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SUBSPACE–HYPERCYCLIC CONDITIONAL WEIGHTED COMPOSITION OPERATORS ON L^p–SPACES

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Abstract. A conditional weighted composition operator $T_u : L^p(\Sigma) \to L^p(\mathscr{A})$ $(1 \leqslant p < \infty)$, is defined by $T_u(f) := E^{\mathscr{A}}(uf \circ \varphi)$, where $\varphi : X \to X$ is a measurable transformation, u is a weight function on X and $E^{\mathscr{A}}$ is the conditional expectation operator with respect to $\mathscr{A}$. In this paper, we study the subspace-hypercyclicity of T_u with respect to $L^p(\mathscr{A})$. First, we show that if φ is a periodic nonsingular transformation, then T_u is not $L^p(\mathscr{A})$-hypercyclic. The necessary conditions for the subspace-hypercyclicity of T_u are obtained when φ is non-singular and finitely non-mixing. For the sufficient conditions, the normality of φ is required. The subspace-weakly mixing and subspace-topologically mixing concepts are also studied for T_u. Finally, we give an example which is subspace-hypercyclic while is not hypercyclic.

1. Introduction and preliminaries

Suppose that T is a bounded linear operator on a topological vector space X. If there is a vector $x \in X$ such that the orbit $orb(T,x) := \{T^n x : n = 0,1,2,\ldots\}$ is dense in X, then T will be hypercyclic and x is called a hypercyclic vector. Here, T^n stands for the n-th iterate of T and T^0 is the identity map I. Let M be a closed and non-trivial subspace of X. An operator T is *subspace-hypercyclic* with respect to M (M-*hypercyclic*), if there is a a vector $x \in X$ such that $orb(T,x) \cap M$ is dense in M. Also an operator T is *subspace-transitive* with respect to M, if for any non-empty open set $U,V \subseteq M$, there exists an $n \in \mathbb{N}$ such that $T^{-n}(U) \cap V$ contains an open non-empty subset of M. An operator T is *subspace-topologically mixing* with respect to M, if for any non-empty open set $U,V \subseteq M$, there exists an $N \in \mathbb{N}$ such that $T^{-n}(U) \cap V$ contains an open non-empty subset of M for each $n \geqslant N$. It is called *subspace-weakly mixing* if $T \oplus T$ is subspace-hypercyclic with respect to $M \oplus M$.

The study of subspace-hypercyclic linear operators was initiated by B. F. Madore and R. A. Martínez-Avendaño [25]. They found out that subspace-hypercyclicity like as hypercyclicity, can occur only on infinite-dimensional spaces and even subspaces. Also, they proved an interesting Kitai's type *subspace-hypercyclicity criterion* on a topological vector space as follows.

Assume that there exist D_1 and D_2, dense subsets of M, and an increasing sequence of positive integers (n_k) such that

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- $T^{n_k}x \to 0$ for all $x \in D_1$;
- for each $y \in D_2$, there exists a sequence $\{x_k\}$ in M such that $x_k \to 0$ and $T^{n_k}x_k \to y$;
- M is an invariant subspace for T^{n_k} for all $k \in \mathbb{N}$.

Then T is subspace-transitive and hence is subspace-hypercyclic [25, Theorem 3.6]. But the converse is not true, see [24, 29] for more details. Further, it is showed that the compact or hyponormal operators are not subspace-hypercyclic.

For the dynamics of linear operators the survey articles [1], [8], [25], [28], [30], [32] and the books [6], [17] are useful.

Let (X,Σ,μ) be a complete σ-finite measure space and $\mathscr{A}$ is a σ-finite subalgebra of Σ. For each $1 \leqslant p < \infty$, the Banach space $L^p(X,\mathscr{A},\mu_{|_{\mathscr{A}}})$ is denoted by $L^p(\mathscr{A})$ simply. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The *support* of any Σ-measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. The *characteristic function* of any set A and the class of all $\mathscr{A}$-measurable and simple functions on X with finite supports will be denoted by χ_A and $S^{\mathscr{A}}(X)$, respectively.

A Σ-measurable transformation $\varphi : X \to X$ is called *non-singular* whenever $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ, which is symbolically shown by $\mu \circ \varphi^{-1} \ll \mu$. In this case, *Radon-Nikodym property* is denoted by $h := \frac{d\mu\circ\varphi^{-1}}{d\mu}$.

A Σ-measurable transformation $\varphi : X \to X$ is called *periodic* if $\varphi^m = I$ for some $m \in \mathbb{N}$. It is called *aperiodic*, if it is not periodic. Also, if for each subset $F \in \Sigma$ with finite measure, there exists an $N \in \mathbb{N}$ such that $F \cap \varphi^n(F) = \emptyset$ for every $n > N$, then φ is called *finitely non-mixing*.

Set $\Sigma_\infty = \bigcap_{n=1}^{\infty} \varphi^{-n}(\Sigma)$ and suppose that h is Σ_∞-measurable. The assumption $\mu \circ \varphi^{-1} \ll \mu$ implies that $\mu \circ \varphi^{-n} \ll \mu$ for all $n \in \mathbb{N}$ and then

$$h_n := \frac{d\mu\circ\varphi^{-n}}{d\mu} = \frac{d\mu\circ\varphi^{-n}}{d\mu\circ\varphi^{-(n-1)}} \cdots \frac{d\mu\circ\varphi^{-1}}{d\mu}$$
$$= (h\circ\varphi^{-(n-1)})\cdots(h\circ\varphi^{0}) = \prod_{i=0}^{n-1} h\circ\varphi^{-i}.$$

Note that always $h\circ\varphi > 0$ and $h_n = h^n$ whenever $h\circ\varphi = h$. When it is restricted to a σ-subalgebra $\mathscr{A}$, is denoted by $h_n^{\mathscr{A}} = \frac{d(\mu\circ\varphi^{-n}|_{\mathscr{A}})}{d(\mu|_{\mathscr{A}})}$.

