{M_0(\mathbf{x},\mathbf{w})}\right|.$$

In the next section, relation (20) will be exploited in establishing a reverse of another important inequality.

## 4. Reverse of the Hölder inequality

Let $(\Omega,\Sigma,\mu)$ be a $\sigma$-finite measure space and let $\sum_{i=1}^{n}\frac{1}{r_i}=1$, $r_i>1$. If $f_i \in L^{r_i}(\Omega)$, $i=1,2,\ldots,n$, are non-negative measurable functions, then there holds the inequality

$$\int_\Omega \prod_{i=1}^{n} f_i(x)d\mu(x) \leqslant \prod_{i=1}^{n} \|f_i\|_{r_i}, \tag{21}$$

where $\|f_i\|_{r_i} = \left(\int_\Omega f_i^{r_i}(x)d\mu(x)\right)^{\frac{1}{r_i}}$. It is well known that the Hölder inequality can be proved in several ways, in particular, via the arithmetic-geometric mean inequality, i.e., the Young inequality (for more details, see [7, 12]). Bearing in mind this fact, relation (20) can be exploited in establishing a new reverse of the Hölder inequality (21). However, to derive the corresponding result, we also have to impose some extra conditions on non-negative measurable functions $f_i \in L^{r_i}(\Omega)$, $i=1,2,\ldots,n$.

THEOREM 5. *Let $(\Omega,\Sigma,\mu)$ be a $\sigma$-finite measure space and let $\sum_{i=1}^{n}\frac{1}{r_i}=1$, $r_i>1$, $i=1,2,\ldots,n$. Further, suppose that $f_i \in L^{r_i}(\Omega)$, $i=1,2,\ldots,n$, are non-negative measurable functions such that*

$$0 < f_i(x) \leqslant b^{\frac{1}{r_i}}\|f_i\|_{r_i},\ x\in\Omega,\ i=1,2,\ldots,n, \tag{22}$$

*where $b>0$. Then there holds the inequality*

$$\begin{aligned}&\prod_{i=1}^{n}\|f_i\|_{r_i} - \int_\Omega \prod_{i=1}^{n} f_i(x)d\mu(x)\\ &\leqslant b\prod_{i=1}^{n}\|f_i\|_{r_i}\sum_{i=1}^{n}\frac{1}{r_i}\int_\Omega\left|\log\left(\frac{f_i^{r_i-1}(x)}{\|f_i\|_{r_i}^{r_i-1}}\prod_{j=1,j\neq i}^{n}\frac{\|f_j\|_{r_j}}{f_j(x)}\right)\right|d\mu(x).\end{aligned} \tag{23}$$

*Proof.* Rewriting (20) with $(\frac{1}{r_1}, \frac{1}{r_2}, \ldots, \frac{1}{r_n})$ instead of the $n$-tuple $\mathbf{w}$, we arrive at the corresponding Young form of inequality:

$$\sum_{i=1}^{n} \frac{x_i}{r_i} - \prod_{i=1}^{n} x_i^{\frac{1}{r_i}} \leqslant b \sum_{i=1}^{n} \frac{1}{r_i} \Big| \log \Big( x_i^{1-\frac{1}{r_i}} \prod_{j=1, j\neq i}^{n} x_j^{-\frac{1}{r_j}} \Big) \Big|.$$

The next step is to substitute $\frac{f_i^{r_i}(x)}{\|f_i\|_{r_i}^{r_i}}$, $x \in \Omega$, instead of $x_i$, $i = 1, 2, \ldots, n$, in the above inequality. Of course, this can be done since $\frac{f_i^{r_i}(x)}{\|f_i\|_{r_i}^{r_i}} \in (0, b]$, due to (22). Consequently, we have that

$$\begin{aligned} &\sum_{i=1}^{n} \frac{f_i^{r_i}(x)}{r_i \|f_i\|_{r_i}^{r_i}} - \prod_{i=1}^{n} \frac{f_i(x)}{\|f_i\|_{r_i}} \\ &\leqslant b \sum_{i=1}^{n} \frac{1}{r_i} \left| \log \left( \frac{f_i^{r_i-1}(x)}{\|f_i\|_{r_i}^{r_i-1}} \prod_{j=1, j\neq i}^{n} \frac{\|f_j\|_{r_j}}{f_j(x)} \right) \right|. \end{aligned}$$

