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CURVATURE ESTIMATES OF A SPACELIKE GRAPH IN A LORENTZIAN PRODUCT SPACE

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Abstract. Let M be an n-dimensional complete Riemannian manifold with the metric $\langle , \rangle_M$ and let $M \times \mathbb{R}_1$ be a Lorentzian product space $M \times \mathbb{R}$ with the metric $\langle , \rangle_M - dt^2$. We first provide Heinz type curvature estimates for the spacelike graph in $M \times \mathbb{R}_1$ of a C^2-function f defined on a closed geodesic ball $\overline{B}_{x_0}(R)$ of radius R centered at x_0 on M. In particular, the estimates are related to the radius R and the value of $\|\nabla f(x_1)\|$ for which $f(x_1) = \max_{\partial \overline{B}_{x_0}(R)} f$. Secondly, we give L^2-estimates of the mean curvature for a spacelike graph defined on a compact Riemannian manifold.

1. Introduction

Curvature estimate for a graphical surface, which is a non-parametric surface, is a natural consideration to analyze the surface. However, a graph with no geometric condition may have curvature as high as desired at some point on a bounded domain although the height is generally bounded on the domain. The following results are related to the estimates of infimums of curvatures for the graph with no geometric condition: Heinz [9] obtained the following estimates for the mean curvature H and Gaussian curvature K of the graph of a function defined on an open disk $D_{x_0}(r)$ of radius r centered at x_0 on $\mathbb{R}^2$:

$$\inf |H| \leqslant \frac{1}{r},$$
$$\inf |K| \leqslant \frac{3e^2}{r^2},$$

where e is the natural number. Chern [3] and Flanders [5] proved independently the above inequalities to higher dimensions. Salavessa [13] extended the above result for the graph of a smooth function $f : M \to N$ to a product space $(M \times N, g \times -h)$ of two Riemannian manifolds (M, g) and (N, h). Coswosck and Fontenele [4] provided similar curvature estimates for graphs on a Riemannian domain according to the infimum of the Ricci curvature of the domain. In particular, Honda, Kawakami, Koiso and Tori [10] provided Heinz-type estimates for spacelike and timelike graphs on a relatively compact domain in the Minkowski space, which are related to the maximum value for the

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norm of the gradient of the function on the domain. We can find related results [2, 6] and the references therein.

In this paper, we mainly consider a spacelike graph over a Riemannian manifold in a Lorentzian product space based on the Riemannian manifold, which is a natural way to construct a spacetime based on a Riemannian manifold. Let M be an n-dimensional Riemannian manifold with the metric $\langle , \rangle_M$ and $M \times \mathbb{R}_1$ be a Lorentzian product space $M \times \mathbb{R}$ with the non-degenerate bilinear symmetirc form

$$\langle , \rangle = \langle , \rangle_M - dt^2.$$

Typical examples are an $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$ and homogeneous product spaces $\mathbb{H}^n \times \mathbb{R}_1$ and $\mathbb{S}^n \times \mathbb{R}_1$ where $\mathbb{H}^n$ and $\mathbb{S}^n$ are n-dimensional hyperbolic space and sphere, respectively. The spacelike graph Γ_f of a function f on $D \subset M$ is defined by the following set:

$$\Gamma_f = \{(x, f(x)) \in M \times \mathbb{R}_1 \mid x \in D \subset M \text{ and } \|\nabla f\| < 1\},$$

where ∇ is the gradient on M. We denote by the sectional curvature, Ricci curvature and scalar curvature of M by K_M, Ric^M and K_M, respectively, and also denote the mean curvature, Ricci curvature, scalar curvature and the second fundamental form of Γ_f by H, Ric, K and B, respectively.

We organize this paper into two parts in terms of curvature estimates for Γ_f: one is Heinz type estimates for Γ_f on a closed geodesic ball in M and the other is L^2-estimates of the mean curvature of a spacelike graph in $M \times \mathbb{R}_1$ where M is a compact manifold using Poincaré and Sobolev inequalities.

In Section 2, we deal with the following Heinz type estimates for H, K and $|B|$ of Γ_f in $M \times \mathbb{R}_1$:

THEOREM 1. (Theorems 7 and 8) *Let $M \times \mathbb{R}_1$ be a Lorentzian product space based on an n-dimensional complete Riemannian manifold M and $\overline{B}_{x_0}(R)$ a closed geodesic ball of radius R centered at x_0 in M with $C = \max_{\overline{B}_{x_0}(R)} K_M$ and $c = \min_{\overline{B}_{x_0}(R)} K_M$. Suppose that there is a point x_1 on $\partial \overline{B}_{x_0}(R)$ such that $f(x_1) = \max_{\partial \overline{B}_{x_0}(R)} f$ and $\|\nabla f(x_1)\| = \xi$ for a function $f \in C^2(\overline{B}_{x_0}(R))$. Then, the spacelike graph Γ_f in $M \times \mathbb{R}_1$ satisfies*

1. $c < 0$

$$\min_{\Gamma_f} |H| \leqslant \frac{n\xi\sqrt{-c}}{\sqrt{1-\xi^2}} \coth(\sqrt{-c}R),$$

$$\min_{\Gamma_f} |\mathrm{K}| \leqslant \frac{2c(2n-1)(n-1)\xi^2}{1-\xi^2} \coth^2(\sqrt{-c}R) + n(n-1)|C|.$$

2. $c \geqslant 0$

$$\min_{\Gamma_f} |H| \leqslant \frac{n\xi}{R\sqrt{1-\xi^2}},$$

$$\min_{\Gamma_f} |\mathrm{K}| \leqslant \frac{2(2n-1)(n-1)\xi^2}{R^2(1-\xi^2)} + n(n-1)C.$$

In particular, the above inequalities hold if $\xi = \max_{\overline{B}_{x_0}(R)} \|\nabla f\|$.

THEOREM 2. (Theorem 9) *Let* $M \times \mathbb{R}_1$ *be a Lorentzian product space based on an* n*-dimensional complete Riemannian manifold* M *and* $\overline{B}_{x_0}(R)$ *a closed geodesic ball of radius* R *centered at* x_0 *in* M *with* $c = \min_{\overline{B}_{x_0}(R)} K_M$. *Suppose that there is a point* x_1 *on* $\partial\overline{B}_{x_0}(R)$ *such that* $f(x_1) = \max_{\partial\overline{B}_{x_0}(R)} f$ *and* $\|\nabla f(x_1)\| = \xi$ *for a function* $f \in C^2(\overline{B}_{x_0}(R))$. *Then, the spacelike graph* Γ_f *in* $M \times \mathbb{R}_1$ *with Ric* $< (n-1)c$ *satisfies*

$$\min_{\Gamma_f} |B| \leqslant 3(n-2)\frac{\xi}{\sqrt{1-\xi^2}}\sqrt{-c}\coth(\sqrt{-c}R), \qquad c < 0,$$

$$\min_{\Gamma_f} |B| \leqslant 3(n-2)\frac{\xi}{R\sqrt{1-\xi^2}}, \qquad c > 0.$$

In particular, the above inequalities hold if $\xi = \max_{\overline{B}_{x_0}(R)} \|\nabla f\|$.

Poincaré and Sobolev inequalities are important inequalities to study the partial differential equation theory, which are related to a function and the norm of its gradient on a domain. In Section 3, let M be a compact manifold and then we obtain three L^2-estimates for the mean curvature H of Γ_f in $M \times \mathbb{R}_1$ from Poincaré and Sobolev inequalities on M.

THEOREM 3. (Theorem 10) *Let* $M \times \mathbb{R}_1$ *be a Lorentzian product space based on an* n*-dimensional compact Riemannian manifold* M *and* Γ_f *the spacelike graph of* $f \in C^2(M)$ *with the mean curvature* H *in* $M \times \mathbb{R}_1$. *Then,* f *satisfies*

$$\frac{1}{c_M^2}\|f - \overline{f}\|_{L^2}^2 \leqslant \|f\|_{L^2}\|H\|_{L^2} + \int_{\partial M}\left\langle \frac{f\nabla f}{\sqrt{1-\|\nabla f\|^2}}, \eta \right\rangle_M ds,$$

where c_M *is a positive constant related to* n *and* M, *namely, Poincaré constant, and* η *is the outward unit normal vector of* ∂M.