The *change of variable formula*

$$\int_{\varphi^{-n}(A)} f\circ\varphi^n d\mu = \int_A h_n f d\mu, \quad A \in \Sigma,\ f \in L^1(\Sigma),$$

will be used frequently.

When $\varphi(\Sigma) \subseteq \Sigma$ and $\mu\circ\varphi \ll \mu$, then a measure μ is called *normal* with respect to φ and in this case $h^\sharp = \frac{d\mu\circ\varphi}{d\mu}$ is defined. Now, consider that

$$h^\sharp = \left(\frac{d\mu}{d\mu\circ\varphi}\right)^{-1} = \left(\frac{d\mu\circ\varphi^{-1}}{d\mu}\circ\varphi\right)^{-1} = \frac{1}{h\circ\varphi}$$

and

$$h_n^\sharp := \frac{d\mu \circ \varphi^n}{d\mu} = (h^\sharp \circ \varphi^{(n-1)}) \cdots (h^\sharp \circ \varphi^0) = \prod_{i=0}^{n-1} h^\sharp \circ \varphi^i = \prod_{i=1}^{n} (h \circ \varphi^i)^{-1},$$

$h_n^\sharp \circ \varphi > 0$, $h_{n+1}^\sharp = h^\sharp h_n^\sharp \circ \varphi$.

Let $1 \leqslant p \leqslant \infty$. For any non-negative Σ-measurable functions f or for any $f \in L^p(\Sigma)$, Radon-Nikodym theorem, ensures the existence of a unique $\mathscr{A}$-measurable function $E^{\mathscr{A}}(f)$ such that

$$\int_A E^{\mathscr{A}}(f) d\mu = \int_A f d\mu, \qquad \text{for all } A \in \mathscr{A}.$$

A contractive projection $E^{\mathscr{A}} : L^p(\Sigma) \to L^p(\mathscr{A})$ is called a *conditional expectation operator* associated with the σ-finite subalgebra $\mathscr{A}$.

Here, we list some useful properties of the conditional expectation operator:

- $E^{\mathscr{A}}(1) = 1$;
- If g is $\mathscr{A}$-measurable, then $E^{\mathscr{A}}(fg) = E^{\mathscr{A}}(f)g$;
- $|E^{\mathscr{A}}(f)|^p \leqslant E^{\mathscr{A}}(|f|^p)$;
- For each $f \geqslant 0$, $\sigma(f) \subseteq \sigma(E^{\mathscr{A}}(f))$;
- Monotonicity: If f and g are real-valued with $f \leqslant g$, then $E^{\mathscr{A}} f \leqslant E^{\mathscr{A}} g$;
- For each $f \geqslant 0, E^{\mathscr{A}}(f) \geqslant 0$.
- $h_{n+1} = hE^{\varphi^{-1}(\Sigma)}(h_n) \circ \varphi^{-1} = h_n E^{\varphi^{-n}(\Sigma)}(h) \circ \varphi^{-1}$ [20].

A detailed information of the condition expectation operator may be found in [19, 23, 26, 27].

A *weighted composition operator* $uC_\varphi : L^p(\Sigma) \to L^p(\Sigma)$ defined by $f \mapsto uf \circ \varphi$ is bounded if and only if $J \in L^\infty(\Sigma)$, where $J := hE^{\mathscr{A}}(|u|^p) \circ \varphi^{-1}$, and in this case $\|uC_\varphi\|^p = \|J\|_\infty$ (see [20, 21, 31]).

Now, we are ready to define a *conditional weighted composition operator* T_u by:

$$T_u : L^p(\Sigma) \to L^p(\mathscr{A})$$

$$T_u f := E^{\mathscr{A}} \circ uC_\varphi(f) = E^{\mathscr{A}}(uf \circ \varphi).$$

For the fundamental properties of the conditional type operators, the reader is refereed to [13, 14, 15, 16].

The hypercyclicity of the well-known operators such as weighted shifts, weighted translations, conditional weighted translations and weighted composition operators in different settings has been studied in [1, 3, 4, 5, 7, 8, 11, 30, 32]. Recently, the spaceability of the set of hypercyclic vectors for shift-like operators has been studied in [12].

Separability and infinite-dimension are two essential objects for the underlying space to admit a hypercyclic vector [6, 17]. To that end, it is important to know that

$L^p(X,\Sigma,\mu)$ is separable if and only if (X,Σ,μ) is separable, i.e., there exists a countable σ-subalgebra $\mathscr{F} \subseteq \Sigma$ such that for each $\varepsilon > 0$ and $A \in \Sigma$ we have $\mu(A\Delta B) < \varepsilon$ for some $B \in \mathscr{F}$. For more details consult [26].

In this paper, we will survey the dynamics of a conditional weighted composition operator $T_u = E^{\mathscr{A}}(uf \circ \varphi)$ on $L^p(\Sigma)$ spaces. First, we prove that T_u cannot be $L^p(\mathscr{A})$-hypercyclic if φ is a periodic non-singular transformation. In addition, the necessary conditions for the subspace-hypercyclicity of T_u are then given provided that φ is non-singular and finitely non-mixing. For the sufficient conditions, we also require that φ is normal. The subspace-weakly mixing and subspace-topologically mixing concepts are also studied for T_u. At the end, about what we argued, an examples is given.

2. Subspace-hypercyclicity of T_u on $L^p(\Sigma)$

In this section, the $L^p(\mathscr{A})$-hypercyclicity of a conditional weighted composition operator T_u is studied. When φ is periodic transformation, it is seen that T_u is not $L^p(\mathscr{A})$-hypercyclic. But, when it is aperiodic, by Kitai's subspace-hypercyclicity criterion we obtain some necessary and then sufficient conditions for T_u to be subspace-hypercyclic. We are thankful to the techniques used in [11, 30].