We proceed with integrating the above inequality over $\Omega$ with respect to the measure $\mu$. More precisely, since $\int_\Omega f_i^{r_i}(x) d\mu(x) = \|f_i\|_{r_i}^{r_i}$, we have that

$$\begin{aligned} &\sum_{i=1}^{n} \frac{1}{r_i} - \frac{\int_\Omega \prod_{i=1}^{n} f_i(x) d\mu(x)}{\prod_{i=1}^{n} \|f_i\|_{r_i}} \\ &\leqslant b \sum_{i=1}^{n} \frac{1}{r_i} \int_\Omega \left| \log \left( \frac{f_i^{r_i-1}(x)}{\|f_i\|_{r_i}^{r_i-1}} \prod_{j=1, j\neq i}^{n} \frac{\|f_j\|_{r_j}}{f_j(x)} \right) \right| d\mu(x), \end{aligned}$$

that is,

$$\begin{aligned} &1 - \frac{\int_\Omega \prod_{i=1}^{n} f_i(x) d\mu(x)}{\prod_{i=1}^{n} \|f_i\|_{r_i}} \\ &\leqslant b \sum_{i=1}^{n} \frac{1}{r_i} \int_\Omega \left| \log \left( \frac{f_i^{r_i-1}(x)}{\|f_i\|_{r_i}^{r_i-1}} \prod_{j=1, j\neq i}^{n} \frac{\|f_j\|_{r_j}}{f_j(x)} \right) \right| d\mu(x), \end{aligned}$$

since $\sum_{i=1}^{n} \frac{1}{r_i} = 1$. Clearly, the last inequality is equivalent to (23), which completes the proof. □

REMARK 7. In particular, if $n = 2$, $r_1 = r_2 = 2$, $f_1 = f$, $f_2 = g$, inequality (23) reduces to

$$\|f\|\|g\| - \int_\Omega f(x)g(x)d\mu(x) \leqslant b\|f\|\|g\| \int_\Omega \left| \log \left( \frac{f(x)}{g(x)} \frac{\|g\|}{\|f\|} \right) \right| d\mu(x), \qquad (24)$$

provided that $0 < f(x) \leqslant \sqrt{b}\|f\|$ and $0 < g(x) \leqslant \sqrt{b}\|g\|$, $x \in \Omega$. Here, $\|\cdot\|$, stands for the usual $L^2$ norm, i.e $\|f\| = \sqrt{\int_\Omega f^2(x) d\mu(x)}$. Clearly, relation (24) represents the reverse of the celebrated Cauchy-Schwarz inequality.

## 5. Further extensions, remarks and applications

The main topic in this section is our basic relation (5) that describes a reverse of convexity in terms of Lipschitzianity. We aim here to extend it in several different ways.

Inequality (5) holds for all $\nu \in [0,1]$. Our first goal is to derive its external version. More precisely, we are going to derive a version of (5) that holds for values of $\nu$ not belonging to the unit interval. In fact, the proof is very familiar to Theorem 1.

THEOREM 6. *Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be $L$-Lipschitzian mapping on $J$ and let $\nu \notin [0,1]$. If $a, b, (1-\nu)a + \nu b \in J$, then holds the inequality*

$$|(1-\nu) f(a) + \nu f(b) - f((1-\nu)a + \nu b)| \leqslant 2L|\nu(1-\nu)|\,|b-a|. \tag{25}$$