The following theorems are related to L^2-norm of the mean curvature of spacelike graphs defined on n-dimensional compact Riemannian manifolds with no boundary or with a smooth boundary. We denote w_n by the volume of the n-dimensional unit sphere in $\mathbb{R}^{n+1}$.

THEOREM 4. (Theorem 11) *Let* $M \times \mathbb{R}_1$ *be a Lorentzian product space based on an* n*-dimensional compact Riemannian manifold* M, $n \geqslant 3$, *with no boundary and* Γ_f *the spacelike graph of* $f \in C^2(M)$ *with the mean curvature* H *in* $M \times \mathbb{R}_1$ *and let* $\frac{1}{p} = \frac{1}{2} - \frac{1}{n}$. *Then, there exists a positive number* A *such that*

$$\|f\|_{L^p}^2 \leqslant \frac{4}{n(n-2)w_n^{\frac{2}{n}}}\|f\|_{L^2}\|H\|_{L^2} + A\|f\|_{L^2}^2.$$

THEOREM 5. (Theorem 12) *Let $M\times\mathbb{R}_1$ be a Lorentzian product space based on an n-dimensional compact Riemannian manifold M, $n\geqslant 3$, with a smooth boundary and Γ_f the spacelike graph of $f\in C^2(M)$ with the mean curvature H in $M\times\mathbb{R}_1$ and let $q=\frac{2(n-1)}{n-2}$. Then, there exists a positive number B such that*

$$\|f\|^2_{L^q(\partial M)}\leqslant\frac{2}{(n-2)w_{n-1}^{\frac{1}{n-1}}}\left(\|f\|_{L^2}\|H\|_{L^2}+\int_{\partial M}\left\langle\frac{f\nabla f}{\sqrt{1-\|\nabla f\|^2}},\eta\right\rangle ds\right)+B\|f\|^2_{L^2(\partial M)}.$$

2. Heinz type curvature estimates of the graph of f

We consider that an n-dimensional Riemannian manifold M with the metric $\langle,\rangle_M$ and an $(n+1)$-dimensional Lorentzian product space $M\times\mathbb{R}_1$ with the non-degenerate bilinear symmetirc form $\langle,\rangle=\langle,\rangle_M-dt^2$. More precisely, let $M\times\mathbb{R}$ be a product set of M and $\mathbb{R}$. Let $\pi_1:M\times\mathbb{R}\to M$ and $\pi_2:M\times\mathbb{R}\to\mathbb{R}$ be the projection maps defined by $\pi_1(x,t)=x$ and $\pi_2(x,t)=t$. We consider a metric $\langle,\rangle=\langle,\rangle_M-dt^2$ on $M\times\mathbb{R}$ as follows: for all $(x,t)\in M\times\mathbb{R}$ and all $v,w\in T_{(x,t)}(M\times\mathbb{R})$, the metric $\langle,\rangle$ is defined by

$$\langle v,w\rangle=\langle d\pi_1(v),d\pi_1(w)\rangle_M-dt^2(d\pi_2(v),d\pi_2(w)).$$

The tilde over a letter is used to denote its lift to $M\times\mathbb{R}_1$. Let f be a C^2-function defined on M and X be the non-parametric form of the graph Γ_f of f in $M\times\mathbb{R}_1$, namely, $X(x)=(x,f(x))$ where $x\in M$. It follows that

$$dX_p(v)=\tilde{v}+\langle\nabla f(p),v\rangle_M\frac{\partial}{\partial t},$$

where ∇ is the gradient on M. Then, we have

$$\|dX_p(v)\|^2=\|\tilde{v}\|^2-\langle\nabla f(p),v\rangle^2_M, \tag{1}$$

Let ν be a unit normal vector field to Γ_f:

$$\begin{aligned}\nu&=\frac{1}{\tilde{W}}\left(\tilde{\nabla}f+\frac{\partial}{\partial t}\right),\\ W&=\sqrt{|\|\nabla f\|^2-1|}=\sqrt{1-\|\nabla f\|^2}.\end{aligned}$$

The second fundamental form B of Γ_f and the Hessian ∇^2 on M are defined as follows: for $v,w\in T_pM$ and $f\in C^2(M)$,

$$\begin{aligned}\langle B(dX_p(v),dX_p(w)),\nu\rangle&=-\frac{1}{W(p)}(\nabla^2 f)_p(v,w), &(2)\\ (\nabla^2 f)_p(v,w)&=\langle\nabla_v\nabla f,w\rangle.\end{aligned}$$

Let $\{e_i\}$ be an orthonormal basis of Γ_f in $M\times\mathbb{R}_1$ and $\{v_i\}$ be a basis on $T_{\tilde{p}}M$ such that $dX_p(v_i)=e_i$. The principal curvatures λ_i for $i=1,\dots,n$ and the mean curvature

H of Γ_f at $\tilde{p}$ are

$$\begin{aligned}\lambda_i(\tilde{p}) &= -\langle \tilde{\nabla}_{e_i} e_i, \nu\rangle = -\langle B(e_i,e_i),\nu\rangle = \frac{1}{W(p)}(\nabla^2 f)_p(v_i,v_i),\\ H(\tilde{p}) &= -\sum_{i=1}^{n}\langle \tilde{\nabla}_{e_i} e_i, \nu\rangle = -\sum_{i=1}^{n}\langle B(e_i,e_i),\nu\rangle = \sum_{i=1}^{n}\frac{1}{W(p)}(\nabla^2 f)_p(v_i,v_i).\end{aligned}$$

LEMMA 1. *Let M be a complete Riemannian manifold and $\overline{B}_{x_0}(R)$ the closed geodesic ball of radius R centered at x_0 in M. Let $s(\cdot) = \mathrm{dist}_M(\cdot, x_0)$ be the distance on M from x_0 and $h : \overline{B}_{x_0}(R) \to \mathbb{R}$ the function defined by*

$$h(x) = \sqrt{s^2(x) + r^2},$$

where r is a positive number. Then, the followings hold:

1. *The gradient of h at $x \in \overline{B}_{x_0}(R)$ is*

$$\nabla h(x) = \begin{cases} 0, & x = x_0,\\ \dfrac{s(x)}{\sqrt{r^2 + s^2(x)}}\nabla s(x), & x \in \overline{B}_{x_0}(R).\end{cases}$$

2. *The Hessian of h at $x \in \overline{B}_{x_0}(R)$ for a unit vector $v \in T_xM$ is*

$$(\nabla^2 h)_x(v,v) = \begin{cases} \dfrac{1}{r}, & x = x_0,\\ \dfrac{r^2}{(r^2+s^2(x))^{\frac{3}{2}}}\langle \nabla s(x), v\rangle^2 + \dfrac{s(x)}{\sqrt{r^2+s^2(x)}}(\nabla^2 s)_x(v,v), & x \in \overline{B}_{x_0}(R).\end{cases}$$

Proof. Let v be a unit tangent vector on $T_{x_0}M$ and $\gamma : (-\varepsilon, \varepsilon) \to M$ be the geodesic with $\gamma(0) = x_0$ and $\gamma'(0) = v$. Then, we have $s(\gamma(t)) = t$ and

$$\begin{aligned}\langle \nabla s(x_0), v\rangle &= \frac{d}{dt}s(\gamma(t))\Big|_{t=0} = \frac{d}{dt}t\Big|_{t=0} = 1,\\ (\nabla^2 s)_{x_0}(v,v) &= \frac{d^2}{dt^2}s(\gamma(t))\Big|_{t=0} = \frac{d^2}{dt^2}t\Big|_{t=0} = 0.\end{aligned}$$