THEOREM 1. *Let φ be a periodic* non-singular *transformation and $\varphi^{-1}\mathscr{A} \subseteq \mathscr{A}$. Then a conditional weighted composition operator $T_u : L^p(\Sigma) \to L^p(\mathscr{A})$ is not subspace-hypercyclic with respect to $L^p(\mathscr{A})$, for each $1 \leqslant p < \infty$.*

Proof. Suppose that there exists an $m \in \mathbb{N}$ such that $\varphi^m = I$. Since $\varphi^{-1}\mathscr{A} \subseteq \mathscr{A}$, the orbit of T_u at each $f \in L^p(\Sigma)$ is written as follows:

$$
\begin{aligned}
&orb(T_u,f) = \{f, T_u f, \cdots, T_u^m f\} \cup \{T_u^{m+1} f, T_u^{m+2} f, \cdots, T_u^{2m} f\} \cup \cdots \\
&\cup \{T_u^{km+1} f, T_u^{km+2} f, \cdots, T_u^{(k+1)m} f\} \cup \cdots \\
&= \Big\{f, E^{\mathscr{A}}(uf\circ\varphi), E^{\mathscr{A}}(u)E^{\mathscr{A}}(uf\circ\varphi)\circ\varphi, \cdots, \prod_{i=0}^{m-2} E^{\mathscr{A}}(u)\circ\varphi^i E^{\mathscr{A}}(uf\circ\varphi)\circ\varphi^{m-1}\Big\} \\
&\cup \Big\{ \prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i E^{\mathscr{A}}(uf\circ\varphi), \prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i E^{\mathscr{A}}(u)E^{\mathscr{A}}(uf\circ\varphi)\circ\varphi, \cdots, \\
&\prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i \prod_{i=0}^{m-2} E^{\mathscr{A}}(u)\circ\varphi^i E^{\mathscr{A}}(uf\circ\varphi)\circ\varphi^{m-1}\Big\} \\
&\cup \Big\{ (\prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i)^2 E^{\mathscr{A}}(uf\circ\varphi), (\prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i)^2 E^{\mathscr{A}}(u)E^{\mathscr{A}}(uf\circ\varphi)\circ\varphi, \cdots, \\
&(\prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i)^2 \prod_{i=0}^{m-2} E^{\mathscr{A}}(u)\circ\varphi^i E^{\mathscr{A}}(uf\circ\varphi)\circ\varphi^{m-1}\Big\} \cup \\
&\vdots
\end{aligned}
$$

Now we consider that $\|\prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i\|_\infty \leqslant 1$. Since $\|T_u\| \leqslant \|J\|_\infty^{1/p}$, $\|T_u^n\| \leqslant \|T_u\|^n \leqslant \|J\|_\infty^{n/p}$, and for each $n\in\mathbb{N}$ we have

$$\begin{aligned}\|T_u^n f\|_p \leqslant \max\{&\|f\|_p, \|E^{\mathscr{A}}(uf\circ\varphi)\|_p, \|E^{\mathscr{A}}(u)E^{\mathscr{A}}(uf\circ\varphi)\circ\varphi\|_p,\cdots,\\ &\|\prod_{i=0}^{m-2} E^{\mathscr{A}}(u)\circ\varphi^i E^{\mathscr{A}}(uf\circ\varphi)\circ\varphi^{m-1}\|_p\}\\ &\leqslant \|f\|_p \max\{1, \|J\|_\infty^{\frac{1}{p}}, \|J\|_\infty^{\frac{2}{p}},\cdots,\|J\|_\infty^{\frac{m-1}{p}}\}.\end{aligned}$$

Therefore, $orb(T_u,f)$ is a bounded subset and cannot be dense in $L^p(\mathscr{A})$.

In the second case $\|\prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i\|_\infty > 1$, assume that T_u is subspace-hypercyclic with respect to $L^p(\mathscr{A})$. Then there exists a subset $F\in\mathscr{A}$ with $0<\mu(F)<\infty$ for each $\varepsilon>0$, such that $|\prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i|>1$. Then there is a subspace-hypercyclic vector $f\in L^p(\mathscr{A})$ and $n\in\mathbb{N}$ such that

$$\|f-2\chi_F\|_p<\varepsilon \quad \text{and} \quad \|(T_u^{m+1})^n f\|_p<\varepsilon.$$

We set $S=\{t\in F: |f(t)|<1\}$ and note that $\chi_S\leqslant\chi_S|f-2|\leqslant\chi_S|f-2\chi_F|$. Thus, $\mu(S)<\varepsilon^p$. On the other hand,

$$\begin{aligned}\varepsilon^p > \|(T_u^m)^n f)\|_p^p &= \int_X |\prod_{i=0}^{mn-1} E^{\mathscr{A}}(u)\circ\varphi^i f\circ\varphi^{mn}|^p d\mu\\ &= \int_X |\prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i|^{np}|f|^p d\mu \geqslant \int_{F-S}|f|^p d\mu \geqslant \mu(\chi_{F-S}).\end{aligned}$$