*Proof.* Utilizing the triangle inequality and relation (4), we have that

$$\begin{aligned}
&|(1-\nu) f(a) + \nu f(b) - f((1-\nu)a + \nu b)| \\
&= |(1-\nu)(f(a) - f((1-\nu)a + \nu b)) + \nu (f(b) - f((1-\nu)a + \nu b))| \\
&\leqslant |1-\nu|\,|f(a) - f((1-\nu)a + \nu b)| + |\nu|\,|f(b) - f((1-\nu)a + \nu b)| \\
&\leqslant 2L|\nu|\,|1-\nu|\,|b-a|,
\end{aligned}$$

so (25) holds. □

REMARK 8. Consider the function $f(x) = x^{2k+1}$, $k \in \mathbb{N}$, on $[a,b] \subset \mathbb{R}$. Since $f'(x) = (2k+1)x^{2k}$, utilizing the Lagrange mean value theorem, it follows that $f$ is $L$-Lipschitzian on $[a,b]$, with $L = (2k+1)\max\left\{a^{2k}, b^{2k}\right\}$. Moreover, if $\alpha, \beta \in [a,b]$ and $\nu \notin [0,1]$ are such that $(1-\nu)\alpha + \nu\beta \in [a,b]$, then holds the inequality

$$\begin{aligned}
&\left|(1-\nu)\alpha^{2k+1} + \nu\beta^{2k+1} - ((1-\nu)\alpha + \nu\beta)^{2k+1}\right| \\
&\leqslant 2(2k+1)\max\left\{a^{2k}, b^{2k}\right\}|\nu(1-\nu)|(\beta-\alpha),
\end{aligned}$$

due to (25).

According to our discussion in Section 2, we also give a differentiable and convex version of the previous theorem.

COROLLARY 6. *Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex mapping and let $L = \sup_{t \in J} |f'(t)| < \infty$. If $a, b \in J$ and $\nu \notin [0,1]$ are such that $(1-\nu)a + \nu b \in J$, then holds the inequality*

$$0 \leqslant f((1-\nu)a + \nu b) - (1-\nu) f(a) - \nu f(b) \leqslant 2L|\nu(1-\nu)|\,|b-a|.$$

Note that inequality (5) includes convex combination $(1-\nu)a + \nu b$, provided that $0 \leqslant \nu \leqslant 1$. Therefore, it can be rewritten as

$$|w_1 f(a_1) + w_2 f(a_2) - f(w_1 a_1 + w_2 a_2)| \leqslant 2L w_1 w_2 |a_1 - a_2|, \tag{26}$$

provided that $w_1, w_2 > 0$ are such that $w_1 + w_2 = 1$. We aim now to establish a variant of (26) in the case when $w_1 + w_2 \leqslant 1$. To do this, we need to impose an additional condition on the corresponding Lipschitzian function.

THEOREM 7. *Let $J \subseteq \mathbb{R}$ be an interval such that $0 \in J$, and let $a_1, a_2 \in J$. Further, let $f : J \to \mathbb{R}$ be $L$-Lipschitzian mapping on $J$ such that $f(0) = 0$. If $w_1, w_2 > 0$ are such that $w_1 + w_2 \leqslant 1$, then holds the inequality*

$$\begin{aligned} &|w_1 f(a_1) + w_2 f(a_2) - f(w_1 a_1 + w_2 a_2)| \\ &\leqslant 2L\Big( (1 - (w_1 + w_2))\, |w_1 a_1 + w_2 a_2| + \frac{w_1 w_2}{w_1 + w_2} |a_1 - a_2| \Big). \end{aligned} \tag{27}$$

*Proof.* By putting $\nu = w_1 + w_2$ and $x = \frac{w_1}{w_1 + w_2} \cdot a_1 + \frac{w_2}{w_1 + w_2} \cdot a_2$ in relation (6), we have that

$$\begin{aligned} &\left| (w_1 + w_2) f\left( \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right) - f(w_1 a_1 + w_2 a_2) \right| \\ &\leqslant 2L(w_1 + w_2)(1 - (w_1 + w_2)) \left| \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right| \\ &= 2L(1 - (w_1 + w_2))\, |w_1 a_1 + w_2 a_2|. \end{aligned} \tag{28}$$