By direct computation, we have

$$\nabla h = \frac{s}{\sqrt{s^2+r^2}}\nabla s, \tag{3}$$

$$\begin{aligned}\nabla_v \nabla h &= v\left(\frac{s}{\sqrt{s^2+r^2}}\right)\nabla s + \frac{s}{\sqrt{s^2+r^2}}\nabla_v\nabla s\\ &= \frac{r^2}{(r^2+s^2)^{\frac{3}{2}}}\langle \nabla s, v\rangle \nabla s + \frac{s}{\sqrt{r^2+s^2}}\nabla_v \nabla s,\end{aligned}$$

$$(\nabla^2 h)(v,v) = \frac{r^2}{(r^2+s^2)^{\frac{3}{2}}}\langle \nabla s, v\rangle^2 + \frac{s}{\sqrt{r^2+s^2}}(\nabla^2 s)(v,v). \tag{4}$$

Since $s(x) = \mathrm{dist}_M(x_0,x)$, the equations (3) and (4) at $x = x_0$ yield

$$\begin{aligned}\nabla h(x_0) &= \frac{s(x_0)}{\sqrt{s^2(x_0)+r^2}}\nabla s(x_0) = 0,\\ (\nabla^2 h)_{x_0}(v,v) &= \frac{r^2}{(r^2+s^2(x_0))^{\frac{3}{2}}}\langle \nabla s(x_0), v\rangle^2 + \frac{s(x_0)}{\sqrt{r^2+s^2(x_0)}}(\nabla^2 s)_{x_0}(v,v)\\ &= \frac{1}{r}. \quad \square\end{aligned}$$

LEMMA 2. *Let M be a complete Riemannian manifold and $\overline{B}_{x_0}(R)$ the closed geodesic ball of radius R centered at x_0 in M. Suppose that there is a point x_1 such that $f(x_1) = \max_{\partial \overline{B}_{x_0}(R)} f$ and $\|\nabla f(x_1)\| = \xi$ for a function $f \in C^2(\overline{B}_{x_0}(R))$ and $g : \overline{B}_{x_0}(R) \to \mathbb{R}$ is a function defined by*

$$g(x) = \begin{cases} \sqrt{s^2(x)+r^2}, & \xi \neq 0,\\ 0, & \xi = 0,\end{cases}$$

where $r = R\frac{\sqrt{1-\xi^2}}{\xi}$. Then, there exist a point $x \in \overline{B}_{x_0}(R)$ and a constant r such that $(g-f)(x) = \min_{\overline{B}_{x_0}(R)}(g-f)$ and $\|\nabla(g-f)(x)\| = 0$.

Proof. Since $\partial \overline{B}_{x_0}(R)$ is compact, there is a point $x_1 \in \partial \overline{B}_{x_0}(R)$ such that $f(x_1) = \max_{\partial \overline{B}_{x_0}(R)} f$ and $\|\nabla f(x_1)\| = \xi$. We define a function $g_t : \overline{B}_{x_0}(R) \to \mathbb{R}$ by

$$g_t(x) = \begin{cases} \sqrt{s^2(x)+r^2}+t, & \xi \neq 0,\\ t, & \xi = 0,\end{cases}$$

where $r = R\frac{\sqrt{1-\xi^2}}{\xi}$. Let Γ_f be the spacelike graph of a function $f \in C^2(\overline{B}_{x_0}(R))$. For a sufficiently large t, the graph Γ_{g_t} of g_t is contained in the above component of $(\overline{B}_{x_0}(R) \times \mathbb{R}) \setminus \Gamma_f$. If t decreases until that Γ_{g_t} touches Γ_f at a first touching point, say $t = t_0$ and $\tilde{p} = (p, f(p)) \in \Gamma_f$, then p is a minimum point of $g - f$, namely, $(g-f)(p) = \min_{\overline{B}_{x_0}(R)}(g-f)$. It is possible to occur that the first touching point $\tilde{p}$ is on the interior or the boundary of Γ_f.

We first assume that $\tilde{p}$ is on the interior of Γ_f. Since t_0 is a maximum value such that $(g_{t_0} - f)(p) = 0$, the tangent space $T_{\tilde{p}}\Gamma_f$ at $\tilde{p}$ to Γ_f coincides to $T_{\tilde{p}}\Gamma_{g_{t_0}}$ at $\tilde{p}$ to $\Gamma_{g_{t_0}}$. Therefore, $g - f$ has a minimum point $p \in \overline{B}_{x_0}(R)$ satisfying $\|\nabla(g-f)(p)\| = 0$.

Secondly, we assume that $\tilde{p}$ is on the boundary of Γ_f, namely, $\tilde{p} = (x_1, f(x_1))$. Let $\gamma : [0,R] \to M$ be a minimizing geodesic segment from x_0 to x_1. Because γ is minimizing, $\gamma(t)$ does not pass through the cut locus $\mathrm{Cut}(x_0)$ of x_0. Note that $\nabla f(x_1)$ is parallel to $\gamma'(R)$ by the assumption of x_1. We consider two cases according to the value of ξ:

1. $\xi \neq 0$
   By Lemma 1, we have

$$\begin{aligned}\frac{d}{dt}(g_{t_0}-f)(\gamma(t))\Big|_{t=R} &= \frac{d}{dt}(g-f)(\gamma(t))\Big|_{t=R}\\ &= \langle \nabla(g-f)(x_1),\gamma'(R)\rangle\\ &= \langle \nabla g(x_1),\gamma'(R)\rangle - \langle \nabla f(x_1),\gamma'(R)\rangle\\ &= \frac{R}{\sqrt{R^2+r^2}}\left\langle \nabla s(x_1),\gamma'(R)\right\rangle - \langle \nabla f(x_1),\gamma'(R)\rangle\\ &= \frac{R}{\sqrt{R^2+r^2}} - \xi\\ &= 0.\end{aligned}$$

2. $\xi = 0$
   By direct computation, we have

$$\begin{aligned}\frac{d}{dt}(g_{t_0}-f)(\gamma(t))\Big|_{t=R} &= \langle \nabla g(x_1),\gamma'(R)\rangle - \langle \nabla f(x_1),\gamma'(R)\rangle\\ &= -\langle \nabla f(x_1),\gamma'(R)\rangle\\ &= 0.\end{aligned}$$

Therefore, the function $g-f$ has a minimum value at the point $x_1 \in \partial\overline{B}_{x_0}(R)$ such that $\|\nabla(g-f)(x_1)\| = 0$. □

We need the following Hessian comparison theorem to compare the principal curvatures of Γ_f and Γ_g (see, [11,14]):

THEOREM 6. (Hessian comparison theorem) *Let M be a complete Riemannian manifold with sectional curvature bounded below by a constant c and $\gamma : [0,t] \to M$ a minimizing geodesic on M with $\gamma(0) = p$. Let $s : [0,t] \to \mathbb{R}$ be the smooth distance function from p to $\gamma(t)$ on M. Then, for any unit vector $v \in T_{\gamma(t)}M$ that is perpendicular to $\gamma'(t)$,*

$$(\nabla^2 s)_{\gamma(t)}(v,v) \leqslant \begin{cases}\sqrt{c}\cot(\sqrt{c}t), & c>0,\\ t^{-1}, & c=0,\\ \sqrt{-c}\coth(\sqrt{-c}t), & c<0.\end{cases}$$

PROPOSITION 1. *Let $M \times \mathbb{R}_1$ be a Lorentzian product space based on an n-dimensional complete Riemannian manifold M and $\overline{B}_{x_0}(R)$ a closed geodesic ball of radius R centered at x_0 in M with $c = \min_{\overline{B}_{x_0}(R)} K_M$. Suppose that there is a point x_1 on $\partial\overline{B}_{x_0}(R)$ such that $f(x_1) = \max_{\partial\overline{B}_{x_0}(R)} f$ and $\|\nabla f(x_1)\| = \xi$ for a function $f \in C^2(\overline{B}_{x_0}(R))$. Then, there exists a point $\tilde{p}$ on the spacelike graph Γ_f of f in $M \times \mathbb{R}_1$*

such that principal curvatures $\lambda_i(\tilde{p})$ for $i=1,\ldots,n$ of Γ_f satisfy

$$\lambda_i(\tilde{p}) \leqslant \frac{\xi}{\sqrt{1-\xi^2}}\sqrt{-c}\coth(\sqrt{-c}R), \qquad c<0,$$
$$\lambda_i(\tilde{p}) \leqslant \frac{\xi}{R\sqrt{1-\xi^2}}, \qquad c\geqslant 0.$$

In particular, the inequalities hold if $\xi = \max_{\overline{B}_{x_0}(R)} \|\nabla f\|$.