Therefore, $\mu(F)=\mu(S)+\mu(F-S)<2\varepsilon^p$, which is a contradiction. □

REMARK 1. If φ is a periodic *non-singular* transformation, $\varphi^{-1}\mathscr{A}\subseteq\mathscr{A}$ and $u=1$, then a conditional composition operator $T_u f=E^{\mathscr{A}}(f\circ\varphi)$ is not subspace-hypercyclic with respect to $L^p(\mathscr{A})$ either. Since its orbit at $f\in L^p(\Sigma)$ i.e., $orb(T_u,f)=\{f,E^{\mathscr{A}}(f\circ\varphi),E^{\mathscr{A}}(f\circ\varphi)\circ\varphi,E^{\mathscr{A}}(f\circ\varphi)\circ\varphi^2\cdots,E^{\mathscr{A}}(f\circ\varphi)\circ\varphi^{m-1}\}$ is a bounded subset. Indeed,

$$\|T_u^n f\|_p \leqslant \|f\|_p \max\{1,\|h\|_\infty^{\frac{1}{p}},\|h\|_\infty^{\frac{2}{p}},\cdots,\|h\|_\infty^{\frac{m-1}{p}}\}.$$

COROLLARY 1. *Suppose that* $\mathscr{A}=\varphi^{-1}\Sigma$ *and* φ *is a periodic* non-singular *transformation. Then*

$$orb(T_u,f)=\{f,E^{\varphi^{-1}\Sigma}(u)f\circ\varphi,E^{\varphi^{-1}\Sigma}(u)E^{\varphi^{-1}\Sigma}(u)\circ\varphi f\circ\varphi^2,\cdots,\prod_{i=0}^{m-1}E^{\varphi^{-1}\Sigma}(u)\circ\varphi^i f\}$$

and hence T_u *is not subspace-hypercyclic with respect to* $L^p(\varphi^{-1}\Sigma)$, *for each* $1\leqslant p<\infty$.

THEOREM 2. *Let $\varphi : X \to X$ be a non-singular and finitely non-mixing transformation and $\varphi^{-1}\mathscr{A} \subseteq \mathscr{A}$. Suppose that $T_u : L^p(\Sigma) \to L^p(\mathscr{A})$ is subspace-hypercyclic with respect to $L^p(\mathscr{A})$. Then for each subset $F \in \mathscr{A}$ with $0 < \mu(F) < \infty$, there exists a sequence of $\mathscr{A}$-measurable sets $\{V_k\} \subseteq F$ such that $\mu(V_k) \to \mu(F)$ as $k \to \infty$, and there is a sequence of integers (n_k) such that*

$$\lim_{k\to\infty} \|(\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i)^{-1}|_{V_k}\|_\infty = 0$$

and

$$\lim_{k\to\infty} \|\sqrt[p]{h_{n_k}^{\mathscr{A}}}[E^{\varphi^{-n_k}(\mathscr{A})}(\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i)]\circ\varphi^{-n_k}|_{V_k}\|_\infty = 0.$$

Proof. Let $F \in \mathscr{A}$ be an arbitrary set with $0 < \mu(F) < \infty$ and let $\varepsilon > 0$ be an arbitrary. A transformation φ is finitely non-mixing and hence, there is an $N \in \mathbb{N}$ such that $F \cap \varphi^n(F) = \emptyset$ for each $n > N$. Choose ε_1 such that $0 < \varepsilon_1 < \frac{\varepsilon}{1+\varepsilon}$. Since the set of all subspace-hypercyclic vectors for T_u, is dense in $L^p(\mathscr{A})$, there exist a subspace-hypercyclic vector $f \in L^p(\mathscr{A})$ and $m \in \mathbb{N}$ with $m > N$ such that

$$\|f - \chi_F\|_p < \varepsilon_1^2 \quad \text{and} \quad \|T_u^m f - \chi_F\|_p < \varepsilon_1^2.$$

Put $P_{\varepsilon_1} = \{t \in F : |f(t) - 1| \geqslant \varepsilon_1\}$ and $R_{\varepsilon_1} = \{t \in X - F : |f(t)| \geqslant \varepsilon_1\}$. Then we have

$$\begin{aligned}
\varepsilon_1^{2p} > \|f-\chi_F\|_p^p &= \int_X |f-\chi_F|^p d\mu \\
&\geqslant \int_{P_{\varepsilon_1}} |f(x)-1|^p d\mu(x) + \int_{R_{\varepsilon_1}} |f(x)|^p d\mu(x) \\
&\geqslant \varepsilon_1^p(\mu(P_{\varepsilon_1}) + \mu(R_{\varepsilon_1})).
\end{aligned}$$

Then, $\max\{\mu(P_{\varepsilon_1}), \mu(R_{\varepsilon_1})\} < \varepsilon_1^p$. Set $S_{m,\varepsilon_1} = \{t \in F : |\prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i f\circ\varphi^m(t) - 1| \geqslant \varepsilon_1\}$ and now consider the following relationships:

$$\begin{aligned}
\varepsilon_1^{2p} &> \|T_u^m f - \chi_F\|_p^p \\
&= \int_X |\prod_{i=0}^{m-2} E^{\mathscr{A}}(u)\circ\varphi^i E^{\mathscr{A}}(uf\circ\varphi)\circ\varphi^{m-1} - \chi_F|^p d\mu \\
&\geqslant \int_{S_{m,\varepsilon_1}} |\prod_{i=0}^{m-2} E^{\mathscr{A}}(u)\circ\varphi^i E^{\mathscr{A}}(uf\circ\varphi)\circ\varphi^{m-1}(t) - 1|^p d\mu(t) \\
&\geqslant \int_{S_{m,\varepsilon_1}} |\prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i f\circ\varphi^m(t) - 1|^p d\mu(t) \\
&\geqslant \varepsilon_1^p \mu(S_{m,\varepsilon_1})
\end{aligned}$$

to deduce that $\mu(S_{m,\varepsilon_1}) < \varepsilon_1^p$. But for an arbitrary $t \in F$, it is readily seen that $\varphi^m(t) \notin F$ because of $F \cap \varphi^{-m}(F) = \emptyset$. Hence, for each $t \in F - (S_{m,\varepsilon_1} \cup \varphi^{-m}(R_{\varepsilon_1}))$, we have

$$|(\prod_{i=0}^{m-1} E^{\mathscr{A}}(u)\circ\varphi^i)^{-1}(t)| < \frac{|f\circ\varphi^m(t)|}{1-\varepsilon_1} < \frac{\varepsilon_1}{1-\varepsilon_1} < \varepsilon.$$