On the other hand, considering (26) with $\frac{w_1}{w_1 + w_2}$ and $\frac{w_2}{w_1 + w_2}$, instead of $w_1$ and $w_2$, respectively, we obtain

$$\begin{aligned} &\left| \frac{w_1}{w_1 + w_2} f(a_1) + \frac{w_2}{w_1 + w_2} f(a_2) - f\left( \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right) \right| \\ &\leqslant L\left( \frac{w_1}{w_1 + w_2} \left| a_1 - \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right| + \frac{w_2}{w_1 + w_2} \left| a_2 - \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right| \right), \end{aligned}$$

that is,

$$\begin{aligned} &\left| w_1 f(a_1) + w_2 f(a_2) - (w_1 + w_2) f\left( \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right) \right| \\ &\leqslant L\left( w_1 \left| a_1 - \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right| + w_2 \left| a_2 - \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right| \right), \end{aligned} \tag{29}$$

after multiplying the previous relation by $w_1 + w_2$ both-sided. Now, utilizing the triangle inequality, as well as relations (28) and (29), we have that

$$\begin{aligned} &|w_1 f(a_1) + w_2 f(a_2) - f(w_1 a_1 + w_2 a_2)| \\ &\leqslant \left| (w_1 + w_2) f\left( \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right) - f(w_1 a_1 + w_2 a_2) \right| \\ &\qquad + \left| w_1 f(a_1) + w_2 f(a_2) - (w_1 + w_2) f\left( \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right) \right| \\ &\leqslant 2L(1 - (w_1 + w_2))\, |w_1 a_1 + w_2 a_2| \\ &\qquad + L\left( w_1 \left| a_1 - \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right| + w_2 \left| a_2 - \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \right| \right). \end{aligned} \tag{30}$$

The last inequality is obviously equivalent to (27), which completes the proof. □

It should be noticed here that if $w_1+w_2=1$, then inequality (27) reduces to (26). Furthermore, if $f$ is convex, then the modulus function on the left-hand side of (27) is redundant. This can be easily deduced from the first inequality in (30). Therefore, a differentiable convex version of the previous theorem reads as follows:

COROLLARY 7. *Let $J\subseteq\mathbb{R}$ be an interval such that $0\in J$, and let $a_1,a_2\in J$. Further, let $f:J\to\mathbb{R}$ be a differentiable convex function such that $f(0)=0$ and $L=\sup_{t\in[a_1,a_2]}|f'(t)|<\infty$. Then,*

$$\begin{aligned}0&\leqslant w_1f(a_1)+w_2f(a_2)-f(w_1a_1+w_2a_2)\\&\leqslant 2L\Big((1-(w_1+w_2))\,|w_1a_1+w_2a_2|+\frac{w_1w_2}{w_1+w_2}\,|a_1-a_2|\Big),\end{aligned}$$

*provided that $w_1+w_2\leqslant 1$ and $w_1,w_2>0$.*

Our next goal is to establish a refinement of inequality (5). To do this, we recall a more precise relation that holds for convex functions. It is well known (see, e.g. [2] or [6]) that if $f:J\to\mathbb{R}$ is a convex function and $a,b\in J$, then for any $0\leqslant\nu\leqslant 1$ holds the inequality

$$f((1-\nu)a+\nu b)\leqslant(1-\nu)f(a)+\nu f(b)-2r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right), \tag{31}$$

where $r=\min\{\nu,1-\nu\}$. Clearly, this inequality represents the refinement of the Jensen inequality. In fact, the left inequality in (16) is a consequence of (31) (for more details, see [6]). Further, since the modulus function is convex on $\mathbb{R}$, we infer that

$$|(1-\nu)a+\nu b|\leqslant(1-\nu)|a|+\nu|b|-r(|a|+|b|-|a+b|), \tag{32}$$

for all $a,b\in\mathbb{R}$ and $0\leqslant\nu\leqslant 1$. It should be noticed here that (32) represents a more accurate triangle inequality. This relation will be crucial in establishing refinement of (5).