Proof. Let $g:\overline{B}_{x_0}(R)\to\mathbb{R}$ be the function defined by

$$g_t(x)=\begin{cases}\sqrt{s^2(x)+r^2}+t, & \xi\neq 0,\\ t, & \xi=0,\end{cases}$$

where $r=R\frac{\sqrt{1-\xi^2}}{\xi}$. In particular, we have the following inequality for $\bar{\xi}=\max_{\overline{B}_{x_0}(R)}\|\nabla f\|$:

$$\frac{\xi}{\sqrt{1-\xi^2}} \leqslant \frac{\bar{\xi}}{\sqrt{1-\bar{\xi}^2}}.$$

Let $\{e_i\}$ be an orthonormal basis of Γ_f in $M\times\mathbb{R}_1$ and $\{v_i\}$ be a basis on $T_{\tilde{p}}M$ such that $dX_p(v_i)=e_i$. By Lemma 2, the function $g-f$ has the global minimum at point $p\in\overline{B}_{x_0}(R)$. If we consider $\xi=0$, then the followings hold: for $v\in T_{\tilde{p}}\Gamma_f$,

$$\nabla f(p) = \nabla g(p)=0,$$
$$(\nabla^2 f)_p(v,v) \leqslant (\nabla^2 g)_p(v,v)=0,$$

and then for $v=v_i$,

$$\lambda_i(\tilde{p})=\frac{1}{W(p)}(\nabla^2 f)_p(v_i,v_i)\leqslant 0.$$

Then, we have only $\xi\neq 0$. There are two cases: $p=x_0$ and $p\neq x_0$. In particular, the case of $p\neq x_0$ can be considered as two possibilities: $p\in \mathrm{Cut}(x_0)$ and $p\notin\mathrm{Cut}(x_0)$.

1. $p=x_0$
   By Lemmas 1 and 2, we have

$$\nabla f(x_0) = \nabla g(x_0)=0,$$
$$(\nabla^2 f)_{x_0}(v,v) \leqslant (\nabla^2 g)_{x_0}(v,v)=\frac{\|v\|^2}{r},$$

   for $v\in T_{x_0}M$. The equations (1) and (2) yield

$$-\langle B(dX_{x_0}(v),dX_{x_0}(v)),\nu\rangle = \frac{1}{W(x_0)}(\nabla^2 f)_{x_0}(v,v)\leqslant\frac{\|v\|^2}{r}=\frac{\|dX_{x_0}(v)\|^2}{r}.$$

For $v = v_i$, we have

$$\lambda_i(\tilde{p}) = -\frac{\langle B(dX_{x_0}(v_i), dX_{x_0}(v_i)), \nu\rangle}{\|dX_{x_0}(v_i)\|^2} \leqslant \frac{1}{r}.$$

Besides, $\frac{1}{r} < \sqrt{-c}\coth(\sqrt{-c}r)$ for any $c < 0$.

2. $p \neq x_0$ and $p \notin \mathrm{Cut}(x_0)$
By Lemmas 1 and 2, we have for any $v \in T_pM$,

$$\begin{aligned}
\nabla f(p) &= \nabla g(p) = \frac{s(p)}{\sqrt{r^2+s^2(p)}}\nabla s(p),\\
(\nabla^2 f)_p(v,v) &\leqslant (\nabla^2 g)_p(v,v)\\
&= \frac{r^2}{(r^2+s^2(p))^{\frac{3}{2}}}\langle \nabla s(p), v\rangle^2 + \frac{s(p)}{\sqrt{r^2+s^2(p)}}(\nabla^2 s)_p(v,v).
\end{aligned}$$

It is easy to consider $\|\nabla s(p)\| = 1$ and $\|v\|^2 = \|v^\perp\|^2 + \langle v, \nabla s(p)\rangle^2$ where $v^\perp$ is the normal component of $\nabla s(p)$. By the equation (1), we obtain

$$\begin{aligned}
\|dX_p(v)\|^2 &= \|v^\perp\|^2 + \langle v, \nabla s(p)\rangle^2 - \langle \nabla f(p), v\rangle^2\\
&= \|v^\perp\|^2 + \langle v, \nabla s(p)\rangle^2 - \frac{s^2(p)}{r^2+s^2(p)}\langle \nabla s(p), v\rangle^2\\
&= \|v^\perp\|^2 + \frac{r^2}{r^2+s^2(p)}\langle \nabla s(p), v\rangle^2.
\end{aligned}$$

It is easy to verify that for $u \in T_pM$ satisfying that u is parallel to $\nabla s(p)$,

$$\nabla_u \nabla s(p) = 0.$$

Thus, we obtain

$$\begin{aligned}
&-\langle B(dX_p(v), dX_p(v)), \nu\rangle\\
&= \frac{1}{W(p)}(\nabla^2 f)_p(v,v)\\
&\leqslant \frac{\sqrt{r^2+s^2(p)}}{r}\left(\frac{r^2\langle \nabla s(p), v\rangle^2}{(r^2+s^2(p))^{\frac{3}{2}}} + \frac{s(p)(\nabla^2 s)_p(v,v)}{\sqrt{r^2+s^2(p)}}\right)\\
&= \frac{r\langle \nabla s(p), v\rangle^2}{r^2+s^2(p)} + \frac{s(p)}{r}(\nabla^2 s)_p(v,v)\\
&= \frac{\|dX_p(v)\|^2}{r} - \frac{\|v^\perp\|^2}{r} + \frac{s(p)}{r}(\nabla^2 s)_p(v^\perp, v^\perp)\\
&= \frac{\|dX_p(v)\|^2}{r} + \frac{\|v^\perp\|^2}{r}\left(s(p)(\nabla^2 s)_p\left(\frac{v^\perp}{\|v^\perp\|}, \frac{v^\perp}{\|v^\perp\|}\right) - 1\right).
\end{aligned}$$

If we assume $v^{\perp} = 0$, then the following inequality holds for $v = v_i$:

$$\lambda_i(\tilde{p}) = -\frac{\langle B(dX_p(v_i), dX_p(v_i)), \nu\rangle}{\|dX_p(v_i)\|^2} \leqslant \frac{1}{r} = \frac{\xi}{R\sqrt{1-\xi^2}}.$$

In particular, the inequality $\frac{1}{R} < \sqrt{-c}\coth(\sqrt{-c}R)$ for each R yields that the first result holds. We assume $v^{\perp} \neq 0$. According to Hessian comparison theorem (Theorem 6), we distinguish two cases as follows:

(a) $c < 0$

By Hessian comparison theorem (Theorem 6), we obtain

$$\begin{aligned} -\langle B(dX_p(v), dX_p(v)), \nu\rangle &\leqslant \frac{\|dX_p(v)\|^2}{r} \\ &\quad + \frac{\|v^{\perp}\|^2}{r}\left(s(p)\sqrt{-c}\coth(\sqrt{-c}s(p)) - 1\right) \\ &\leqslant \frac{\|dX_p(v)\|^2}{r} \\ &\quad + \frac{\|dX_p(v)\|^2}{r}\left(s(p)\sqrt{-c}\coth(\sqrt{-c}s(p)) - 1\right) \\ &= \frac{\|dX_p(v)\|^2}{r} s(p)\sqrt{-c}\coth(\sqrt{-c}s(p)) \\ &\leqslant \frac{\|dX_p(v)\|^2}{r} R\sqrt{-c}\coth(\sqrt{-c}R) \\ &= \frac{\xi}{\sqrt{1-\xi^2}}\|dX_p(v)\|^2\sqrt{-c}\coth(\sqrt{-c}R). \end{aligned}$$