Now, let $U_{m,\varepsilon_1}=\varphi^{-m}(\{t\in F:\sqrt[p]{h_m^{\mathscr{A}}(t)}\ |E^{\varphi^{-m}(\mathscr{A})}(\prod_{i=0}^{m-1}E^{\mathscr{A}}(u)\circ\varphi^i)\circ\varphi^{-m}(t)f(t)|\geqslant\varepsilon_1\})$. Here, we remind that $\prod_{i=0}^{m-1}E^{\mathscr{A}}(u)\circ\varphi^i\circ\varphi^{-m}=\prod_{i=1}^{m}E^{\mathscr{A}}(u)\circ\varphi^{-i}$ on $\sigma(h_m^{\mathscr{A}})$. Use the change of variable formula to obtain that

$$\begin{aligned}\varepsilon_1^{2p}&>\|T_u^m f-\chi_F\|_p^p\\&=\int_X|\prod_{i=0}^{m-1}E^{\mathscr{A}}(u)\circ\varphi^i f\circ\varphi^m-\chi_F|^p d\mu\\&\geqslant\int_X|E^{\varphi^{-m}(\mathscr{A})}(\prod_{i=0}^{m-1}E^{\mathscr{A}}(u)\circ\varphi^i)f\circ\varphi^m-E^{\varphi^{-m}(\mathscr{A})}(\chi_F)|^p d\mu\\&\geqslant\int_{U_{m,\varepsilon_1}}|E^{\varphi^{-m}(\mathscr{A})}(\prod_{i=0}^{m-1}E^{\mathscr{A}}(u)\circ\varphi^i)f\circ\varphi^m|^p d\mu\\&\geqslant\int_{\varphi^m(U_{m,\varepsilon_1})}|E^{\varphi^{-m}(\mathscr{A})}(\prod_{i=0}^{m-1}E^{\mathscr{A}}(u)\circ\varphi^i)\circ\varphi^{-m}f|^p h_m^{\mathscr{A}}d\mu\\&\geqslant\varepsilon_1^p\mu(\varphi^m(U_{m,\varepsilon_1})),\end{aligned}$$

which implies in turn that $\mu(\varphi^m(U_{m,\varepsilon_1}))<\varepsilon_1^p$. That $E^{\varphi^{-m}(\mathscr{A})}(\chi_F)=0$ is concluded of the fact that $F\cap\varphi^{-m}(F)=\emptyset$. Note that for each $t\in F-(\varphi^m(U_{m,\varepsilon_1})\cup P_{\varepsilon_1})$, we have

$$\sqrt[p]{h_m(t)}\ |E^{\varphi^{-m}(\mathscr{A})}(\prod_{i=0}^{m-1}E^{\mathscr{A}}(u)\circ\varphi^i)\circ\varphi^{-m}(t)f(t)|<\frac{\varepsilon_1}{1-\varepsilon_1}<\varepsilon.$$

Finally, put $V_{m,\varepsilon_1}:=F-(P_{\varepsilon_1}\cup\varphi^{-m}(R_{m,\varepsilon_1})\cup S_{m,\varepsilon_1}\cup\varphi^m(U_{m,\varepsilon_1}))$. Then, clearly $\mu(F-V_{m,\varepsilon_1})<4\varepsilon_1^p$, $\|(\prod_{i=0}^{m-1}E^{\mathscr{A}}(u)\circ\varphi^i)^{-1}|_{V_{m,\varepsilon_1}}\|_\infty<\varepsilon$ and

$$\|\sqrt[p]{h_m}[E^{\varphi^{-m}(\mathscr{A})}(\prod_{i=0}^{m-1}E^{\mathscr{A}}(u)\circ\varphi^i)]\circ\varphi^{-m}|_{V_{m,\varepsilon_1}}\|_\infty<\varepsilon.$$

By induction, for each $k\in\mathbb{N}$ we get a measurable subset $V_k\subseteq F$ and an increasing subsequence (n_k) such that $\mu(F-V_k)<4(\frac{1}{k})^p$, $\|(\prod_{i=0}^{n_k-1}E^{\mathscr{A}}(u)\circ\varphi^i)^{-1}|_{V_k}\|_\infty<\varepsilon$ and $\|\sqrt[p]{h_{n_k}^{\mathscr{A}}}[E^{\varphi^{-n_k}(\mathscr{A})}(\prod_{i=0}^{n_k-1}E^{\mathscr{A}}(u)\circ\varphi^i)]\circ\varphi^{-n_k}|_{V_k}\|_\infty<\varepsilon$. □

THEOREM 3. *Let $T_u:L^p(\Sigma)\to L^p(\mathscr{A})$ be bounded with $\sigma(u)=X$, and let φ be a normal and finitely non-mixing transformation provided that $\varphi^{-1}\mathscr{A}\subseteq\mathscr{A}\subseteq\Sigma_\infty$ and $\sup_n\|h_n^{\mathscr{A}\sharp}\|_\infty<\infty$. If for each subset $F\in\mathscr{A}$ with $0<\mu(F)<\infty$, there exists a sequence of $\mathscr{A}$-measurable sets $\{V_k\}\subseteq F$ such that $\mu(V_k)\to\mu(F)$ as $k\to\infty$, and there is a sequence of integers (n_k) such that*

$$\lim_{k\to\infty}\|(\prod_{i=0}^{n_k-1}E^{\mathscr{A}}(u)\circ\varphi^i)^{-1}|_{V_k}\|_\infty=0$$

and

$$\lim_{k\to\infty}\|\sqrt[p]{h_{n_k}}\ [\prod_{i=0}^{n_k-1}E^{\mathscr{A}}(u)\circ\varphi^i]\circ\varphi^{-n_k}|_{V_k}\|_\infty=0,$$

then T_u is subspace-hypercyclic with respect to $L^p(\mathscr{A})$.