THEOREM 8. *Let $J\subseteq\mathbb{R}$ be an interval such that $a,b\in J$, and let $f:J\subseteq\mathbb{R}\to\mathbb{R}$ be a $L$-Lipschitzian mapping on $J$. Then the inequality*

$$\begin{aligned}&|(1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b)|\\&\leqslant 2L(1-\nu)\nu|b-a|-r\Big(|f(a)-f((1-\nu)a+\nu b)|\\&\qquad+|f(b)-f((1-\nu)a+\nu b)|-|f(a)+f(b)-2f((1-\nu)a+\nu b)|\Big),\end{aligned} \tag{33}$$

*where $r=\min\{\nu,1-\nu\}$, holds for all $0\leqslant\nu\leqslant 1$.*

*Proof.* By virtue of the improved triangle inequality (32) and relation (4), it follows that

$$\begin{aligned}
&|(1-v)f(a)+vf(b)-f((1-v)a+vb)| \\
&=|(1-v)(f(a)-f((1-v)a+vb))+v(f(b)-f((1-v)a+vb))| \\
&\leqslant (1-v)|f(a)-f((1-v)a+vb)|+v|f(b)-f((1-v)a+vb)| \\
&\quad -r\big(|f(a)-f((1-v)a+vb)|+|f(b)-f((1-v)a+vb)| \\
&\qquad\qquad -|f(a)+f(b)-2f((1-v)a+vb)|\big) \\
&\leqslant 2L(1-v)v|b-a|-r\big(|f(a)-f((1-v)a+vb)|+|f(b)-f((1-v)a+vb)| \\
&\qquad\qquad -|f(a)+f(b)-2f((1-v)a+vb)|\big),
\end{aligned}$$

so (33) holds. □

REMARK 9. In particular, if $0\in J$ and $f(0)=0$, then (33) implies the inequality

$$|vf(x)-f(vx)|\leqslant 2L(1-v)v|x|-r\big(|f(vx)|+|f(x)-f(vx)|-|f(x)-2f(vx)|\big),$$

that holds for any $x\in J$ and $0\leqslant v\leqslant 1$.

REMARK 10. According to our discussion in Section 2, a differentiable and convex version of Theorem 8 follows easily. Moreover, taking into account the multivariable version of relation (31) (see, e.g., [2] or [6]), one can establish a refinement of inequality (13) that corresponds to (33). These results are omitted here and left to the reader.

In order to conclude this paper, we will exploit (5) in establishing a reverse of another celebrated inequality. The Hermite-Hadamard inequality (see, e.g., [7] or [12]) asserts that if $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is a convex function, then there holds the following double inequality:

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{f(a)+f(b)}{2}. \tag{34}$$

By virtue of (5), we are able to derive a reverse of the right inequality in (34).

THEOREM 9. *Let $J\subseteq\mathbb{R}$ be an interval such that $a,b\in J$, and let $f:J\subseteq\mathbb{R}\to\mathbb{R}$ be a $L$-Lipschitzian mapping on $J$. Then there holds the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leqslant\frac{L}{3}|b-a|. \tag{35}$$

*Proof.* Integrating relation (5) over the unit integral, with respect to variable $v$, we have that

$$\begin{aligned}
\int_0^1|(1-v)f(a)+vf(b)-f((1-v)a+vb)|dv &\leqslant 2L|b-a|\int_0^1(1-v)vdv \\
&=\frac{L}{3}|b-a|.
\end{aligned}$$

On the other hand, since

$$\int_0^1 ((1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b))\,d\nu = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx,$$

we have that

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \int_0^1 |(1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b)|\,d\nu,$$

due to the triangle inequality for integrals. Clearly, the previous two inequalities yield (35), as claimed. □

Of course, if $f$ is a convex function, then the modulus function on the left-hand side of (35) is redundant, so we obtain the reverse of the right inequality in (34).

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