Therefore, for $v = v_i$, we have

$$\lambda_i(\tilde{p}) = -\frac{\langle B(dX_p(v_i), dX_p(v_i)), \nu\rangle}{\|dX_p(v_i)\|^2} \leqslant \frac{\xi\sqrt{-c}}{\sqrt{1-\xi^2}}\coth(\sqrt{-c}R).$$

(b) $c \geqslant 0$

Hessian comparison theorem (Theorem 6) implies

$$\lambda_i(\tilde{p}) = -\frac{\langle B(dX_p(v_i), dX_p(v_i)), \nu\rangle}{\|dX_p(v_i)\|^2} \leqslant \frac{1}{r} = \frac{\xi}{R\sqrt{1-\xi^2}}.$$

3. $p \neq x_0$ and $p \in \mathrm{Cut}(x_0)$

Let $\gamma : [0, s(p)] \to M$ be a minimizing geodesic segment from x_0 to p passing through a point $z = \gamma(\varepsilon)$ sufficiently close to x_0. We consider the distance function $\bar{s}(x)$ from z to x. Then, we have $p \notin \mathrm{Cut}(z)$ and define the following function $\bar{g} : \overline{B}_p(R_p) \to \mathbb{R}$ where $R_p = \min\{s(p), R - s(p)\}$:

$$\bar{g}(x) = \begin{cases} \sqrt{(\bar{s}(x)+\varepsilon)^2 + r^2}, & \xi \neq 0 \\ 0 & \xi = 0, \end{cases}$$

where $r = R\frac{\sqrt{1-\xi^2}}{\xi}$. In particular, for all $x \in \overline{B}_p(R_p)$, $\bar{g}$ satisfies $\bar{g}(p) = g(p)$ and $\bar{g}(x) \geqslant g(x)$ and then

$$(\bar{g}-f)(x) \geqslant (g-f)(x) \geqslant (g-f)(p) = (\bar{g}-f)(p).$$

It is easy to verify that if $\xi = 0$, then $(\nabla f)(p) = 0$ and $(\nabla^2 f)_p(v,v) \leqslant 0$. By direct computation, we have for all $v \in T_pM$,

$$\begin{aligned}
\nabla f(p) &= \nabla \bar{g}(p) = \frac{s(p)}{\sqrt{s^2(p)+r^2}} \nabla \bar{s}(p), \\
(\nabla^2 f)_p(v,v) &\leqslant (\nabla^2 \bar{g})_p(v,v) = \frac{r^2 \langle \nabla \bar{s}(p), v\rangle^2}{(s^2(p)+r^2)^{\frac{3}{2}}} + \frac{s(p)(\nabla^2 \bar{s})_p(v,v)}{\sqrt{s^2(p)+r^2}}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\|v\|^2 &= \|v^\perp\|^2 + \langle v, \nabla \bar{s}\rangle^2, \\
\|dX_p(v)\|^2 &= \|v^\perp\|^2 + \frac{r^2\langle \nabla \bar{s}(p), v\rangle^2}{s^2(p)+r^2}.
\end{aligned}$$

Combining the above equations yields

$$\begin{aligned}
&-\langle B(dX_p(v), dX_p(v)), \nu\rangle \\
&\leqslant \frac{\|dX_p(v)\|^2}{r} + \frac{\|v^\perp\|^2}{r}\left(s(p)(\nabla^2 \bar{s})_p\left(\frac{v^\perp}{\|v^\perp\|}, \frac{v^\perp}{\|v^\perp\|}\right) - 1\right).
\end{aligned}$$

In particular, if we assume $v^\perp = 0$, then

$$-\frac{\langle B(dX_p(v), dX_p(v)), \nu\rangle}{\|dX_p(v)\|^2} \leqslant \frac{1}{r} = \frac{\xi}{R\sqrt{1-\xi^2}} < \frac{\xi}{\sqrt{1-\xi^2}}\sqrt{-c}\coth(\sqrt{-c}R).$$

According to Hessian comparison theorem (Theorem 6), we have two cases:

(a) $c < 0$

It follows that

$$\begin{aligned}
-\langle B(dX_p(v), dX_p(v)), \nu\rangle &\leqslant \frac{\|dX_p(v)\|^2}{r}(\bar{s}(p)+\varepsilon)\sqrt{-c}\coth(\sqrt{-c}\,\bar{s}(p)) \\
&\leqslant \frac{\|dX_p(v)\|^2}{r}(R+\varepsilon)\sqrt{-c}\coth(\sqrt{-c}R).
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\lambda_i(\tilde{p}) = -\frac{\langle B(dX_p(v_i), dX_p(v_i)), \nu\rangle}{\|dX_p(v_i)\|^2} \leqslant \frac{\xi\sqrt{-c}}{\sqrt{1-\xi^2}}\coth(\sqrt{-c}R).$$

(b) $c \geqslant 0$
We have $K_M \geqslant 0$ on $\overline{B}_{x_0}(R)$ and obtain

$$-\langle B(dX_p(v), dX_p(v)), \nu\rangle \leqslant \frac{\|dX_p(v)\|^2}{r}\left(1+\frac{\varepsilon}{\overline{s}(p)}\right).$$

Letting $\varepsilon \to 0$, we arrive at the following result:

$$\lambda_i(\tilde{p}) = -\frac{\langle B(dX_p(v_i), dX_p(v_i)), \nu\rangle}{\|dX_p(v_i)\|^2} \leqslant \frac{\xi}{R\sqrt{1-\xi^2}}. \quad \square$$

THEOREM 7. *Let $M \times \mathbb{R}_1$ be a Lorentzian product space based on an n-dimensional complete Riemannian manifold M and $\overline{B}_{x_0}(R)$ a closed geodesic ball of radius R centered at x_0 in M with $c = \min_{\overline{B}_{x_0}(R)} K_M$. Suppose that there is a point x_1 on $\partial\overline{B}_{x_0}(R)$ such that $f(x_1) = \max_{\partial\overline{B}_{x_0}(R)} f$ and $\|\nabla f(x_1)\| = \xi$ for a function $f \in C^2(\overline{B}_{x_0}(R))$. Then, the spacelike graph Γ_f of f in $M \times \mathbb{R}_1$ satisfies*

$$\min_{\Gamma_f}|H| \leqslant \frac{n\xi\sqrt{-c}}{\sqrt{1-\xi^2}}\coth(\sqrt{-c}R), \quad c<0,$$
$$\min_{\Gamma_f}|H| \leqslant \frac{n\xi}{R\sqrt{1-\xi^2}}, \quad c \geqslant 0$$

In particular, the above inequalities hold if $\xi = \max_{\overline{B}_{x_0}(R)}\|\nabla f\|$.