Proof. Since, $S^{\mathscr{A}}(X)$ is dense in $L^p(\mathscr{A})$, we may take $D_1 = D_2 = S^{\mathscr{A}}(X)$ in the subspace-hypercyclicity's criterion. For an arbitrary $f \in S^{\mathscr{A}}(X)$, one can easily find $\{V_k\} \subseteq \sigma(f)$ such that $\mu(V_k) \to \mu(\sigma(f))$ and finds an N_1 such that $\sigma(f) \cap \varphi^n(\sigma(f)) = \emptyset$ for each $n > N_1$. Now, for each $n_k > N_1$ define the vector $f_k = \frac{f\circ\varphi^{-n_k}}{[\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i]\circ\varphi^{-n_k}}$. Since $\varphi^{-1}\mathscr{A} \subseteq \mathscr{A} \subseteq \Sigma_\infty$, then $f_k \in L^p(\mathscr{A})$ and the simple computations show that $T_u^{n_k} f_k = f$. Now, we will show that $\|T_u^{n_k} f\|_p \to 0$ and $\|f_k\|_p \to 0$ as $k \to \infty$. For an arbitrary $\varepsilon > 0$, there exist $M, N_1 \in \mathbb{N}$, sufficiently large such that $V_{N_1} \subseteq \sigma(f)$ and

$$\mu(\sigma(f) - V_{N_1}) < \frac{\varepsilon}{M\|f\|_\infty^p}.$$

By Egoroff's theorem, there exists an N_2 such that for each $n_k > N_2$, $\| \sqrt[p]{h_{n_k}^{\mathscr{A}}}[\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u) \circ\varphi^i]\circ\varphi^{-n_k}\|_\infty^p < \frac{\varepsilon}{\|f\|_\infty^p}$ on V_{N_1}. So, there exists a non-negative real number M such that $\| \sqrt[p]{h_{n_k}^{\mathscr{A}}}[\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i]\circ\varphi^{-n_k}\|_\infty^p \leqslant M < \infty$ on $\sigma(f)$. Now, by the change of variable formula, for each $n_k > N = \max\{N_1, N_2\}$ we have

$$\begin{aligned}
\|T_u^{n_k} f\|_p^p &= \int_X |\prod_{i=0}^{n_k-2} E^{\mathscr{A}}(u)\circ\varphi^i E^{\mathscr{A}}(uf\circ\varphi)\circ\varphi^{n_k-1}|^p d\mu \\
&= \int_X |\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i f\circ\varphi^{n_k}|^p d\mu \\
&= \int_{\sigma(f)} |\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i\circ\varphi^{-n_k} f|^p h_{n_k} d\mu \\
&= \int_{\sigma(f)-V_N} |\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i\circ\varphi^{-n_k} f|^p h_{n_k} d\mu \\
&\quad + \int_{V_N} |\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i\circ\varphi^{-n_k} f|^p h_{n_k} d\mu \\
&< \| \sqrt[p]{h_{n_k}} \prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i\circ\varphi^{-n_k}\|_\infty^p \|f\|_\infty^p\, \mu(\sigma(f) - V_N) \\
&\quad + \frac{\varepsilon}{\|f\|_\infty^p}\|f\|_\infty^p < 2\varepsilon.
\end{aligned}$$

By taking into account that $\sup_n \|h_n^{\mathscr{A}\sharp}\|_\infty < \infty$, we have

$$\begin{aligned}
\lim_{k\to\infty} \|f_k\|_p^p &= \lim_{k\to\infty} \int_X \left|\frac{f\circ\varphi^{-n_k}}{\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i\circ\varphi^{-n_k}}\right|^p d\mu \\
&= \lim_{k\to\infty} \int_{\sigma(f)} \left|\frac{f}{\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i}\right|^p h_{n_k}^\sharp d\mu
\end{aligned}$$

$$\leqslant \sup_k \|h_{n_k}^{\mathscr{A}\sharp}\|_\infty \Big(\lim_{k\to\infty}\int_{\sigma(f)-V_N} \Big|\frac{f}{\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i}\Big|^p d\mu$$
$$+\lim_{k\to\infty}\int_{V_N} \Big|\frac{f}{\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i}\Big|^p d\mu\Big)$$
$$=0.$$

Finally, it is clear that $T_u^{n_k} L^p(\mathscr{A}) \subseteq L^p(\mathscr{A})$ for all $k \in \mathbb{N}$, because of $\varphi^{-1}\mathscr{A} \subseteq \mathscr{A}$ and hence T_u satisfies in the subspace-hypercyclicity criterion and is subspace-hypercyclic. □

PROPOSITION 1. Suppose that $\varphi : X \to X$ is a normal and finitely non-mixing transformation with $\varphi^{-1}(\mathscr{A}) \subseteq \mathscr{A} \subseteq \Sigma_\infty$. Let $\sup_n \|h_n^\sharp\|_\infty < \infty$ and $\sigma(u) = X$. Then the following conditions are equivalent:

(i) T_u satisfies the subspace-hypercyclic criterion.

(ii) T_u is subspace-hypercyclic with respect to $L^p(\mathscr{A})$.

(iii) $T_u \oplus T_u$ is subspace-hypercyclic with respect to $L^p(\mathscr{A}) \oplus L^p(\mathscr{A})$.