Proof. If H changes sign, then the result follows trivially. Suppose that H does not vanish at any point on $\overline{B}_{x_0}(R)$. By Proposition 1, we have for $i = 1, \ldots, n$,

$$\lambda_i(\tilde{p}) \leqslant \frac{\xi\sqrt{-c}}{\sqrt{1-\xi^2}}\coth(\sqrt{-c}R), \quad c<0,$$
$$\lambda_i(\tilde{p}) \leqslant \frac{\xi}{R\sqrt{1-\xi^2}}, \quad c \geqslant 0.$$

Since $\overline{B}_{x_0}(R)$ is compact and the functions f and g in Lemma 1 is of $C^2(\overline{B}_{x_0}(R))$, the following inequalities are obtained:

$$\min_{\Gamma_f}|H| \leqslant H(\tilde{p}) = \sum_{i=1}^{n}\lambda_i(\tilde{p}) \leqslant \frac{n\xi\sqrt{-c}}{\sqrt{1-\xi^2}}\coth(\sqrt{-c}R), \quad c<0,$$
$$\min_{\Gamma_f}|H| \leqslant H(\tilde{p}) = \sum_{i=1}^{n}\lambda_i(\tilde{p}) \leqslant \frac{n\xi}{R\sqrt{1-\xi^2}}, \quad c \geqslant 0. \quad \square$$

REMARK 1. Salavessa [13] proved an estimate of minimum value of mean curvature for a spacelike graph $\Gamma_f \subset M \times \mathbb{R}_1$ of a function f defined on a compact domain $D \subset M$ with $b_D = \max_{\overline{D}}\|\nabla f\|$:

$$\min_{\Gamma_f}|H| \leqslant \frac{b_D}{\sqrt{1-b_D^2}}\frac{\mathrm{Vol}_g(\partial D)}{\mathrm{Vol}_g(D)}, \tag{5}$$

where H is the mean curvature not divided by n. Let M_c be a space form with constant sectional curvature c and $D = \overline{B}_{x_0}(R)$ a closed ball of radius R centered at x_0 in M_c. Although $\xi \leqslant b_D$, we assume $\xi = b_D$ to compare Theorem 7 to the inequality (5). Then, if $c = -1$ or 1, then the inequality (5) is sharper than Theorem 7 and if $c = 0$, then the estimates are the same.

THEOREM 8. *Let $M \times \mathbb{R}_1$ be a Lorentzian product space based on an n-dimensional complete Riemannian manifold M and $\overline{B}_{x_0}(R)$ a closed geodesic ball of radius R centered at x_0 in M with $C = \max_{\overline{B}_{x_0}(R)} K_M$ and $c = \min_{\overline{B}_{x_0}(R)} K_M$. Suppose that there is a point x_1 on $\partial\overline{B}_{x_0}(R)$ such that $f(x_1) = \max_{\partial\overline{B}_{x_0}(R)} f$ and $\|\nabla f(x_1)\| = \xi$ for a function $f \in C^2(\overline{B}_{x_0}(R))$. Then, the spacelike graph Γ_f of f in $M \times \mathbb{R}_1$ satisfies*

$$\min_{\Gamma_f} |\mathrm{K}| \leqslant \frac{2c(2n-1)(n-1)\xi^2}{1-\xi^2}\coth^2(\sqrt{-c}R) + n(n-1)|C|, \quad c < 0,$$

$$\min_{\Gamma_f} |\mathrm{K}| \leqslant \frac{2(2n-1)(n-1)\xi^2}{R^2(1-\xi^2)} + n(n-1)C, \quad c \geqslant 0.$$

In particular, the above inequalities hold if $\xi = \max_{\overline{B}_{x_0}(R)} \|\nabla f\|$.

Proof. If K changes sign, then the result is trivial. Suppose that K does not change sign and then, we first assume $\mathrm{K} < 0$ on M. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $T_{\tilde{p}}\Gamma_f$ in which the second fundamental form is diagonal. Since Γ_f is an n-dimensional spacelike hypersurface, the Gauss equation is

$$K_{\tilde{p}}(e_i, e_j) = -\lambda_i(\tilde{p})\lambda_j(\tilde{p}) + \overline{K}_{\tilde{p}}(e_i, e_j), \tag{6}$$

for $i \neq j$ where $\overline{K}_{\tilde{p}}$ is the sectional curvature of $M \times \mathbb{R}_1$ at $\tilde{p}$, which yields by taking sum on j

$$\begin{aligned} \mathrm{Ric}_{\tilde{p}}(e_i, e_i) &= -\sum_{j=1, i\neq j}^{n} \lambda_i(\tilde{p})\lambda_j(\tilde{p}) + \overline{\mathrm{Ric}}_{\tilde{p}}(e_i, e_i) \\ &= -\sum_{j=1}^{n} \lambda_i(\tilde{p})\lambda_j(\tilde{p}) + \lambda_i^2(\tilde{p}) + \overline{\mathrm{Ric}}_{\tilde{p}}(e_i, e_i) \\ &= -H(\tilde{p})\lambda_i(\tilde{p}) + \lambda_i^2(\tilde{p}) + \overline{\mathrm{Ric}}_{\tilde{p}}(e_i, e_i), \end{aligned}$$

where $\overline{\mathrm{Ric}}_{\tilde{p}}$ is Ricci curvature of $M \times \mathbb{R}_1$ at $\tilde{p}$. By direct computation, we have

$$\begin{aligned} \langle e_i, e_j \rangle &= \left\langle \tilde{v}_i + \langle \nabla f, v_i \rangle_M \frac{\partial}{\partial t}, \tilde{v}_j + \langle \nabla f, v_j \rangle_M \frac{\partial}{\partial t} \right\rangle \\ &= \langle v_i, v_j \rangle_M - \langle \nabla f, v_i \rangle_M \langle \nabla f, v_j \rangle_M, \end{aligned}$$

and then,

$$\|v_i\|^2 \|v_j\|^2 - \langle v_i, v_j \rangle^2 = 1 + \langle \nabla f, v_i \rangle_M^2 + \langle \nabla f, v_j \rangle_M^2 \geqslant 1.$$

Thus, we have

$$\begin{aligned}\overline{\mathrm{Ric}}_{\tilde{p}}(e_i,e_i) &= \sum_{i\neq j}\langle R(e_i,e_j)e_i,e_j\rangle \\ &= \sum_{i\neq j}\langle R_M(v_i,v_j)v_i,v_j\rangle \\ &= \sum_{i\neq j}\frac{\langle R_M(v_i,v_j)v_i,v_j\rangle}{\|v_i\|^2\|v_j\|^2-\langle v_i,v_j\rangle^2}\left(\|v_i\|^2\|v_j\|^2-\langle v_i,v_j\rangle^2\right) \\ &= \sum_{i\neq j}K_M(v_i,v_j)\left(\|v_i\|^2\|v_j\|^2-\langle v_i,v_j\rangle^2\right) \\ &\geqslant \sum_{i\neq j}K_M(v_i,v_j).\end{aligned}$$

Thus, we have

$$\mathrm{Ric}_{\tilde{p}}(e_i,e_i) \geqslant -H(\tilde{p})\lambda_i(\tilde{p})+\lambda_i^2(\tilde{p})+(n-1)c. \tag{7}$$

Taking again sum on i yields

$$\mathrm{K}_{\tilde{p}} \geqslant -H^2(\tilde{p})+|B(\tilde{p})|^2+n(n-1)c \geqslant -H^2(\tilde{p})+n(n-1)c, \tag{8}$$

which implies

$$|\mathrm{K}_{\tilde{p}}| \leqslant |-H^2(\tilde{p})+n(n-1)c| \leqslant H^2(\tilde{p})+n(n-1)|c|.$$

By Theorem 7, we obtain

$$\begin{aligned}\min_{\Gamma_f}|\mathrm{K}| &\leqslant \min_{\Gamma_f}|H|^2-n(n-1)c \leqslant \frac{cn^2\xi^2}{1-\xi^2}\coth^2(\sqrt{-c}R)-n(n-1)c, \quad c<0, \\ \min_{\Gamma_f}|\mathrm{K}| &\leqslant \min_{\Gamma_f}|H|^2+n(n-1)c \leqslant \frac{n^2\xi^2}{R^2(1-\xi^2)}+n(n-1)c, \quad c\geqslant 0.\end{aligned}$$

Secondly, we assume $\mathrm{K}>0$ and take the first touching point $\tilde{p}$. Since $\mathrm{K}_{\tilde{p}}>0$ on M, we can assume

$$\lambda_1(\tilde{p}) \leqslant \cdots \leqslant \lambda_l(\tilde{p}) < 0 \leqslant \lambda_{l+1}(\tilde{p}) \leqslant \cdots \leqslant \lambda_n(\tilde{p}).$$

By the Gauss equation (6), we have as follows:

$$\begin{aligned}\mathrm{K}_{\tilde{p}} &= -2\sum_{1\leqslant i<j\leqslant n}\lambda_i(\tilde{p})\lambda_j(\tilde{p})+\overline{\mathrm{K}}_{\tilde{p}} \\ &= -2\sum_{1\leqslant i<j\leqslant l}\lambda_i(\tilde{p})\lambda_j(\tilde{p})-2\sum_{l+1\leqslant i<j\leqslant n}\lambda_i(\tilde{p})\lambda_j(\tilde{p})-2\sum_{i=1}^{l}\sum_{j=l+1}^{n}\lambda_i(\tilde{p})\lambda_j(\tilde{p})+\overline{\mathrm{K}}_p \\ &\leqslant -2\sum_{i=1}^{l}\sum_{j=l+1}^{n}\lambda_i(\tilde{p})\lambda_j(\tilde{p})+n(n-1)C.\end{aligned}$$

We obtain

$$0 < \mathrm{K}_{\tilde{p}} \leqslant -2\sum_{i=1}^{l}\sum_{j=l+1}^{n}\lambda_i(\tilde{p})\lambda_j(\tilde{p}) + n(n-1)C$$
$$= -2\left(H(\tilde{p}) - \sum_{i=l+1}^{n}\lambda_i(\tilde{p})\right)\sum_{j=l+1}^{n}\lambda_j(\tilde{p}) + n(n-1)C.$$

Then, the following inequalities hold:

$$|\mathrm{K}_{\tilde{p}}| \leqslant 2\left|-H(\tilde{p}) + \sum_{i=l+1}^{n}\lambda_i(\tilde{p})\right|\sum_{j=l+1}^{n}\lambda_j(\tilde{p}) + n(n-1)|C|$$
$$\leqslant 2\left(|H(\tilde{p})| + \sum_{i=l+1}^{n}\lambda_i(\tilde{p})\right)\sum_{j=l+1}^{n}\lambda_j(\tilde{p}) + n(n-1)|C|.$$

By Proposition 1 and Theorem 7, we have

$$\min_{\Gamma_f}|\mathrm{K}| \leqslant \frac{2c(2n-1)(n-1)\xi^2}{1-\xi^2}\coth^2(\sqrt{-c}R) + n(n-1)|C|, \quad c<0,$$
$$\min_{\Gamma_f}|\mathrm{K}| \leqslant \frac{2(2n-1)(n-1)\xi^2}{R^2(1-\xi^2)} + n(n-1)C, \quad c \geqslant 0. \quad \square$$

THEOREM 9. *Let $M\times\mathbb{R}_1$ be a Lorentzian product space based on an n-dimensional complete Riemannian manifold M and $\overline{B}_{x_0}(R)$ a closed geodesic ball of radius R centered at x_0 in M with $c = \min_{\overline{B}_{x_0}(R)} K_M$. Suppose that there is a point x_1 on $\partial\overline{B}_{x_0}(R)$ such that $f(x_1) = \max_{\partial\overline{B}_{x_0}(R)} f$ and $\|\nabla f(x_1)\| = \xi$ for a function $f \in C^2(\overline{B}_{x_0}(R))$. Then, the spacelike graph Γ_f of f in $M\times\mathbb{R}_1$ with Ric $< (n-1)c$ satisfies*

$$\min_{\Gamma_f}|B| \leqslant 3(n-2)\frac{\xi}{\sqrt{1-\xi^2}}\sqrt{-c}\coth(\sqrt{-c}R), \qquad c<0,$$
$$\min_{\Gamma_f}|B| \leqslant 3(n-2)\frac{\xi}{R\sqrt{1-\xi^2}}, \qquad c>0.$$

In particular, the above inequalities hold if $\xi = \max_{\overline{B}_{x_0}(R)}\|\nabla f\|$.

Proof. By the equation (7), we have

$$0 > \mathrm{Ric}_{\tilde{p}}(e_i,e_i) - (n-1)c \geqslant \lambda_i(\tilde{p})(\lambda_i(\tilde{p}) - H(\tilde{p})). \tag{9}$$

We first consider that all principal curvatures $\lambda_i(\tilde{p})$ at $\tilde{p}$ are positive by replacing f by $-f$ if necessary. Then, we have

$$|B|^2 = \sum_{k=1}^{n}\lambda_k(\tilde{p})^2 \leqslant \left(\sum_{k=1}^{n}|\lambda_k(\tilde{p})|\right)^2.$$

By Proposition 1, we have

$$\min_{\Gamma_f}|B| \leqslant n\frac{\xi}{\sqrt{1-\xi^2}}\sqrt{-c}\coth(\sqrt{-c}R), \qquad c<0,$$
$$\min_{\Gamma_f}|B| \leqslant n\frac{\xi}{R\sqrt{1-\xi^2}}, \qquad c>0.$$

Secondly, we assume the sign of $\lambda_i(\tilde{p})$ at $\tilde{p}$ for each $i=1,\dots,n$ such that $\lambda_1(\tilde{p}) \leqslant \cdots \leqslant \lambda_l(\tilde{p}) < 0 < \lambda_{l+1}(\tilde{p}) \leqslant \cdots \leqslant \lambda_n(\tilde{p})$ for $2 \leqslant l \leqslant n-1$. By the equation (9), we have

$$\lambda_i(\tilde{p})\left(\sum_{k=1}^{n}\lambda_k(\tilde{p})-\lambda_i(\tilde{p})\right)>0.$$

From the above inequality, we obtain

$$\sum_{k=l+1}^{n}\lambda_k(\tilde{p})-\lambda_i(\tilde{p})>-\sum_{k=1}^{l}\lambda_k(\tilde{p}).$$

Taking sum on $i=1,\dots,l$ yields

$$\sum_{k=1}^{l}|\lambda_k(\tilde{p})|<\frac{l}{l-1}\sum_{k=l+1}^{n}\lambda_k(\tilde{p}).$$

Then, we have

$$\sum_{k=1}^{n}|\lambda_k(\tilde{p})|=\sum_{k=1}^{l}|\lambda_k(\tilde{p})|+\sum_{k=l+1}^{n}|\lambda_k(\tilde{p})|<\left(2+\frac{1}{l-1}\right)\sum_{k=l+1}^{n}\lambda_k(\tilde{p})<3\sum_{k=l+1}^{n}\lambda_k(\tilde{p}),$$

which implies

$$|B|^2=\sum_{k=1}^{n}\lambda_k(\tilde{p})^2 \leqslant \left(\sum_{k=1}^{n}|\lambda_k(\tilde{p})|\right)^2<3^2\left(\sum_{k=l+1}^{n}\lambda_k(\tilde{p})\right)^2.$$

By Proposition 1, we have

$$\begin{aligned}|B| &\leqslant 3(n-l)\frac{\xi}{\sqrt{1-\xi^2}}\sqrt{-c}\coth(\sqrt{-c}R)\\ &< 3(n-2)\frac{\xi}{\sqrt{1-\xi^2}}\sqrt{-c}\coth(\sqrt{-c}R), \qquad c<0,\\ |B| &\leqslant 3(n-l)\frac{\xi}{R\sqrt{1-\xi^2}}<3(n-2)\frac{\xi}{R\sqrt{1-\xi^2}}, \qquad c>0.\end{aligned}$$

Therefore, we obtain

$$\min_{\Gamma_f}|B| \leqslant 3(n-2)\frac{\xi}{\sqrt{1-\xi^2}}\sqrt{-c}\coth(\sqrt{-c}R), \qquad c<0,$$
$$\min_{\Gamma_f}|B| \leqslant 3(n-2)\frac{\xi}{R\sqrt{1-\xi^2}}, \qquad c>0. \quad \square$$