(iv) T_u is subspace-weakly mixing.

Proof. $(i) \Rightarrow (ii)$. Note that if an operator satisfies the subspace-hypercyclic criterion, then it is subspace-transitive and hence is subspace-hypercyclic [25, Theorem 3.5]. For the implication $(ii) \Rightarrow (iii)$, we show that $T_u \oplus T_u$ is subspace-topologically transitive, according [25, Theorem 3.3]. To begin, pick two pairs of non-empty open sets (A_1, B_1) and (A_2, B_2) in $L^p(\mathscr{A}) \oplus L^p(\mathscr{A})$ arbitrarily. For $j = 1,2$, choose the functions $f_j, g_j \in S^{\mathscr{A}}(X)$ with $f_j \in A_j$ and $g_j \in B_j$. Let $F = \sigma(f_1) \cup \sigma(f_2) \cup \sigma(g_1) \cup \sigma(g_2)$. Then $\mu(F) < \infty$. Assume that $\{V_k\} \subseteq F$, $\{(\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i)^{-1}\}$ and $\{\sqrt[p]{h_{n_k}^{\mathscr{A}}}\, E^{\varphi^{-n_k}(\mathscr{A})}(\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i)\circ\varphi^{-n_k}\}$ are as provided by Theorem 2. There is an $N_1 \in \mathbb{N}$, such that for all $n > N_1$, $F \cap \varphi^n(F) = \emptyset$. Moreover, for each $\varepsilon > 0$ there exists $N_2 \in \mathbb{N}$, such that for each $k > N_2$ and $n_k > N_1$, $\| \sqrt[p]{h_{n_k}^{\mathscr{A}}}\, E^{\varphi^{-n_k}(\mathscr{A})}(\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i)\circ\varphi^{-n_k}|_{V_k}\|_\infty^p < \frac{\varepsilon}{\|f_j\|_p^p}$ on V_k. Hence, for $k > N_2$, we get that

$$\|T_u^{n_k}(f_j\chi_{V_k})\|_p^p = \int_X |T_u^{n_k}(f_j\chi_{V_k})|^p d\mu$$
$$= \int_X \Big|\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i (f_j\chi_{V_k})\circ\varphi^{n_k}\Big|^p \, d\mu$$
$$= \int_{V_k} \Big|[\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i]\circ\varphi^{-n_k} f_j\Big|^p h_{n_k} d\mu < \varepsilon.$$

Now, define a map $D_\varphi(f) = \frac{f\circ\varphi^{-1}}{E^{\mathscr{A}}(u)\circ\varphi^{-1}}$ on the subspace $S^{\mathscr{A}}(X)$. Then for each $f \in S^{\mathscr{A}}(X)$, $T_u^{n_k} D_\varphi^{n_k}(f) = f$. Again, we may find an $N_3 \in \mathbb{N}$ such that for each $k > N_3$ and

$n_k > N_1$, $\|(\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i)^{-1}\|_\infty^p < \frac{\varepsilon}{M\|g_j\|_\infty^p}$ on V_k, where $M = \sup_n \|h_n^{\mathscr{A}\sharp}\|_\infty < \infty$. On the other hand, for each $k > N_3$ note that

$$\|D_\varphi^{n_k}(g_j\chi_{V_k})\|_p^p = \int_{\varphi^{n_k}(V_k)} \Big|\frac{g_j\circ\varphi^{-n_k}}{[\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i]\circ\varphi^{-n_k}}\Big|^p d\mu$$
$$= \int_{V_k} \Big|\frac{g_j}{\prod_{i=0}^{n_k-1} E^{\mathscr{A}}(u)\circ\varphi^i}\Big|^p\, h_n^\sharp d\mu < \varepsilon.$$

For each $k\in\mathbb{N}$, let $f_{j,k}^\natural = f_j\chi_{V_k} + D_\varphi^{n_k}(g_j\chi_{V_k})$. Then we have $f_{j,k}^\natural \in L^p(\mathscr{A})$,

$$\|f_{j,k}^\natural - f_j\|_p^p \leqslant \|f_j\|_\infty^p\,\mu(F-V_k) + \|D_\varphi^{n_k}(g_j\chi_{V_k})\|_p^p$$

and

$$\|T_u^{n_k} f_{j,k}^\natural - g_j\|_p^p \leqslant \|g_j\|_\infty^p\,\mu(F-V_k) + \|T_u^{n_k}(f_j\chi_{V_k})\|_p^p.$$

Hence, $\lim_{k\to\infty} f_{j,k}^\natural = f_j$, $\lim_{k\to\infty} T_u^{n_k} f_{j,k}^\natural = g_j$ and $T_u^{n_k}(A_j)\cap B_j \neq \emptyset$ for some $k\in\mathbb{N}$. Moreover, since $\varphi^{-1}(\mathscr{A})\subseteq\mathscr{A}$ then $T_u^{n_k}(L^p(\mathscr{A}))\subseteq L^p(\mathscr{A})$. So $T_u\oplus T_u$ is subspace-hypercyclic on $L^p(\mathscr{A})\oplus L^p(\mathscr{A})$.