3. Estimates from Poincaré inequality and Sobolev inequality

Let (M,g) be an n-dimensional compact Riemannian manifold and let $L^p(M)$ be the Lebesgue space on M consisting of all measurable functions f satisfying

$$\|f\|_{L^p} = \left(\int_M |f|^p \, dV_g\right)^{\frac{1}{p}} < \infty.$$

Because M is compact, we consider the Sobolev space $H_1^q(M)$ for $q \in [1,n)$ defined as the completion of $C^\infty(M)$ with respect to the norm

$$\|f\|_{H_1^q} = \sqrt{\|f\|_{L^q}^2 + \|\nabla f\|_{L^q}^2}.$$

There are many types of Poincaré and Sobolev inequalities (see, [1, 7, 8, 12] and references therein). In this section, we consider Poincaré and Sobolev inequalities on a compact manifold. The Poincaré inequality (see, [7] for example) is that for an n-dimensional compact Riemannian manifold M and $q \in [1,n)$, there exists a positive constant c_M such that for any function $f \in H_1^q(M)$,

$$\|f - \bar{f}\|_{L^q} \leqslant c_M \|\nabla f\|_{L^q}, \tag{10}$$

where

$$\bar{f} = \frac{1}{\mathrm{Vol}_g(M)} \int_M f \, dV_g.$$

Hebey and Vaugon [8] proved that for an n-dimensional compact Riemannian manifold M, $n \geqslant 3$, with no boundary and $\frac{1}{p} = \frac{1}{2} - \frac{1}{n}$, there exists a positive number A such that for any $f \in H_1^2(M)$,

$$\|f\|_{L^p}^2 \leqslant K\|\nabla f\|_{L^2}^2 + A\|f\|_{L^2}^2, \tag{11}$$

where

$$K = \frac{4}{n(n-2)w_n^{\frac{2}{n}}}.$$

Here, we denote w_n by the volume of the n-dimensional unit sphere in $\mathbb{R}^{n+1}$. On the other hand, Li and Zhu [12] proved that for an n-dimensional compact Riemannian manifold, $n \geqslant 3$, with a smooth boundary and $q = \frac{2(n-1)}{n-2}$, there exists a positive number B such that for any $f \in H_1^2(M)$,

$$\|f\|_{L^q(\partial M)}^2 \leqslant S\|\nabla f\|_{L^2}^2 + B\|f\|_{L^2(\partial M)}^2, \tag{12}$$

where

$$S = \frac{2}{(n-2)w_{n-1}^{\frac{1}{n-1}}}.$$

Note that the constants K and S in the inequalities (11) and (12) are optimal in such inequalities.

The inequality (10) gives the following estimate for the L^2-norm of the mean curvature of a spacelike graph defined on a compact manifold, which is inspired by [15]:

THEOREM 10. *Let $M \times \mathbb{R}_1$ be a Lorentzian product space based on an n-dimensional compact Riemannian manifold M and Γ_f the spacelike graph of $f \in C^2(M)$ with the mean curvature H in $M \times \mathbb{R}_1$. Then, f satisfies*

$$\frac{1}{c_M^2}\|f-\bar{f}\|_{L^2}^2 \leqslant \|f\|_{L^2}\|H\|_{L^2} + \int_{\partial M}\left\langle \frac{f\nabla f}{\sqrt{1-\|\nabla f\|^2}}, \eta \right\rangle_M ds,$$

where c_M is a positive constant related to n and M, namely, Poincaré constant, and η is the outward unit normal vector of ∂M.

Proof. Suppose that f satisfies the Poincaré inequality (10):

$$\|f-\bar{f}\|_{L^2} \leqslant c_M \|\nabla f\|_{L^2}.$$

Then we have

$$\|f-\bar{f}\|_{L^2}^2 \leqslant c_M^2 \|\nabla f\|_{L^2}^2 \leqslant c_M^2 \int_M \frac{||\nabla f||^2}{\sqrt{1-||\nabla f||^2}} dV_g. \tag{13}$$

The mean curvature H of Γ_f in $M \times \mathbb{R}_1$ is

$$H = \mathrm{div}_M\left(\frac{\nabla f}{\sqrt{1-\|\nabla f\|^2}}\right),$$

where div_M is the divergence on M. The divergence theorem yields

$$\int_M \frac{\|\nabla f\|^2}{\sqrt{1-\|\nabla f\|^2}} dV_g = \int_M fH\, dV_g + \int_{\partial M}\left\langle \frac{f\nabla f}{\sqrt{1-\|\nabla f\|^2}}, \eta \right\rangle ds, \tag{14}$$

where η is the outward unit normal vector of ∂M. Using Hölder's inequality, we obtain the following inequalities:

$$\int_M fH\, dV_g \leqslant \int_M |f||H|\, dV_g \leqslant \|f\|_{L^2}\|H\|_{L^2}. \tag{15}$$

Combining inequalities (13), (14) and (15), the following inequality is obtained:

$$\frac{1}{c_M^2}\|f-\bar{f}\|_{L^2(M)}^2 \leqslant \|f\|_{L^2}\|H\|_{L^2} + \int_{\partial M}\left\langle \frac{f\nabla f}{\sqrt{1-\|\nabla f\|^2}}, \eta \right\rangle ds. \quad \square$$

REMARK 2. We assume that the integral in the right hand side of the above inequality becomes zero for some suitable boundary condition. Then, we obtain $\|f-\bar{f}\|_{L^2}^2 \leqslant c_M^2\|H\|_{L^2}^2$. In particular, if Γ_f is maximal, then f is a constant function on M, so Γ_f is isomorphic to M, namely, Γ_f is a slice in $M\times\mathbb{R}_1$.

The following theorems are considered two types of Sobolev inequalities:

THEOREM 11. *Let $M\times\mathbb{R}_1$ be a Lorentzian product space based on an n-dimensional compact Riemannian manifold M, $n\geqslant 3$, with no boundary and Γ_f the spacelike graph of $f\in C^2(M)$ with the mean curvature H in $M\times\mathbb{R}_1$ and let $\frac{1}{p}=\frac{1}{2}-\frac{1}{n}$. Then, there exists a positive number A such that*

$$\|f\|_{L^p}^2 \leqslant \frac{4}{n(n-2)w_n^{\frac{2}{n}}}\|f\|_{L^2}\|H\|_{L^2}+A\|f\|_{L^2}^2.$$

Proof. We follow the proof of Theorem 10 for the inequality (11):

$$\begin{aligned}\|f\|_{L^p}^2 &\leqslant K\|\nabla f\|_{L^2}^2+A\|f\|_{L^2}^2\\ &\leqslant K\int_M \frac{\|\nabla f\|^2}{\sqrt{1-\|\nabla f\|^2}}dV_g+A\|f\|_{L^2}^2.\end{aligned}$$

By the equations (14) and (15) with the fact that M has no boundary, we have

$$\|f\|_{L^p}^2 \leqslant K\|f\|_{L^2}\|H\|_{L^2}+A\|f\|_{L^2}^2. \quad \square$$

As the similar process, we have the following theorem for a compact Riemannian manifold with a smooth boundary using the equation (12):

THEOREM 12. *Let $M\times\mathbb{R}_1$ be a Lorentzian product space based on an n-dimensional compact Riemannian manifold M, $n\geqslant 3$, with a smooth boundary and Γ_f the spacelike graph of $f\in C^2(M)$ with the mean curvature H in $M\times\mathbb{R}_1$ and let $q=\frac{2(n-1)}{n-2}$. Then, there exists a positive number B such that*

$$\|f\|_{L^q(\partial M)}^2 \leqslant \frac{2}{(n-2)w_{n-1}^{\frac{1}{n-1}}}\left(\|f\|_{L^2}\|H\|_{L^2}+\int_{\partial M}\left\langle \frac{f\nabla f}{\sqrt{1-\|\nabla f\|^2}},\eta\right\rangle ds\right)+B\|f\|_{L^2(\partial M)}^2.$$

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