To prove the implication $(iv)\Rightarrow(i)$, we use Bès-Peris's approach stated in [6, Theorem 4.2]. Assume that $T_u\oplus T_u$ is subspace-hypercyclic on $L^p(\mathscr{A})\oplus L^p(\mathscr{A})$ with subspace-hypercyclic vector $f\oplus g$. Note that for each $n\in\mathbb{N}$, the operator $I\oplus T_u^n$ has dense range and commutes with $T_u\oplus T_u$, therefore $orb(I\oplus T_u^n, f\oplus g) = (I\oplus T_u^n)orb(T_u\oplus T_u, f\oplus g)$. Eventually $f\oplus T_u^n g$ is subspace-hypercyclic vector as well. We show that the subspace-hypercyclic criterion is satisfied by $D_1 = D_2 = orb(T_u\oplus T_u, f\oplus g)$. Let U be an arbitrary open neighborhood of 0 in $L^p(\mathscr{A})$. Hence, one can find a sequence $(g_k)\subseteq U$ and an increasing sequence of integers (n_k) such that $T_u^{n_k} f\oplus T_u^{n_k} g_k \to 0\oplus g$ and $g_k\to 0$. Clearly, $T_u^{n_k}(L^p(\mathscr{A}))\subseteq L^p(\mathscr{A})$. □

COROLLARY 2. *Under the assumptions of Proposition 1, the following conditions are equivalent:*

(i) T_u is subspace-topologically mixing on $L^p(\mathscr{A})$.

(ii) For each $\mathscr{A}$-measurable subset $F\subseteq X$ with $0<\mu(F)<\infty$, there exists a sequence of $\mathscr{A}$-measurable sets $\{V_n\}\subseteq F$ such that $\mu(V_n)\to\mu(F)$ as $n\to\infty$ and $\lim_{n\to\infty}\|(\prod_{i=0}^{n-1} E^{\mathscr{A}}(u)\circ\varphi^i)^{-1}|_{V_n}\|_\infty = \lim_{n\to\infty}\|\sqrt[p]{h_n^{\mathscr{A}}}(\prod_{i=0}^{n-1} E^{\mathscr{A}}(u)\circ\varphi^i\circ\varphi^{-n})|_{V_n}\|_\infty = 0$.

Proof. By Theorem 3 and Proposition 1 the implication $(ii)\Rightarrow(i)$ is established, just use the full sequences instead of subsequences. For the implication $(i)\Rightarrow(ii)$, let $\varepsilon>0$ and $F\in\mathscr{A}$ with $0<\mu(F)<\infty$ be arbitrary. Consider a non-empty and open subset $U = \{f\in L^p(\mathscr{A}) : \|f-\chi_F\|_p<\varepsilon\}$. Since T_u is subspace-topologically mixing and φ is finitely non-mixing, one may find $N\in\mathbb{N}$ such that for all $n>N$, $T_u^n(U)\cap U\neq\emptyset$ and $F\cap\varphi^n(F)=\emptyset$. Hence, for each $n>N$, we can choose a function $f_n\in U$ such that $T_u^n f_n\in U$. Then $\|f_n-\chi_F\|_p<\varepsilon$ and $\|T_u^n f_n-\chi_F\|_p<\varepsilon$. The rest of the proof can be proceed like as Theorem 2. □

EXAMPLE 1. Let $X=\mathbb{R}$ be the real line with Lebesgue measure μ on the σ-algebra Σ of all Lebesgue measurable subsets of $\mathbb{R}$. Let $\mathscr{A}$ be the σ-subalgebra generated by the symmetric intervals about the origin. For a positive real number t define the transformation $\varphi:\mathbb{R}\to\mathbb{R}$ by $\varphi(x)=x+t,\ x\in\mathbb{R}$. Clearly, $\varphi^{-1}\mathscr{A}\subseteq\mathscr{A}\subseteq\Sigma_\infty$ and in this setting, $E^{\mathscr{A}}(f)=\frac{f(x)+f(-x)}{2}$, which is the even part of $f\in L^p(\Sigma)$. Fix $r>1$ and define the weight function u on $\mathbb{R}$ by

$$u(x)=\begin{cases}2x+r, & 1\leqslant x,\\ -x^2-\frac{x}{2}+2, & -1<x<1,\\ x^3+\frac{1}{r}, & x\leqslant -1.\end{cases}$$

Then, we have

$$E^{\mathscr{A}}(u)(x)=\begin{cases}r, & 1\leqslant x,\\ -\frac{x^2}{2}+2, & -1<x<1,\\ \frac{1}{r}, & x\leqslant -1.\end{cases}$$

For an arbitrary $F=[-a,a]$, take $V_k=(-a+\frac{1}{k},a-\frac{1}{k})$. In this case, one may easily find a sequence (n_k) such that both quantities $\|(\prod_{i=0}^{n_k-1}E^{\mathscr{A}}(u)\circ\varphi^i)^{-1}|_{V_k}\|_\infty$ and $\|\sqrt[p]{h_{n_k}^{\mathscr{A}}}\ [\prod_{i=0}^{n_k-1}E^{\mathscr{A}}(u)\circ\varphi^i]\circ\varphi^{-n_k}|_{V_k}\|_\infty$ tend zero as $k\to\infty$. Because, $h_{n_k}^{\mathscr{A}}=h_{n_k}^{\mathscr{A}^\sharp}=1$ and $[\prod_{i=0}^{n_k-1}E^{\mathscr{A}}(u)\circ\varphi^i]\circ\varphi^{-n_k}=\prod_{i=1}^{n_k}E^{\mathscr{A}}(u)\circ\varphi^{-i}$, since φ is onto (or $\sigma(h_{n_k}^{\mathscr{A}})=\mathbb{R}$). Therefore, by Theorem 3, T_u is subspace-hypercyclic with respect to $L^p(\mathscr{A})$ while it is not hypercyclic on $L^p(\Sigma)$ [5, Theorem 2.3]. For this, just consider that $\|\sqrt[p]{h_{n_k}}[E_{n_k}(\prod_{i=0}^{n_k-1}u\circ\varphi^i)]\circ\varphi^{-n_k}|_{V_k}\|_\infty=\|\prod_{i=1}^{n_k}u\circ\varphi^{-i}|_{V_k}\|_\infty\nrightarrow 0$.

Conflict of interest statement

The authors have no conflicts of interest to declare. We certify that the submission is original work and is not under review at any other publication.

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